SCHUR FUNCTION AT GENERAL POINTS AND LIMIT SHAPE OF PERFECT MATCHINGS ON CONTRACTING SQUARE HEXAGON LATTICES WITH PIECEWISE BOUNDARY CONDITIONS

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Abstract. We obtain a new formula to relate the value of a Schur polynomial with variables $(x_1, \ldots, x_N)$ with values of Schur polynomials at $(1, \ldots, 1)$. This allows to study the limit shape of perfect matchings on a square hexagon lattice with periodic weights and piecewise boundary conditions. In particular, when the edge weights satisfy certain conditions, asymptotics of the Schur function imply that the liquid region of the model in the scaling limit has multiple connected components, while the frozen boundary consists of disjoint cloud curves.

1. Introduction

Schur polynomials, named after Issai Schur, are a class of symmetric polynomials indexed by decreasing sequences of non-negative integers, which form a linear basis for the space of all symmetric polynomials; see [27]. Besides their applications in representation theory, Schur polynomials also play an important role in the study of integrable lattice models in statistical mechanics (see [1, 2]). One example of such a model is the dimer model, or equivalently, random tiling model.

A *dimer configuration*, or a *perfect matching*, is a subset of the set of edges of a graph in which each vertex is incident to exactly one edge. A *two-dimensional dimer model* is a probability measure on dimer configurations of a plane graph. Two-dimensional dimer models are exactly solvable models, in the sense that one can exactly compute the number of configurations and the local statistics by algebraic methods. Such a property and the connection of this model with several other models in statistical mechanics, including the Ising model ([9, 24, 22]) and the 1-2 model ([23, 25, 11, 12]) put the dimer model at the intersection of several branches of mathematics (probability, combinatorics, representation theory, algebraic geometry), as well as statistical physics and computer science.

The weighted dimer model has been studied extensively by developing the techniques initiated by Temperley, Fisher and Kasteley ([14, 15]) and analyzing the weighted adjacency matrix of the underlying graph, and spectacular results were obtained including the phase transition ([21, 19]), conformal invariance ([16, 17, 18, 26]), and the limit shape ([30, 20]). Recently the uniform dimer models on the hexagonal lattice or the square grid were studied by analyzing Schur polynomials. As a determinantal process, the correlation kernel for the uniform dimer model can be computed explicitly as a double integral (see [28, 32]) - this implies the limit shape result (law of large numbers; see [31]) and the convergence of height fluctuations to a Gaussian free field (central limit theorem; see [32]) in the scaling limit. The asymptotics of Schur polynomials in a neighborhood of $(1, \ldots, 1)$ were studied in ([10, 7, 6]), and the limit shape and height fluctuations were obtained for the uniform dimer model on the hexagonal lattice, the uniform dimer model on the square grid ([8]), and certain periodic dimer model on the square-hexagon lattice with period of edge weights $1 \times n$ ([5]).

The main aim of this paper is to study questions concerning limit shapes of two-dimensional dimer models. More precisely, to each random perfect matching we associate a height function - a function that assigns an integer to each face of the plane graph. When the plane graphs become larger and larger, we rescale these graphs such that the rescaled graphs approximate a certain simply-connected domain in the plane, then evidence has been amassed that the rescaled height functions are governed by "laws of large numbers", and converge to some naturally defined shapes. These questions have origins from the observations that the uniform random domino tilings of a large Aztec diamond (a subgraph of the 2D square grid consisting of all squares whose centers (x, y) satisfy $|x| + |y| \leq n$) tends to be non-random outside a circle tangent to the boundary of the graph. This circle is called the "arctic circle", which is an example of a frozen boundary.

This paper is a continuation of [5]. In [5], we studied the $1 \times n$ periodic dimer model on the square-hexagon lattice where the boundary condition is also periodic in the sense that each remaining vertex on the boundary is followed by $(m-1)$ removed vertices, where $m \geq 1$ is a fixed positive integer. One difference between the uniform and the $1 \times n$ periodic dimer model is when computing their partition function (weighted sum of all the configurations), the former can computed by the value of Schur functions at $(1, \ldots, 1)$, and the later can be computed by the value of Schur function at a point depending on edge weights. When the boundary condition satisfies the condition that each remaining vertex on the boundary is followed by $(m - 1)$ removed vertices, there is an explicit formula to compute the corresponding Schur function at a generic point. The dimer model on similar graphs were also studied in [3, 4].

In this paper, we study the dimer model on a contracting square-hexagon lattice with piecewise boundary conditions. More precisely, the boundary can be divided into finitely many segments; each segment consists of either only remaining vertices or only removed vertices; the segments consisting of only remaining vertices and the segments consisting of only removed vertices are alternate; the length of each segment grows linearly as the size of the graph grows. The main tool used to study such a model is a formula we obtained to relate the value of a Schur function at a generic point to the values of Schur functions at $(1, \ldots, 1)$, which gives the asymptotics of the Schur function at a generic point when the boundary condition is piecewise and the edge weights are periodic by finding a leading term in the formula. When the edge weights satisfy certain conditions, from the asymptotics of the Schur function, we obtain the surprising results that the liquid region of the model in the scaling limit has multiple connected components, whose boundary consists of disjoint cloud curves.

The organization of the paper is as follows. In Section 2, we review the definitions and summarize the main results proved in the paper. In Section 3, we prove a combinatorial formula which relate the value of a Schur function at a generic point to the values of Schur functions at $(1, \ldots, 1)$. In Section 4, we study the asymptotics of Schur polynomials at a generic point by analyzing the combinatorial formula proved in Section 3. In Section 5, we obtain an explicit integral formula for the moments of the limit of counting measure for the $1 \times n$ periodic dimer model on a contracting square-hexagon lattice with piecewise boundary conditions. In Section 6, we obtain the limit shape of the height function of the model. In Section 7, we discuss the existence of the frozen region, which is the region where each type of edges has either probability 0 or probability 1 to occur. For certain special cases, we find explicitly the parametric equation of frozen boundary (which is the boundary of the frozen region), and show that the frozen boundary is a union of n disjoint cloud curves, where n is the size of a period. In Section 8, we give concrete examples to illustrate combinatorial formulas to compute Schur functions proved in Section 3.

2. Main Results

In this section, we define the main objects to be studied in this paper, including the Schur function, the square-hexagon lattice and the perfect matching. Then we state the main results of this paper.

2.1. **Partitions, counting measure and Schur functions.**

Let l be a positive integer. Throughout this paper, we shall use the following notation:

$$[l] = \{1, 2, \ldots, l\}$$

Definition 2.1. *A* partition *of length N is a sequence of nonincreasing, nonnegative integers $\mu = (\mu_1 \geq \mu_2 \geq \ldots \geq \mu_N \geq 0)$. Each μ_k is a* component *of the partition μ. The length N of the partition μ is denoted by $l(\mu)$. The* size *of a partition μ is*

$$|\mu| = \sum_{i=1}^{N} \mu_i.$$

We denote by $\mathbb{GT}_N^+$ the subset of length-N partitions.

A graphic way to represent a partition μ is through its *Young diagram* Y_μ, a collection of $|\mu|$ boxes arranged on non-increasing rows aligned on the left: with μ_1 boxes on the first row, μ_2 boxes on the second row,... μ_N boxes on the Nth row. Some rows may be empty if the corresponding μ_k is equal to 0. The correspondence between partitions of length N and Young diagrams with N (possibly empty) rows is a bijection.

Definition 2.2. *Let Y, W be two Young diagrams. We say that $Y \subset W$* differ by a horizontal strip *if the collection of boxes in $Z = W \setminus Y$ contains at most one box in every column. We say that they* differ by a vertical strip *if Z contains at most one box in every row.*

We say that two non-negative signatures λ and μ interlace, *and write $\lambda \prec \mu$ if $Y_\lambda \subset Y_\mu$ differ by a horizontal strip. We say they* cointerlace *and write $\lambda \prec' \mu$ if $Y_\lambda \subset Y_\mu$ differ by a vertical strip.*

Definition 2.3. *Let $\lambda \in \mathbb{GT}_N^+$. The* rational Schur function *s_λ associated to λ is the homogeneous symmetric function of degree $|\lambda|$ in N variables defined as follows*

(1) If $N = 1$, and $\lambda = (\lambda_1)$ then

$$s_\lambda(u_1) = u_1^{\lambda_1}.$$

(2) If $N \geq 2$, and $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N)$, then

$$s_\lambda(u_1, \ldots, u_N) = \frac{\det_{i,j=1,\ldots,N}(u_i^{\lambda_j+N-j})}{\prod_{1\leq i<j\leq N}(u_i - u_j)}. \tag{2.1}$$

The Schur function defined by (2.1) is a symmetric function because the numerator and denominator are both alternating, and a polynomial since all alternating polynomials are divisible by the Vandermonde determinant.

Let $\lambda \in \mathbb{GT}_N^+$ be a partition of length N. We define the *counting measure* $m(\lambda)$ corresponding to λ as follows.

$$m(\lambda) = \frac{1}{N}\sum_{i=1}^{N} \delta\left(\frac{\lambda_i + N - i}{N}\right). \tag{2.2}$$

Let $\lambda(N) \in \mathbb{GT}_N^+$. Let Σ_N be the permutation group of N elements and let $\sigma \in \Sigma_N$. Let

$$X = (x_1, \ldots, x_N).$$

Assume that there exists n between 1 and N such that $x_1, ..., x_n$ are pairwise distinct and $\{x_1, ..., x_n\} = \{x_1, ..., x_N\}$. For $j \in [N]$, let

$$\eta_j^\sigma(N) = |\{k : k > j, x_{\sigma(k)} \neq x_{\sigma(j)}\}|. \tag{2.3}$$

For $1 \leq i \leq n$, let

$$\Phi^{(i,\sigma)}(N) = \{\lambda_j(N) + \eta_j^\sigma(N) : x_{\sigma(j)} = x_i\} \tag{2.4}$$

and let $\phi^{(i,\sigma)}(N)$ be the partition with length $|\{1 \leq j \leq N : x_j = x_i\}|$ obtained by decreasingly ordering all the elements in $\Phi^{(i,\sigma)}(N)$. Let Σ_N^X be the subgroup Σ_N that preserves the value of X; more precisely

$$\Sigma_N^X = \{\sigma \in \Sigma_N : x_{\sigma(i)} = x_i, \text{ for } i \in [N]\}.$$

Let $[\Sigma/\Sigma_N^X]^r$ be the collection of all the right cosets of Σ_N^X in Σ_N. More precisely,

$$[\Sigma/\Sigma_N^X]^r = \{\Sigma_N^X \sigma : \sigma \in \Sigma_N\},$$

where for each $\sigma \in \Sigma_N$

$$\Sigma_N^X \sigma = \{\xi\sigma : \xi \in \Sigma_N^X\}$$

and $\xi\sigma \in \Sigma_N$ is defined by

$$\xi\sigma(k) = \xi(\sigma(k)), \text{ for } k \in [N].$$

Below is a combinatorial formula which relates the value of a Schur function at a general point to the values of Schur functions at $(1, \ldots, 1)$. The formula will be used to study limit shape of perfect matchings on a square-hexagon lattice, and moreover, the formula may also be of independent interest.

Theorem 2.4. *Under the assumptions above, the Schur function can be computed by the following formula*

$$\begin{aligned} s_\lambda(x_1, \ldots, x_N) \quad = \quad & \sum_{\overline{\sigma} \in [\Sigma_N/\Sigma_N^X]^r} \left(\prod_{i=1}^{n} x_i^{|\phi^{(i,\sigma)}(N)|}\right)\left(\prod_{i=1}^{n} s_{\phi^{(i,\sigma)}(N)}(1, \ldots, 1)\right) \\ & \times \left(\prod_{i<j, x_{\sigma(i)} \neq x_{\sigma(j)}} \frac{1}{x_{\sigma(i)} - x_{\sigma(j)}}\right) \end{aligned} \tag{2.5}$$

where $\sigma \in \overline{\sigma} \cap \Sigma_N$ is a representative.

Theorem 2.4 will be proved in Section 3; it can also lead to asymptotic results for Schur functions at a general point $(x_1, \ldots, x_N)$; see Section 4. In the appendix, we give concrete examples to verify Theorem 2.4. It is straightforward to check that when $N \geq 2$ and $x_1, \ldots, x_N$ are pairwise distinct, the righthand side of (2.5) recovers (2.1).

For simplicity, we make the following assumptions.

Assumption 2.5. *Let $(x_1, \ldots, x_N)$ be an N-tuple of real numbers at which we evaluate the Schur polynomial.*

- *$x_1 > x_2 > \ldots > x_n$; and*
- *N is an integral multiple of n; and.*
- *$\{x_i\}_{i=1}^N$ are periodic with period n, i.e., $x_i = x_j$ for $1 \leq i, j \leq N$ and $[i \mod n] = [j \mod n]$.*

Let $\overline{\sigma}_0 \in [\Sigma_N/\Sigma_N^X]^r$ be the unique element in $[\Sigma_N/\Sigma_N^X]^r$ satisfying the condition that for any representative $\sigma_0 \in \overline{\sigma}_0$, we have

$$x_{\sigma_0(1)} \geq x_{\sigma_0(2)} \geq \ldots \geq x_{\sigma_0(N)}. \tag{2.6}$$

Assumption 2.6. *Assume $x_1, \ldots, x_N$ satisfy Assumption 2.5.*

Let $s \in [N]$. Assume there exists positive integers $K_1, K_2, \ldots K_s$, such that

(1) $\sum_{t=1}^s K_t = N$;

(2)

$$\mu_1 > \ldots > \mu_s \tag{2.7}$$

are all the distinct elements in $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$.

(3)

$$\begin{aligned}
&\lambda_1 = \lambda_2 = \ldots = \lambda_{K_s} = \mu_1;\\
&\lambda_{K_s+1} = \lambda_{K_s+2} = \ldots = \lambda_{K_s+K_{s-1}} = \mu_2;\\
&\ldots\\
&\lambda_{\sum_{t=2}^s K_t} = \lambda_{1+\sum_{t=2}^s K_t} = \ldots = \lambda_{\sum_{t=1}^s K_t} = \mu_s;
\end{aligned}$$

(4) Let

$$J_i = \{t \in [s] : \exists p \in [n], \text{ s.t. } x_{\sigma_0(p)} = x_i, \text{and } \lambda_p = \mu_t\} \tag{2.8}$$

- *(a) If $1 \leq i < j \leq n$, $l \in J_i$, and $t \in J_j$, then $l < t$.*
- *(b) For any p, q satisfying $1 \leq p \leq s$ and $1 \leq q \leq s$, and $q > p$*
$$C_1 N \leq \mu_p - \mu_q \leq C_2 N$$
where C_1, C_2 are constants independent of N.
- *(c) s and n are fixed as $N \to \infty$.*

Assumption 2.6(4)(a) may also be interpreted as follows. First of all, we note the following elementary lemma:

Lemma 2.7. *Let J_i be defined as in (2.8). Then for any $1 \leq i < j \leq n$, $l \in J_i$ and $t \in J_j$, we have $l \leq t$.*

Proof. By Assumption 2.5, if $i < j$, then $x_i > x_j$. For any $l \in J_i$ and $t \in J_j$, there exists $1 \leq p \leq N$ and $1 \leq q \leq N$, such that

$$x_{\sigma_0}(p) = x_i > x_j = x_{\sigma_0}(q);$$

and

$$\lambda_p = \mu_l; \qquad \lambda_q = \mu_t$$

By the definition of σ_0 in (2.6), we obtain $p < q$. By the definition of partition we have $\lambda_p \geq \lambda_q$, and therefore $\mu_l \geq \mu_t$. By (2.7) $l \leq t$. □

Assumption 2.6(4)(a) actually assumes the strict inequality $l < t$ when l, t satisfy the conditions of Lemma 2.7. We write $\lambda_1, \ldots, \lambda_N$ in decreasing order, and $x_{\sigma_0(1)}, \ldots, x_{\sigma_0(N)}$ in decreasing order, and obtain a $2 \times N$ array as follows:

$$\begin{array}{cccc} \lambda_1 & \lambda_2 & \ldots & \lambda_N \\ x_{\sigma_0(1)} & x_{\sigma_0(2)} & \cdots & x_{\sigma_0(N)} \end{array}. \tag{2.9}$$

In the 1st row, there are exactly s distinct values; while in the 2nd row, there are exactly n distinct values. From (2.6) and Assumption 2.5 we can see that the 2nd row of (2.9) is the same as

$$x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_n \ldots, x_n$$

where the 1st $\frac{N}{n}$ entries are x_1's, the next $\frac{N}{n}$ entries are x_2's, and so on. Similarly, by Assumption (2.6), we can see that the 1st row of (2.9) is the same as

$$\mu_1, \ldots, \mu_1, \mu_2, \ldots, \mu_2 \ldots, \mu_s, \ldots, \mu_s,$$

where the 1st K_s entries are μ_1's, the next K_{s-1} entries are μ_2's, etc. Then J_i consists of all the indices $t \in [s]$ such that there exists a value μ_t in the 1st row of (2.9), which is in the same column as a value x_i in the second row. Then Assumption 2.6(4)(a) says that for any $j \in [N-1]$, if $x_{\sigma_0(j)} > x_{\sigma_0(j+1)}$, then $\lambda_j > \lambda_{j+1}$. In other words, no index $t \in [s]$ can appear in more than one J_i's for $i \in [n]$; or the collection of sets $\{J_i\}_{i\in[n]}$ are pairwise disjoint. In particular, this implies that $s \geq n$.

As we shall see later in Lemma 4.8, under Assumptions 2.5 and 2.6, as $N \to \infty$, the counting measures of $\phi^{(i,\sigma_0)}(N)$ converges weakly to a limit measure $\mathbf{m}_i$.

Let

$$H_{\mathbf{m}_i}(u) = \int_0^{\ln(u)} R_{\mathbf{m}_i}(t)dt + \ln\left(\frac{\ln(u)}{u-1}\right) \tag{2.10}$$

and $\mathbf{R}_{\mathbf{m}_i}$ is the Voiculescu R-transform of $\mathbf{m}_i$ given by

$$R_{\mathbf{m}_i} = \frac{1}{S^{(-1)}_{\mathbf{m}_i}(z)} - \frac{1}{z};$$

Where $S_{\mathbf{m}_i}$ is the moment generating function for $\mathbf{m}_i$ given by

$$S_{\mathbf{m}_i}(z) = z + M_1(\mathbf{m}_i)z^2 + M_2(\mathbf{m}_i)z^3 + \ldots; \tag{2.11}$$

$M_k(\mathbf{m}_i) = \int_{\mathbb{R}} x^k \mathbf{m}_i(dx)$; and $S^{-1}_{\mathbf{m}_i}(z)$ is the inverse series of $S_{\mathbf{m}_i}(z)$. See also Section 2.2 of [7] for details.

We may further make the assumptions below

Assumption 2.8. *Assume $x_{1,N} = x_1 > 0$ and $(x_{2,N}, \ldots, x_{n,N})$ changes with N. Assume that for each fixed N, $(x_{1,N}, \ldots, x_{n,N})$ satisfy Assumption 2.5. Suppose that Assumption 2.6 holds. Moreover, assume that*

$$\liminf_{N\to\infty} \frac{\log\left(\min_{1\leq i<j\leq n} \frac{x_{i,N}}{x_{j,N}}\right)}{N} \geq \alpha > 0,$$

where α is a sufficiently large positive constant independent of N.

Theorem 2.9. *Under Assumptions 2.8 and 2.6, for each given* $\{a_i, b_i\}_{i=1}^n$*, when* α *in Assumption 2.8 is sufficiently large,* $u_1, \ldots, u_k$ *are in an open complex neighborhood of* 1*, we have*

$$\lim_{N\to\infty} \frac{1}{N} \log \frac{s_{\lambda(N)}(u_1 x_{1,N}, \ldots, u_k x_{k,N}, x_{k+1,N}, \ldots, x_{N,N})}{s_{\lambda(N)}(x_{1,N}, \ldots, x_{N,N})} = \sum_{i=1}^{k} [Q_i(u_i)] \tag{2.12}$$

where for $1 \leq i \leq k$*,*

(1) if $[i \mod n] \neq 0$*,*

$$Q_i(u) = \frac{H_{\mathbf{m}_{i \mod n}}(u)}{n} - \frac{(n - [i \mod n]) \log(u)}{n}.$$

(2) if $[i \mod n] = 0$*,*

$$Q_i(u) = \frac{H_{\mathbf{m}_n}(u)}{n}.$$

Moreover, the convergence of (2.12) is uniform when $u_1, \ldots, u_k$ *are in an open complex neighborhood of* 1*.*

Theorem 2.9 is proved in Section 4.

2.2. **Square-hexagon lattice.** Consider a doubly-infinite binary sequence indexed by integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

$$\check{a} = (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots) \in \{0, 1\}^{\mathbb{Z}}. \tag{2.13}$$

We now define a bipartite plane graph $\mathrm{SH}(\check{a})$, called *whole-plane square-hexagon lattice* associated with the sequence $\check{a}$. The vertex set of $\mathrm{SH}(\check{a})$ is a subset of $\frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2}$. Each vertex of $\mathrm{SH}(\check{a})$ is either black or white, and we identify the vertices with points on the plane. For $m \in \mathbb{Z}$, the black vertices have y-coordinate m; while the white vertices have y-coordinate $m - \frac{1}{2}$. We will label all the vertices with y-coordinate t $\left(t \in \frac{\mathbb{Z}}{2}\right)$ as vertices in the $(2t)$th row. We further require that for each $m \in \mathbb{Z}$,

- each black vertex on the $(2m)$th row is adjacent to two white vertices in the $(2m+1)$th row; and
- if $a_m = 1$, each white vertex on the $(2m-1)$th row is adjacent to exactly one black vertex in the $(2m)$th row; if $a_m = 0$, each white vertex on the $(2m-1)$th row is adjacent to two black vertices in the $(2m)$th row.

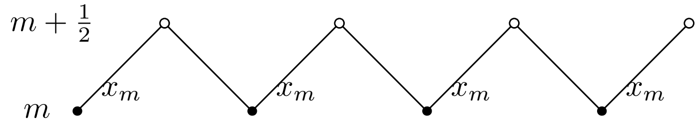
See Figure 2.1.

The square-hexagon lattice defined above is related to the rail-yard graph; see [4].

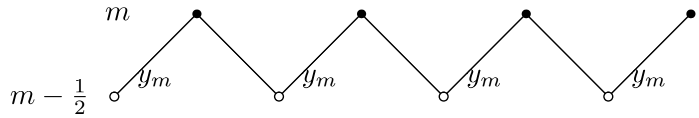
We shall assign edge weights to the whole-plane square-hexagon lattice $\mathrm{SH}(\check{a})$ in the following way.

Assumption 2.10. *For* $m \in \mathbb{Z}$*, we assign weight* $x_m > 0$ *to each NE-SW edge joining the* $(2m)$*th row to the* $(2m+1)$*th row of* $\mathrm{SH}(\check{a})$*. We assign weight* $y_m > 0$ *to each NE-SW edge joining the* $(2m-1)$*th row to the* $(2m)$*th row of* $\mathrm{SH}(\check{a})$*, if such an edge exists. We assign weight* 1 *to all the other edges.*

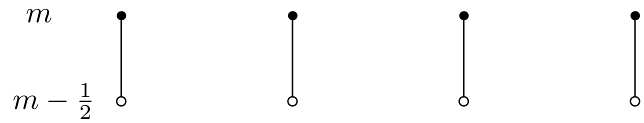
It is straightforward to check that in the graph $\mathrm{SH}(\check{a})$, either all the faces on a row are hexagons, or all the faces on a row are squares, depending on the corresponding entry of $\check{a}$. A *contracting square-hexagon lattice* is built from a whole-plane square-hexagon lattice as follows:



(A) Structure of SH($\check{a}$) between the $(2m)$th row and the $(2m+1)$th row



(B) Structure of SH($\check{a}$) between the $(2m-1)$th row and the $(2m)$th row when $a_m = 0$



(C) Structure of SH($\check{a}$) between the $(2m-1)$th row and the $(2m)$th row when $a_m = 1$

FIGURE 2.1. Graph structures of the square-hexagon lattice on the $(2m-1)$th, $(2m)$th, and $(2m+1)$th rows depend on the values of (a_m). Black vertices are along the $(2m)$th row, while white vertices are along the $(2m-1)$th and $(2m+1)$th row.

Definition 2.11. *Let $N \in \mathbb{N}$. Let $\Omega = (\Omega_1, \ldots, \Omega_N)$ be an N-tuple of positive integers, such that $1 = \Omega_1 < \Omega_2 < \cdots < \Omega_N$. Set $m = \Omega_N - N$. The contracting square-hexagon lattice $\mathcal{R}(\Omega, \check{a})$ is a subgraph of* SH($\check{a}$) *with $2N$ or $2N+1$ rows of vertices. We shall now enumerate the rows of $\mathcal{R}(\Omega, \check{a})$ inductively, starting from the bottom as follows:*

- *The first row consists of vertices (i, j) with $i = \Omega_1 - \frac{1}{2}, \ldots, \Omega_N - \frac{1}{2}$ and $j = \frac{1}{2}$. We call this row the boundary row of $\mathcal{R}(\Omega, \check{a})$.*
- *When $k = 2s$, for $s = 1, \ldots N$, the kth row consists of vertices (i, j) with $j = \frac{k}{2}$ and incident to at least one vertex in the $(2s-1)$th row of the whole-plane square-hexagon lattice* SH($\check{a}$) *lying between the leftmost vertex and rightmost vertex of the $(2s-1)$th row of $\mathcal{R}(\Omega, \check{a})$*
- *When $k = 2s+1$, for $s = 1, \ldots N$, the kth row consists of vertices (i, j) with $j = \frac{k}{2}$ and incident to two vertices in the $(2s)$th row of of $\mathcal{R}(\Omega, \check{a})$.*

The transition from an odd row to the next even row in a contracting square-hexagon lattice can be of two kinds depending on whether vertices are connected to one or two vertices of the row above them.

Definition 2.12. *Let I_1 (resp. I_2) be the set of indices j such that vertices of the $(2j-1)$th row are connected to one vertex (resp. two vertices) of the $(2j)$th row. In terms of the sequence $\check{a}$,*

$$I_1 = \{k \in \{1, \ldots, N\} \mid a_k = 1\}, \quad I_2 = \{k \in \{1, \ldots, N\} \mid a_k = 0\}.$$

The sets I_1 and I_2 form a partition of $\{1, \ldots, N\}$, and we have $|I_1| = N - |I_2|$.

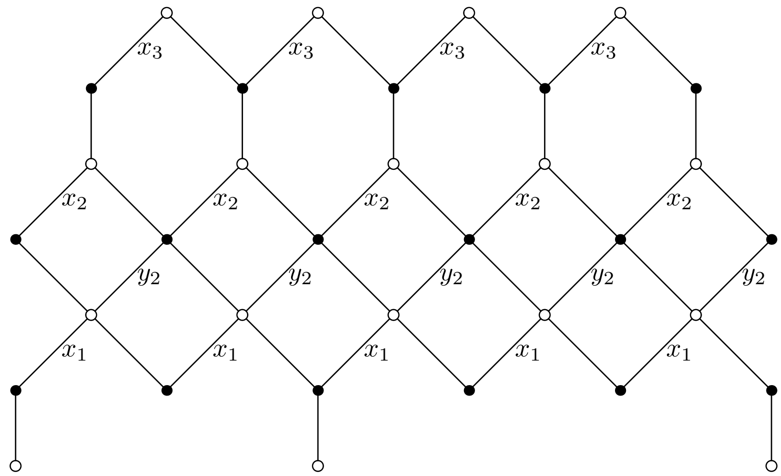


FIGURE 2.2. Contracting square-hexagon lattice with $N = 3$, $m = 3$, $\Omega = (1, 3, 6), (a_1, a_2, a_3) = (1, 0, 1)$.

2.3. **Dimer model.**

Definition 2.13. *A dimer configuration, or a perfect matching M of a contracting square-hexagon lattice $\mathcal{R}(\Omega, \check{a})$ is a set of edges $((i_1, j_1), (i_2, j_2))$, such that each vertex of $\mathcal{R}(\Omega, \check{a})$ belongs to an unique edge in M. The set of perfect matchings of $\mathcal{R}(\Omega, \check{a})$ is denoted by $\mathcal{M}(\Omega, \check{a})$.*

Definition 2.14. *The partition function of the dimer model of a finite graph G with edge weights $(w_e)_{e\in E(G)}$ is given by*

$$Z = \sum_{M\in\mathcal{M}} \prod_{e\in M} w_e,$$

where $\mathcal{M}$ is the set of all perfect matchings of G. The Boltzmann dimer probability measure on M induced by the weights w is thus defined by declaring that probability of a perfect matching is equal to

$$\frac{1}{Z} \prod_{e\in M} w_e.$$

Definition 2.15. *Let $M \in \mathcal{M}(\Omega, \check{a})$ be a perfect matching of $\mathcal{R}(\Omega, \check{a})$. We call an edge $e = ((i_1, j_1), (i_2, j_2)) \in M$ a* V-edge *if* $\max\{j_1, j_2\} \in \mathbb{N}$ *(i.e. if its higher extremity is black) and we call it a* Λ-edge *otherwise. In other words, the edges going upwards starting from an odd row are V-edges and those ones starting from an even row are Λ-edges. We also call the corresponding vertices-(i_1, j_1) and (i_2, j_2) V-vertices and Λ-vertices accordingly.*

We shall associate to each perfect matching in $\mathcal{M}(\Omega, \check{a})$ a sequence of partitions, one for each row of the graph.

Construction 2.16. *To the boundary row $\Omega = (\Omega_1 < \cdots < \Omega_N)$ of a contracting square-hexagon lattice is naturally associated a partition ω of length N by:*

$$\omega = (\Omega_N - N, \ldots, \Omega_1 - 1).$$

Let $j \in \{2, \ldots, 2N + 1\}$. Assume that the jth row of $\mathcal{R}(\Omega, \check{a})$ has n_j V-vertices and m_j Λ-vertices. The a dimer configuration at the jth row of $\mathcal{R}(\Omega, \check{a})$ corresponds to a partition $\mu \in \mathbb{GT}^+_{n_j}$, such that

- $\mu = (\mu_1, \ldots, \mu_{n_j})$;
- *We label all the* V*-vertices on the* j*th row by the 1st* V*-vertex, the 2nd* V*-vertex,* $\ldots$*, the* n_j*th* V*-vertex, such that the 1st* V*-vertex is the rightmost* V*-vertex on the* j*th row. for* $1 \leq k \leq n_j$*,* μ_k *is the number of* Λ*-vertices to the left of the* k*th* V*-vertex.*

Then we have

Theorem 2.17 ([5] Theorem 2.13)**.** *For given* Ω*,* $\check{a}$*, let* ω *be the partition associated to* Ω*. Then the construction 2.16 defines a bijection between the set of perfect matchings* $\mathcal{M}(\Omega, \check{a})$ *and the set* $S(\omega, \check{a})$ *of sequences of partitions*

$$\{(\mu^{(N)}, \nu^{(N)}, \ldots, \mu^{(1)}, \nu^{(1)}, \mu^{(0)}\}$$

where the partitions satisfy the following properties:

- *All the parts of* $\mu^{(0)}$ *are equal to 0;*
- *The partition* $\mu^{(N)}$ *is equal to* ω*;*
- *For* $0 \leq i \leq N$*,* $\mu^{(i)} \in \mathbb{GT}_i^+$*.*
- *The signatures satisfy the following (co)interlacement relations:*

$$\mu^{(N)} \prec' \nu^{(N)} \succ \mu^{(N-1)} \prec' \cdots \mu^{(1)} \prec' \nu^{(1)} \succ \mu^{(0)}.$$

Moreover, if $a_m = 1$*, then* $\mu^{(N+1-k)} = \nu^{(N+1-k)}$*.*

For $N \geq 1$, let $\lambda(N) \in \mathbb{GT}_N^+$ be the boundary partition satisfying Assumption 2.6. Let

$$\Omega = (\Omega_1 < \Omega_2 < \ldots < \Omega_N) = (\lambda_N(N) + 1, \lambda_{N-1}(N) + 2, \ldots, \lambda_1(N) + N)$$

Indeed, $\Omega_1, \ldots, \Omega_N$ are the locations of the N remaining vertices on the bottom boundary of the contracting square-hexagon lattice. Under Assumption 2.6, we may assume

$$\begin{aligned} \Omega \;=\; & (A_1, A_1 + 1, \ldots . B_1 - 1, B_1, \\ & A_2, A_2 + 1, \ldots, B_2 - 1, B_2, \ldots, A_s, A_s + 1, \ldots, B_s - 1, B_s). \end{aligned} \tag{2.14}$$

where

$$B_i - A_i + 1 = K_i.$$

and

$$\sum_{i=1}^{s} (B_i - A_i + 1) = N.$$

Suppose as $N \to \infty$,

$$A_i(N) = a_i N + o(N), \qquad B_i(N) = b_i N + o(N), \qquad \text{for } i \in [s], \tag{2.15}$$

and $a_1 < b_1 < \ldots < a_s < b_s$ are fixed parameters independent of N and satisfy $\sum_{i=1}^{s}(b_i - a_i) = 1$. Under Assumption 2.6, it is straightforward to check that for $i \in [s]$

$$\begin{aligned} b_i &= \lim_{N \to \infty} \frac{\mu_{s-i+1} + \sum_{t=1}^{i} K_t}{N} \\ a_i &= \lim_{N \to \infty} \frac{\mu_{s-i+1} + \sum_{t=1}^{i-1} K_t}{N} \end{aligned}$$

Here are the main theorems concerning the limit counting measures of partitions corresponding to dimer configurations on all the horizontal levels of a contracting square-hexagon lattice. In Theorem 2.18 we give explicit integral formulas for all the moments of limit counting measures at all horizontal levels, from which we can see that the limit counting measure at each horizontal level is deterministic.

Theorem 2.18. *Suppose Assumptions 2.5, 2.8 and 2.6 hold. Let $\kappa \in (0,1)$ be a positive number. Let $\rho_{\lfloor(1-\kappa)N\rfloor}$ be a probability measure on $\mathbb{GT}^+_{\lfloor(1-\kappa)N\rfloor}$, which is the probability measure for partitions corresponding to the random V-edges incident to the $\lfloor(1-\kappa)N\rfloor$th row (counting from the top) of white vertices in a dimer configuration of a contracting square-hexagon lattice $\mathcal{R}(\Omega,\check{a})$. Let $\mathbf{m}[\rho_{\lfloor(1-\kappa)N\rfloor}]$ be the corresponding random counting measure. Then as $N\to\infty$, $\mathbf{m}[\rho_{\lfloor(1-\kappa)N\rfloor}]$ converge in probability, in the sense of moments to a deterministic measure $\mathbf{m}^\kappa$, whose moments are given by*

$$\int_{\mathbb{R}} x^p \boldsymbol{m}^\kappa(dx) = \frac{1}{2(p+1)\pi\mathbf{i}}\sum_{i=1}^{n}\oint_{C_1}\frac{dz}{z}\left(zQ'_{i,\kappa}(z)+\frac{n-i}{n}+\frac{z}{n(z-1)}\right)^{p+1}$$

where for $i\in[n]$

$$Q_{i,\kappa}(z)=\begin{cases}\frac{1}{(1-\kappa)n}\left[H_{\mathbf{m}_i}(z)-(n-i)\log z+\kappa\sum_{l\in[n]\cap I_2}\log\frac{1+y_l z x_1}{1+y_l x_1}\right] & \text{if } i=1\\ \frac{1}{(1-\kappa)n}\left[H_{\mathbf{m}_i}(z)-(n-i)\log z\right] & \text{otherwise}\end{cases}$$

and for $i\geq n+1$,

$$Q_{i,\kappa}(z)=\begin{cases}Q_{(i \mod n),\kappa}(z), & \text{if } (i \mod n)\neq 0\\ Q_{n,\kappa}(z), & \text{if } (i \mod n)=0\end{cases}$$

Theorem 2.18 is proved in Section 5.

The *frozen boundary* of the limit shape is the boundary curve of the region where each type of edge has probability 0 and 1 to occur in the perfect matching (frozen region). The algebraic curve we obtain for the frozen boundary has special properties, that can be read from its dual curve, as described in the definition and the theorem below:

Definition 2.19 ([20])**.** *A degree d real algebraic curve $C\subset\mathbb{R}P^2$ is* winding *if:*

(1) it intersects every line $L\subset\mathbb{R}P^2$ in at least $d-2$ points counting multiplicity; and
(2) there exists a point $p_0\in\mathbb{R}P^2$ called center, such that every line through p_0 intersects C in d points.

The dual curve of a winding curve is called a cloud curve.

Theorem 2.20. *Suppose Assumptions 2.5, 2.8 and 2.6 hold. If $|I_2\cap[n]|\in\{0,1\}$, then the frozen boundary consists of n disjoint cloud curves.*

Theorem 2.20 is proved in Section 7.

2.4. **Height function.** The planar dual graph $\mathrm{SH}^*(\check{a})$ of the square-hexagon lattice $\mathrm{SH}(\check{a})$ is obtained by placing a vertex of $\mathrm{SH}^*(\check{a})$ inside each face of $\mathrm{SH}(\check{a})$; two vertices of $\mathrm{SH}^*(\check{a})$ are adjacent, or joined by an edge in $\mathrm{SH}^*(\check{a})$, if and only if the two corresponding faces of $\mathrm{SH}(\check{a})$ share an edge of $\mathrm{SH}(\check{a})$.

By placing a vertex of $\mathrm{SH}^*(\check{a})$ at the center of each face of $\mathrm{SH}(\check{a})$, we obtain an embedding of $\mathrm{SH}^*(\check{a})$ into the plane. Each face of $\mathrm{SH}^*(\check{a})$ is either a triangle or a square, depending on whether the corresponding vertex of $\mathrm{SH}(\check{a})$ inside the dual face in $\mathrm{SH}^*(\check{a})$ is degree-3 or degree-4.

Consider the contracting square-hexagon lattice $\mathcal{R}(\Omega,\check{a})$. Let $\mathcal{R}^*(\Omega,\check{a})$ be a finite triangle-square lattice such that

- $\mathcal{R}^*(\Omega,\check{a})$ is a finite subgraph of $\mathrm{SH}^*(\check{a})$ as constructed above; and

- $\mathcal{R}(\Omega, \breve{a})$ is the interior dual graph of $\mathcal{R}^*(\Omega, \breve{a})$.

In other words, $\mathcal{R}^*(\Omega, \breve{a})$ is the subgraph of $\mathrm{SH}^*(\breve{a})$ consisting of all the faces of $\mathrm{SH}^*(\breve{a})$ corresponding to vertices of $\mathcal{R}(\Omega, \breve{a})$; see Figure 2.3.

Definition 2.21. *Let $M \in \mathcal{M}(\Omega, \breve{a})$ be a perfect matching of a contracting square-hexagon lattice $\mathcal{R}(\Omega, \breve{a})$. A height function h_M is an integer-valued function on vertices of $\mathcal{R}^*(\Omega, \breve{a})$ satisfying following properties.*

Let f_1, f_2 be a pair of adjacent vertices of $\mathcal{R}^*(\Omega, \breve{a})$. Let (f_1, f_2) denote the non-oriented edge of $\mathcal{R}^*(\Omega, \breve{a})$ with endpoints f_1 and f_2; and let $[f_1, f_2\rangle$ (resp. $[f_2, f_1\rangle$) denote the oriented edge starting from f_1 (resp. f_2) and ending in f_2 (resp. f_1).

- *if (f_1, f_2) is a dual edge crossing a NW-SE edge or a NE-SW edge of $\mathrm{SH}(\breve{a})$,*
  - *if the oriented dual edge $[f_1, f_2\rangle$ crosses an absent edge e of $\mathrm{SH}(\breve{a})$ in M then $h_M(f_2) = h_M(f_1) + 1$ if $[f_1, f_2\rangle$ has the white vertex or e on the left, and $h_M(f_2) = h_M(f_1) - 1$ otherwise.*
  - *if an oriented dual edge $[f_1, f_2\rangle$ crosses a present edge e of $\mathrm{SH}(\breve{a})$ in M then $h_M(f_2) = h_M(f_1) - 3$ if $[f_1, f_2\rangle$ has the white vertex of e on the left, and $h_M(f_2) = h_M(f_1) + 3$ otherwise.*
- *if (f_1, f_2) is a dual edge crossing a vertical edge of $\mathrm{SH}(\breve{a})$.*
  - *If an oriented dual edge $[f_1, f_2\rangle$ crosses an absent edge e of $\mathrm{SH}(\breve{a})$ in M, then $h_M(f_2) = h_M(f_1) + 2$ if $[f_1, f_2\rangle$ has the white vertex of e on the left, and $h_M(f_2) = h_M(f_1) - 2$ otherwise.*
  - *If an oriented dual edge $[f_1, f_2\rangle$ crosses a present edge e of $\mathrm{SH}(\breve{a})$ in M then $h_M(f_2) = h_M(f_1) - 2$ if $[f_1, f_2\rangle$ has the white vertex of e on the left, and $h_M(f_2) = h_M(f_1) + 2$ otherwise.*
- *$h_M(f_0) = 0$, where f_0 is the lexicographic smallest vertex of $\mathcal{R}^*(\Omega, \breve{a})$.*

It is straightforward to verify that the height function above is well-defined, by checking that around each degree-3 face or degree-4 face of $\mathcal{R}^*(\Omega, \breve{a})$, the total height change is 0. Moreover, since none of the boundary edges of $\mathcal{R}(\Omega, \breve{a})$ (by boundary edges we mean edges of $\mathrm{SH}(\breve{a})$ joining exactly one vertex of $\mathcal{R}(\Omega, \breve{a})$ and one vertex outside $\mathcal{R}(\Omega, \breve{a})$) are present in any perfect matching of $\mathcal{R}(\Omega, \breve{a})$, the height function restricted on the boundary vertices of $\mathcal{R}^*(\Omega, \breve{a})$ is fixed and independent of the random perfect matching on $\mathcal{R}(\Omega, \breve{a})$; see Figure 2.3.

We will prove the following limit shape theorem concerning the height function in Section 6.

Theorem 2.22. *(Law of large numbers for the height function.) Consider $N \to \infty$ asymptotics such that all the dimensions of a contracting square-hexagon lattice $\mathcal{R}(\Omega(N), \breve{a})$ linearly grow with N. Assume that*

- *the edge weights are assigned as in Assumption 2.10 (see Figure 2.2 for an example) and satisfy Assumptions 2.5 and 2.8 and 2.6, such that for each $1 \leq i \leq n$ and $i \in I_2$, $y_i > 0$ are fix and independent of N.*

For $\kappa \in (0,1)$, let $\mathbf{m}^\kappa$ be the limit in probability of the counting measure for random partitions corresponding to dimer configurations at level κ of contracting square-hexagon lattices as $N \to \infty$ (where the bottom boundary is level 0, and the top boundary is level 1). Recall that the moments of $\mathbf{m}^\kappa$ are given by Theorem 2.18.

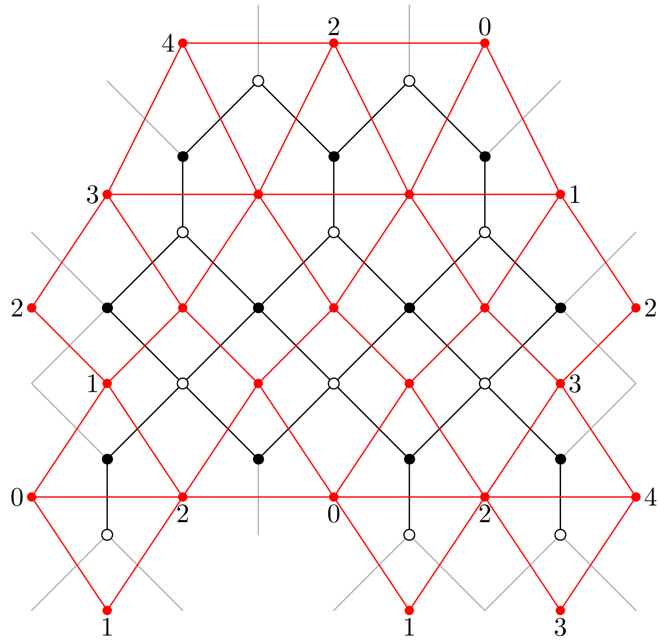


FIGURE 2.3. Contracting square-hexagon lattice $\mathcal{R}(\Omega, \check{a})$, dual graph $\mathcal{R}^*(\Omega, \check{a})$ and height function on the boundary. The black lines represent the graph $\mathcal{R}(\Omega, \check{a})$, the gray lines represent boundary edges of $\mathcal{R}(\Omega, \check{a})$, the red lines represent the dual graph $\mathcal{R}^*(\Omega, \check{a})$, and the height function is defined on vertices of the dual graph. The values of the height function on the boundary vertices of $\mathcal{R}^*(\Omega, \check{a})$ are also shown in the figure.

Define

$$\mathbf{h}(\chi, \kappa) := 2\left(2(1-\kappa)\int_0^{\frac{\chi - \frac{\kappa r}{2n}}{1-\kappa}} d\mathbf{m}^{\kappa} - 2\chi + 2\kappa\right)$$

Then the random height function h_M associate to a random perfect matching M, as defined by Definition 2.21, has the following law of large numbers

$$\frac{h_M([\chi N], [\kappa N])}{N} \to \mathbf{h}(\chi, \kappa), a.s., \text{when } N \to \infty$$

where χ, κ are new continuous parameters of the domain.

3. Schur Polynomial at $(x_1, \ldots, x_N)$ and Schur polynomials at $(1, \ldots, 1)$: combinatorial results

In this section, we prove the combinatorial formula to compute the Schur function at $(x_1, \ldots, x_N)$ by the values of Schur functions at $(1, \ldots, 1)$, as stated in Theorem 2.4. We shall prove a general formula to compute the Schur polynomial (Proposition 3.4) at $(w_1, \ldots, w_N) \in \mathbb{C}^N$, where the variable $(w_1, \ldots, w_N)$ differs from $(x_1, \ldots, x_N)$ by at most k components, and then obtain Theorem 2.4 as a special case when $(w_1, \ldots, w_N) = (x_1, \ldots, x_N)$. Proposition 3.4 will also be used to obtain asymptotical results of Schur polynomials (Theorem 2.9). We start with the following lemma.

Lemma 3.1. *For any $\xi \in \Sigma_N^X$, $\sigma \in \Sigma_N$ and $1 \leq j \leq N$, we have*

$$\eta_j^{\xi\sigma}(N) = \eta_j^{\sigma}(N).$$

Proof. Since $\xi \in \Sigma_N^X$, we have

$$\eta_j^{\xi\sigma}(N) = |\{k : k > j, x_{\xi\sigma(k)} \neq x_{\xi\sigma(j)}\}| = |\{k : k > j, x_{\sigma(k)} \neq x_{\sigma(j)}\}| = \eta_j^{\sigma}(N).$$

□

Lemma 3.2. *For any $\xi \in \Sigma_N^X$, $\sigma \in \Sigma_N$ and $1 \leq i \leq n$, we have*

$$\phi^{(i,\sigma)}(N) = \phi^{(i,\xi\sigma)}(N),$$

as elements in $\mathbb{GT}_{|\{j\in[N]:x_j=x_i\}|}$.

Proof. By Lemma 3.1, we have

$$\Phi^{(i,\sigma)}(N) = \{\lambda_j(N) + \eta_j^{\sigma}(N) : x_{\sigma(j)} = x_i\} = \{\lambda_j(N) + \eta_j^{\xi\sigma}(N) : x_{\xi\sigma(j)} = x_i\} = \Phi^{(i,\xi\sigma)}(N).$$

Then the lemma follows from the fact that $\phi^{(i,\sigma)}(N)$ (respectively. $\phi^{(i,\xi\sigma)}(N)$) is the partition obtained by decreasingly ordering all the elements in $\Phi^{(i,\sigma)}(N)$ (respectively. $\Phi^{(i,\xi\sigma)}(N)$) □

Let $k \in [N]$. Let

$$w_i = \begin{cases} u_i & \text{if } 1 \leq i \leq k \\ x_i & \text{if } k+1 \leq i \leq N \end{cases} \tag{3.1}$$

Assume

$$k = qn + r, \qquad \text{where } r < n, \tag{3.2}$$

and q, r are positive integers.

Lemma 3.3. *Let $\overline{\sigma} \in [\Sigma_N/\Sigma_N^X]^r$. Assume that $\sigma_1, \sigma_2 \in \overline{\sigma}$ be two distinct representatives. Then we have*

$$\prod_{i<j, x_{\sigma_1(i)} \neq x_{\sigma_1(j)}} \frac{1}{w_{\sigma_1(i)} - w_{\sigma_1(j)}} = \prod_{i<j, x_{\sigma_2(i)} \neq x_{\sigma_2(j)}} \frac{1}{w_{\sigma_2(i)} - w_{\sigma_2(j)}} \tag{3.3}$$

Proof. Assume $\sigma_1 = \xi\sigma_2$, where $\xi \in \Sigma_N^X$. We claim ξ is a product of transpositions in Σ_N^X. Indeed, since Σ_N^X is a direct product of $\mathcal{S}_{T_1}$, $\mathcal{S}_{T_2}$,..., $\mathcal{S}_{T_n}$, we have

$$\xi = \prod_{k=1}^{n} \xi_k,$$

where $\xi_k \in \mathcal{S}_{T_k}$. For each $k \in [n]$, since $\mathcal{S}_{T_k}$ is isomorphic to $\Sigma_{|T_k|}$ (the permutation group of $|T_k|$ elements), ξ_k is the product of transpositions in $\mathcal{S}_{T_k} \subset \Sigma_N^X$. Therefore for each $\xi \in \Sigma_N^X$, ξ is a product of transpositions in Σ_N^X.

Now it suffices to show that for each transposition $\eta \in \Sigma_N^X$ and $\sigma \in \Sigma_N$, we have

$$\prod_{i<j, x_{\eta\sigma(i)} \neq x_{\eta\sigma(j)}} \frac{1}{w_{\eta\sigma(i)} - w_{\eta\sigma(j)}} = \prod_{i<j, x_{\sigma(i)} \neq x_{\sigma(j)}} \frac{1}{w_{\sigma(i)} - w_{\sigma(j)}} \tag{3.4}$$

If η is a transposition of two elements not in $\{u_1, u_2, \ldots, u_k\}$, then

$$\eta \in \Sigma_N^W := \{\sigma \in \Sigma_N : w_i = w_{\sigma(i)}, \forall\ i \in [N]\}.$$

In this case (3.4) obviously holds.

We now check that if η is a transposition involving elements in $\{u_1,\ldots,u_k\}$, the identity (3.4) still holds. Without loss of generality, assume that $\eta=(u_{\sigma(a)},w_{\sigma(b)})$, where $a\in[k]$, $b>a$ and $b\in[N]$. Given that $\eta\in\Sigma_N^X$, we must have $x_{\sigma(a)}=x_{\sigma(b)}$. Then

$$\prod_{i<j,x_{\eta\sigma(i)}\neq x_{\eta\sigma(j)}}\frac{1}{w_{\eta\sigma(i)}-w_{\eta\sigma(j)}}:=D_1^{(\eta\sigma)}D_2^{(\eta\sigma)}$$
$$\prod_{i<j,x_{\sigma(i)}\neq x_{\sigma(j)}}\frac{1}{w_{\sigma(i)}-w_{\sigma(j)}}:=D_1^{(\sigma)}D_2^{(\sigma)}$$

where

$$\begin{aligned}D_1^{(\sigma)} \quad:=\quad &\left[\prod_{i<j,\{i,j\}\cap\{a,b\}=\emptyset,x_{\sigma(i)}\neq x_{\sigma(j)}}\frac{1}{w_{\sigma(i)}-w_{\sigma(j)}}\right]\times\left[\prod_{i<a,x_{\sigma(i)}\neq x_{\sigma(a)}}\frac{1}{w_{\sigma(i)}-w_{\sigma(a)}}\right]\\ &\times\left[\prod_{i<a,x_{\sigma(i)}\neq x_{\sigma(b)}}\frac{1}{w_{\sigma(i)}-w_{\sigma(b)}}\right]\times\left[\prod_{j>b,x_{\sigma(j)}\neq x_{\sigma(a)}}\frac{1}{w_{\sigma(a)}-w_{\sigma(j)}}\right]\\ &\times\left[\prod_{j>b,x_{\sigma(j)}\neq x_{\sigma(b)}}\frac{1}{w_{\sigma(b)}-w_{\sigma(j)}}\right]=D_1^{(\eta\sigma)};\end{aligned}$$

and

$$D_2^{(\sigma)}\quad:=\quad\prod_{a<i<b}\left[\frac{1}{(w_{\sigma(a)}-w_{\sigma(i)})(w_{\sigma(i)}-w_{\sigma(b)})}\right]=D_2^{(\eta\sigma)}$$

Then the lemma follows. □

Proposition 3.4. *Let $\{w_i\}_{i\in[N]}$ and k be given by (3.1) and (3.2), respectively. Then we have the following formula*

$$\begin{aligned}(3.5)\qquad &s_\lambda(w_1,\ldots,w_N)\\ =\ &\sum_{\overline{\sigma}\in[\Sigma_N/\Sigma_N^X]^r}\left(\prod_{i=1}^n x_i^{|\phi^{(i,\sigma)}(N)|}\right)\left(\prod_{i=1}^r s_{\phi^{(i,\sigma)}(N)}\left(\frac{u_i}{x_i},\frac{u_{n+i}}{x_i}\ldots,\frac{u_{qn+i}}{x_i},1,\ldots,1\right)\right)\\ &\times\left(\prod_{i=r+1}^n s_{\phi^{(i,\sigma)}(N)}\left(\frac{u_i}{x_i},\frac{u_{n+i}}{x_i}\ldots,\frac{u_{(q-1)n+i}}{x_i},1,\ldots,1\right)\right)\\ &\times\left(\prod_{i<j,x_{\sigma(i)}\neq x_{\sigma(j)}}\frac{1}{w_{\sigma(i)}-w_{\sigma(j)}}\right)\end{aligned}$$

where $\sigma\in\overline{\sigma}\cap\Sigma_N$ is a representative.

Proof. First of all, by Lemmas 3.2 and 3.3, the right hand side of (3.5) is independent of the choice of the representative in $\overline{\sigma}\in[\Sigma_N/\Sigma_N^X]^r$.

For $i\in[n]$, let

$$(3.6)\qquad T_i=\{j:j\in[N],x_j=x_i\}.$$

Let $\mathcal{S}_{T_i}$ be the subgroup of Σ_N consisting of all the permutations within I_i while preserving all the elements outside I_i. Note that

$$\begin{aligned} \mathcal{S}_{T_i} \subset \Sigma_N^X, &\qquad \forall\ i \in [n] \\ \mathcal{S}_{T_i} \cap \mathcal{S}_{T_j} = \{id\}, &\qquad \forall\ i \neq j, \end{aligned}$$

where id is the identity in Σ_N. Indeed, Σ_N^X is the direct product of $\mathcal{S}_{T_n}$, $\mathcal{S}_{T_{n-1}}, \ldots, \mathcal{S}_{T_1}$. Hence we have

$$\overline{\sigma} = \Sigma_N^X \sigma = \mathcal{S}_{T_n}\mathcal{S}_{T_{n-1}} \cdot \ldots \cdot \mathcal{S}_{T_1}\sigma.$$

Recall the well-known formula to compute the Schur function

$$\begin{aligned} s_\lambda(w_1, \ldots, w_N) &= \sum_{\sigma \in \Sigma_N} \left(w_{\sigma(1)}^{\lambda_1} \cdots w_{\sigma(N)}^{\lambda_N} \prod_{i<j} \frac{w_{\sigma(i)}}{w_{\sigma(i)} - w_{\sigma(j)}} \right). \\ &= \sum_{\overline{\sigma} \in [\Sigma_N/\Sigma_N^X]^r} \sum_{\sigma \in \overline{\sigma}} \left(w_{\sigma(1)}^{\lambda_1} \cdots w_{\sigma(N)}^{\lambda_N} \prod_{i<j} \frac{w_{\sigma(i)}}{w_{\sigma(i)} - w_{\sigma(j)}} \right). \end{aligned}$$

If some $w_i = w_j$ for some $i \neq j$, the right hand side is computed by the limit $\lim_{w_i \to w_j}$. To prove (3.5), it suffices to show that for each $\overline{\sigma} \in [\Sigma_N/\Sigma_N^X]^r$, we have

$$\begin{aligned} &\sum_{\sigma \in \overline{\sigma}} \left(w_{\sigma(1)}^{\lambda_1} \cdots w_{\sigma(N)}^{\lambda_N} \prod_{i<j} \frac{w_{\sigma(i)}}{w_{\sigma(i)} - w_{\sigma(j)}} \right) \\ = &\left(\prod_{i=1}^n x_i^{|\phi^{(i,\sigma)}(N)|} \right) \left(\prod_{i=1}^r s_{\phi^{(i,\sigma)}(N)} \left(\frac{u_i}{x_i}, \frac{u_{n+i}}{x_i} \ldots, \frac{u_{qn+i}}{x_i}, 1, \ldots, 1 \right) \right) \\ &\times \left(\prod_{i=r+1}^n s_{\phi^{(i,\sigma)}(N)} \left(\frac{u_i}{x_i}, \frac{u_{n+i}}{x_i} \ldots, \frac{u_{(q-1)n+i}}{x_i}, 1, \ldots, 1 \right) \right) \\ &\times \left(\prod_{i<j, x_{\sigma(i)} \neq x_{\sigma(j)}} \frac{1}{w_{\sigma(i)} - w_{\sigma(j)}} \right) \end{aligned}$$

We have

$$\begin{aligned} &\sum_{\sigma \in \overline{\sigma}} \left(w_{\sigma(1)}^{\lambda_1} \cdots w_{\sigma(N)}^{\lambda_N} \prod_{i<j} \frac{w_{\sigma(i)}}{w_{\sigma(i)} - w_{\sigma(j)}} \right) \\ = &\sum_{\xi_n \in \mathcal{S}_{T_n}} \cdot \ldots \cdot \sum_{\xi_1 \in \mathcal{S}_{T_1}} \left(w_{\xi_n \ldots \xi_1 \sigma(1)}^{\lambda_1} \cdots w_{\xi_n \ldots \xi_1 \sigma(N)}^{\lambda_N} \prod_{i<j} \frac{w_{\xi_n \ldots \xi_1 \sigma(i)}}{w_{\xi_n \ldots \xi_1 \sigma(i)} - w_{\xi_n \ldots \xi_1 \sigma(j)}} \right) \\ = &\sum_{\xi_n \in \mathcal{S}_{T_n}} \cdot \ldots \cdot \sum_{\xi_1 \in \mathcal{S}_{T_1}} \left(w_{\xi_n \ldots \xi_1 \sigma(1)}^{\lambda_1} \cdots w_{\xi_n \ldots \xi_1 \sigma(N)}^{\lambda_N} \prod_{i<j, x_{\sigma(i)} \neq x_{\sigma(j)}} \frac{w_{\xi_n \ldots \xi_1 \sigma(i)}}{w_{\xi_n \ldots \xi_1 \sigma(i)} - w_{\xi_n \ldots \xi_1 \sigma(j)}} \right) \\ &\times \left(\prod_{h=1}^n \prod_{i<j, x_{\sigma(i)} = x_{\sigma(j)} = x_h} \frac{w_{\xi_n \ldots \xi_1 \sigma(i)}}{w_{\xi_n \ldots \xi_1 \sigma(i)} - w_{\xi_n \ldots \xi_1 \sigma(j)}} \right) \end{aligned}$$

Note that if $x_{\sigma(g)} = x_h$, for some $h \in [n]$ and $\xi_i \in \mathcal{S}_{T_i}$, for all $i \in [n]$,

$$\xi_n \ldots \xi_1 \sigma(g) = \xi_h(\sigma(g))$$

Then by Lemma 3.3, we obtain

$$\sum_{\sigma \in \overline{\sigma}} \left(w_{\sigma(1)}^{\lambda_1} \cdots w_{\sigma(N)}^{\lambda_N} \prod_{i<j} \frac{w_{\sigma(i)}}{w_{\sigma(i)} - w_{\sigma(j)}} \right) = \prod_{i<j, x_{\sigma(i)} \neq x_{\sigma(j)}} \frac{1}{w_{\sigma(i)} - w_{\sigma(j)}}$$
$$\left[\prod_{h=1}^{n} \left(\sum_{\xi_h \in \mathcal{S}_{T_h}} \prod_{g: x_{\sigma(g)} = x_h} w_{\xi_h \sigma(g)}^{\lambda_g + \eta_g^{\sigma}} \prod_{i<j, x_{\sigma(i)} = x_{\sigma(j)} = x_h} \frac{w_{\xi_h \sigma(i)}}{w_{\xi_h \sigma(i)} - w_{\xi_h \sigma(j)}} \right) \right]$$

Note that

$$\sum_{\xi_h \in \mathcal{S}_{T_h}} \prod_{g: x_{\sigma(g)} = x_h} w_{\xi_h \sigma(g)}^{\lambda_g + \eta_g^{\sigma}} \prod_{i<j, x_{\sigma(i)} = x_{\sigma(j)} = x_h} \frac{w_{\xi_h \sigma(i)}}{w_{\xi_h \sigma(i)} - w_{\xi_h \sigma(j)}}$$
$$= x_h^{|\phi^{(h,\sigma)}(N)|} s_{\phi^{(h,\sigma)}(N)} \left(\frac{u_h}{x_h}, \frac{u_{n+h}}{x_h} \ldots, \frac{u_{tn+h}}{x_h}, 1, \ldots, 1 \right).$$

where

$$t = \begin{cases} q & \text{if } 1 \leq i \leq r \\ q-1 & \text{if } r+1 \leq i \leq n \end{cases}$$

Then the proposition follows.. □

Proof of Theorem 2.4. Theorem 2.4 follows from Proposition 3.4 by letting $u_j = x_j$ for all $j \in [k]$. □

4. Asymptotics of the Schur polynomial at a general point

In this section, we use Proposition 3.4 to study the asymptotics of Schur functions at a general point. The main goal is to prove Theorem 2.9. Note that Proposition 3.4 expresses the Schur polynomial at a general point as a sum of Schur polynomials at $(1, \ldots, 1)$; when $(x_1, \ldots, x_N)$ are periodic with a fixed finite period n as in Assumption 2.5, the number of summands to compute the Schur function at a general point in the formula as given in Proposition 3.4, is exponential in N. The idea is to find a leading term among all these summands, and then use the leading term to study the asymptotics of the Schur polynomial at a general point.

Recall that evaluating the Schur function $s_{\phi^{(i,\sigma)}(N)}$ at $(1, \ldots, 1)$ can be done by using the Weyl character formula

$$s_{\phi^{(i,\sigma)}(N)}(1, \ldots, 1) = \prod_{1 \leq j < k \leq |T_i|} \frac{\phi_j^{(i,\sigma)}(N) - \phi_k^{(i,\sigma)}(N) + k - j}{k - j}$$

where T_i is defined by (3.6) for $i \in [n]$.

With the help of the Weyl character formula, (2.5) can also be written as

$$s_\lambda(x_1,\ldots,x_N) = \sum_{\overline{\sigma}\in[\Sigma_N/\Sigma_N^X]^r}\left(\prod_{i=1}^{N} x_{\sigma(i)}^{\lambda_i(N)}\right)\left(\prod_{i=1}^{n}\prod_{1\leq j<k\leq|T_i|}\frac{\phi_j^{(i,\sigma)}-\phi_k^{(i,\sigma)}+k-j}{k-j}\right) \times\left(\prod_{i<j,x_{\sigma(i)}\neq x_{\sigma(j)}}\frac{x_{\sigma(i)}}{x_{\sigma(i)}-x_{\sigma(j)}}\right)$$

Let $\sigma\in\Sigma_N$. Under Assumption 2.6 we have

$$\left|\frac{x_{\sigma_0(1)}^{\lambda_1}\cdots x_{\sigma_0(N)}^{\lambda_N}}{x_{\sigma(1)}^{\lambda_1}\cdots x_{\sigma(N)}^{\lambda_N}}\right| = \left|\frac{\left[\prod_{i=1}^{K_s}x_{\sigma_0(i)}\right]^{\mu_1}\left[\prod_{i=K_s+1}^{K_s+K_{s-1}}x_{\sigma_0(i)}\right]^{\mu_2}\cdot\ldots\cdot\left[\prod_{i=1+\sum_{t=2}^{s}K_t}^{\sum_{t=1}^{s}K_t}x_{\sigma_0(i)}\right]^{\mu_s}}{\left[\prod_{i=1}^{K_s}x_{\sigma(i)}\right]^{\mu_1}\left[\prod_{i=K_s+1}^{K_s+K_{s+1}}x_{\sigma(i)}\right]^{\mu_2}\cdot\ldots\cdot\left[\prod_{i=1+\sum_{t=2}^{s}K_t}^{\sum_{t=1}^{s}K_t}x_{\sigma(i)}\right]^{\mu_s}}\right| \tag{4.1}$$

For $i,j\in[n]$, $t\in[s]$, $\sigma\in\Sigma_N$, define

$$I_{i,j,t}^{\sigma}=\{p:p\in[N],x_{\sigma_0(p)}=x_i,x_{\sigma(p)}=x_j,\lambda_p=\mu_t\}. \tag{4.2}$$

We may interpret $I_{i,j,t}$ as follows. Consider a $3\times N$ array

$$B:=\begin{pmatrix}\lambda_1 & \lambda_2 & \ldots & \lambda_N\\ x_{\sigma_0(1)} & x_{\sigma_0(2)} & \ldots & x_{\sigma_0(N)}\\ x_{\sigma(1)} & x_{\sigma(2)} & \ldots & x_{\sigma(N)}\end{pmatrix}.$$

Then $I_{i,j,t}^{\sigma}$ consists of all the column indices p, such that

$$B(1,p)=\mu_t,\ \text{and}$$
$$B(2,p)=x_i,\ \text{and} \tag{4.3}$$
$$B(3,p)=x_j \tag{4.4}$$

We use $\left|I_{i,j,t}^{\sigma}\right|$ to denote the cardinality of the set $I_{i,j,t}^{\sigma}$.

Recall that for $i\in[n]$, J_i is defined in (2.8). Let

$$I_{i,j}^{\sigma}=\cup_{t\in[s]}I_{i,j,t}^{\sigma}=\cup_{t\in J_i}I_{i,j,t}^{\sigma}. \tag{4.5}$$

That is, $I_{i,j}^{\sigma}$ consists of all the column indices $p\in[N]$ such that (4.3) and (4.4) hold. The last identity follows from the fact that $I_{i,j,t}^{\sigma}=\emptyset$ unless $t\in J_i$. Since the right hand side of (4.5) is a disjoint union, we have

$$\left|I_{i,j}^{\sigma}\right|=\sum_{t\in J_i}\left|I_{i,j,t}^{\sigma}\right|=\sum_{t\in[s]}\left|I_{i,j,t}^{\sigma}\right| \tag{4.6}$$

Recall also that T_i is defined by (3.6), we have

$$\sigma_0^{-1}(T_i)=\cup_{j\in[n]}I_{i,j}^{\sigma}=\cup_{j\in[n]}\cup_{t\in J_i}I_{i,j,t}^{\sigma} \tag{4.7}$$

$$\sigma^{-1}(T_j)=\cup_{i\in[n]}I_{i,j}^{\sigma}=\cup_{i\in[n]}\cup_{t\in J_i}I_{i,j,t}^{\sigma} \tag{4.8}$$

where for $j\in[n]$ and $\sigma\in\Sigma_N$,

$$\sigma^{-1}(T_j)=\{l\in[N]:x_{\sigma(l)}=x_j\}.$$

Lemma 4.1. *Let $\mathcal{P}$ be a nonempty, proper subset of $[n]$, i.e.*

$$\mathcal{P} \subset [n]; \qquad \mathcal{P} \neq \emptyset; \qquad \mathcal{P} \neq [n].$$

Let $\mathcal{P}^c = [n] \setminus \mathcal{P}$ be the complement of $\mathcal{P}$ in $[n]$. Then

$$\sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}^c} \left|I_{i,j}^{\sigma}\right| = \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}^c} \left|I_{j,i}^{\sigma}\right|. \tag{4.9}$$

Proof. Let

$$X_{\mathcal{P}} = \{x_i : i \in \mathcal{P}\}; \qquad X_{\mathcal{P}^c} = \{x_i : i \in \mathcal{P}^c\}$$

By (4.2), the left hand side of (4.9) is equal to

$$\begin{aligned} &|\{p : p \in [N], x_{\sigma_0(p)} \in X_{\mathcal{P}}, x_{\sigma(p)} \in X_{\mathcal{P}^c}\}|; \qquad (4.10)\\ = \ &|\{p : p \in [N], x_{\sigma_0(p)} \in X_{\mathcal{P}}\}| - |\{p : p \in [N], x_{\sigma_0(p)} \in X_{\mathcal{P}}, x_{\sigma(p)} \in X_{\mathcal{P}}\}| \end{aligned}$$

while the right hand side of (4.9) is equal to

$$\begin{aligned} &|\{p : p \in [N], x_{\sigma_0(p)} \in X_{\mathcal{P}^c}, x_{\sigma(p)} \in X_{\mathcal{P}}\}|; \qquad (4.11)\\ = \ &|\{p : p \in [N], x_{\sigma(p)} \in X_{\mathcal{P}}\}| - |\{p : p \in [N], x_{\sigma_0(p)} \in X_{\mathcal{P}}, x_{\sigma(p)} \in X_{\mathcal{P}}\}| \end{aligned}$$

Since both σ and σ_0 are bijections from $[N]$ to $[N]$, both (4.10) and (4.11) are equal to

$$|\{q : q \in [N], x_q \in X_{\mathcal{P}}\}| - |\{p : p \in [N], x_{\sigma_0(p)} \in X_{\mathcal{P}}, x_{\sigma(p)} \in X_{\mathcal{P}}\}|;$$

then the lemma follows. □

Lemma 4.2. *Assume (2.7) holds, and $\sigma \in \Sigma_N \setminus \overline{\sigma}_0$. For $p \in [n]$, let*

$$m_p = \max\{l : l \in J_p\}$$

Let

$$L : \ = \sum_{1\leq i<j\leq n}\sum_{t=1}^{s} \mu_t \left[\left|I_{i,j,t}^{\sigma}\right| - \left|I_{j,i,t}^{\sigma}\right|\right](j-i) \tag{4.12}$$

Then we have

$$L = \sum_{p=1}^{n}\left[\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{t\in J_i}(\mu_t - \mu_{m_p})\left|I_{i,j,t}^{\sigma}\right| + \sum_{i=p+1}^{n}\sum_{j=1}^{p}\sum_{t\in J_i}(\mu_{m_p} - \mu_t)\left|I_{i,j,t}^{\sigma}\right|\right]$$

In particular $L \geq 0$. Moreover, if Assumption 2.6 holds, then

$$\begin{aligned} L \ &\geq \ \frac{1}{2}\min_{p<q}(\mu_p - \mu_q)\sum_{i=1}^{n}\sum_{j\in[n]\setminus\{i\}} |I_{i,j}^{\sigma}|. \qquad (4.13)\\ &\geq \ \frac{C_1}{2} N \sum_{i=1}^{n}\sum_{j\in[n]\setminus\{i\}} |I_{i,j}^{\sigma}| \end{aligned}$$

Proof. We shall rewrite the right hand side of (4.12) as follows:

$$L = \sum_{t=1}^{s}\mu_t\left[\sum_{1\leq i<j\leq n}\left|I_{i,j,t}^{\sigma}\right|(j-i) - \sum_{1\leq j<i\leq n}\left|I_{i,j,t}^{\sigma}\right|(i-j)\right], \tag{4.14}$$

which is also equal to

$$
\begin{aligned}
(4.15)\qquad L &= \sum_{p=1}^{n}\left[\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{t=1}^{s}\mu_t\left|I^{\sigma}_{i,j,t}\right| - \sum_{i=p+1}^{n}\sum_{j=1}^{p}\sum_{t=1}^{s}\mu_t\left|I^{\sigma}_{i,j,t}\right|\right]\\
&= \sum_{p=1}^{n}\left[\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{t\in J_i}\mu_t\left|I^{\sigma}_{i,j,t}\right| - \sum_{i=p+1}^{n}\sum_{j=1}^{p}\sum_{t\in J_i}\mu_t\left|I^{\sigma}_{i,j,t}\right|\right]
\end{aligned}
$$

To see why (4.15) follows from (4.14), note that for each given $(a,b)\in[n]^2$, $a<b$, in $\sum_{p=1}^{n}\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{t=1}^{s}\mu_t\left|I^{\sigma}_{i,j,t}\right|$, the monomial $\mu_t\left|I^{\sigma}_{a,b,t}\right|$ is added when $p=a,a+1,\ldots,b-1$, which is exactly $(b-a)$ times.

By Lemma 2.7, we have

$$
L\geq\sum_{p=1}^{n}\mu_{m_p}\left[\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{t\in J_i}\left|I^{\sigma}_{i,j,t}\right| - \sum_{i=p+1}^{n}\sum_{j=1}^{p}\sum_{t\in J_i}\left|I^{\sigma}_{i,j,t}\right|\right]
$$

Let

$$
\mathcal{P}=[p]\subseteq[n].
$$

By Lemma 4.1 and (4.6), we obtain

$$
\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{t\in J_i}\left|I^{\sigma}_{i,j,t}\right| = \sum_{i=p+1}^{n}\sum_{j=1}^{p}\sum_{t\in J_i}\left|I^{\sigma}_{i,j,t}\right|
$$

Therefore $L\geq 0$.

Moreover, we have

$$
\begin{aligned}
(4.16)\qquad L &= L-0\\
&= \sum_{p=1}^{n}\left[\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{t=1}^{s}\mu_t\left|I^{\sigma}_{i,j,t}\right| - \sum_{i=p+1}^{n}\sum_{j=1}^{p}\sum_{t=1}^{s}\mu_t\left|I^{\sigma}_{i,j,t}\right|\right]\\
&\quad -\sum_{p=1}^{n}\mu_{m_p}\left[\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{t=1}^{s}\left|I^{\sigma}_{i,j,t}\right| - \sum_{i=p+1}^{n}\sum_{j=1}^{p}\sum_{t=1}^{s}\left|I^{\sigma}_{i,j,t}\right|\right]\\
&= \sum_{p=1}^{n}\left[\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{t\in J_i}(\mu_t-\mu_{m_p})\left|I^{\sigma}_{i,j,t}\right|\right.\\
&\quad \left.+\sum_{i=p+1}^{n}\sum_{j=1}^{p}\sum_{t\in J_i}(\mu_{m_p}-\mu_t)\left|I^{\sigma}_{i,j,t}\right|\right]
\end{aligned}
$$

Under the assumption that $\sigma\notin\overline{\sigma}_0$, there exist $i,j\in[n]$, $i\neq j$ and $t\in[s]$, such that $\left|I^{\sigma}_{i,j,t}\right|>0$. The following cases might occur

- there exist $i<j$ such that $\sum_{t=1}^{s}\left|I_{i,j,t}^{\sigma}\right|>0$, then by (4.9) we have for any positive integer p satisfying $i\leq p<j$,

$$\sum_{a=1}^{p}\sum_{b=p+1}^{n}\sum_{t=1}^{s}\left|I_{a,b,t}^{\sigma}\right|=\sum_{a'=p+1}^{n}\sum_{b'=1}^{p}\sum_{t=1}^{s}\left|I_{a',b',t}^{\sigma}\right|$$

Then there exist $i'>j'$, such that $\sum_{t=1}^{s}\left|I_{i',j',t}^{\sigma}\right|>0$.

- there exist $i>j$ such that $\sum_{t=1}^{s}\left|I_{i,j,t}^{\sigma}\right|>0$, then by Assumption 2.6 for any $i<j$, $l\in J_i$, $t\in J_j$, we have $l<t$, then by (4.16), we obtain

$$L\ \geq\ \min_{p<q}(\mu_p-\mu_q)\sum_{r=1}^{n}\sum_{i=r+1}^{n}\sum_{j=1}^{r}\sum_{t\in J_i}\left|I_{i,j,t}^{\sigma}\right|=\min_{p<q}(\mu_p-\mu_q)\sum_{r=1}^{n}\sum_{i=r+1}^{n}\sum_{j=1}^{r}\left|I_{i,j}^{\sigma}\right|$$

By Lemma 4.1, we have

$$\begin{aligned}
L\ &\geq\ \min_{p<q}(\mu_p-\mu_q)\sum_{r=1}^{n}\sum_{i=1}^{r}\sum_{j=r+1}^{n}\left|I_{i,j}^{\sigma}\right|\\
&=\ \frac{\min_{p<q}(\mu_p-\mu_q)}{2}\left[\sum_{1\leq j<i\leq n}\left|I_{i,j}^{\sigma}\right|(i-j)+\sum_{1\leq i<j\leq n}\left|I_{i,j}^{\sigma}\right|(j-i)\right]\\
&\geq\ \frac{\min_{p<q}(\mu_p-\mu_q)}{2}\sum_{i=1}^{n}\sum_{j\in[n]\setminus\{i\}}\left|I_{i,j}^{\sigma}\right|\\
&\geq\ \frac{C_1N}{2}\sum_{i=1}^{n}\sum_{j\in[n]\setminus\{i\}}\left|I_{i,j}^{\sigma}\right|
\end{aligned}$$

where the last inequality follows from Assumption 2.6. Then the Lemma follows.

□

Then

$$\begin{aligned}
(4.17)\qquad &\left|\frac{x_{\sigma_0(1)}^{\lambda_1}\cdots x_{\sigma_0(N)}^{\lambda_N}}{x_{\sigma(1)}^{\lambda_1}\cdots x_{\sigma(N)}^{\lambda_N}}\right|\\
&=\ \prod_{1\leq i<j\leq n}\prod_{t=1}^{s}\left(\frac{x_i}{x_j}\right)^{\mu_t\left[\left|I_{i,j,t}^{\sigma}\right|-\left|I_{j,i,t}^{\sigma}\right|\right]}\\
&=\ \prod_{1\leq i<j\leq n}\left(\frac{x_i}{x_{i+1}}\cdot\frac{x_{i+1}}{x_{i+2}}\cdot\ldots\cdot\frac{x_{j-1}}{x_j}\right)^{\sum_{t=1}^{s}\mu_t\left[\left|I_{i,j,t}^{\sigma}\right|-\left|I_{j,i,t}^{\sigma}\right|\right]}\\
&\geq\ \left(\min_{1\leq i<j\leq n}\frac{x_i}{x_j}\right)^{\sum_{1\leq i<j\leq n}\sum_{t=1}^{s}\mu_t\left[\left|I_{(i,j,t)}^{\sigma}\right|-\left|I_{(j,i,t)}^{\sigma}\right|\right](j-i)}
\end{aligned}$$

If Assumption 2.6 holds, then by (4.13), we have

$$\left|\frac{x_{\sigma_0(1)}^{\lambda_1}\cdots x_{\sigma_0(N)}^{\lambda_N}}{x_{\sigma(1)}^{\lambda_1}\cdots x_{\sigma(N)}^{\lambda_N}}\right|\geq\left(\min_{1\leq i<j\leq n}\frac{x_i}{x_j}\right)^{\min_{p<q}(\mu_p-\mu_q)}$$

Lemma 4.3. *Let σ_0 be defined as in (2.6) and $\sigma \in \Sigma_N$. Then*

$$\left|\frac{\prod_{i<j, x_{\sigma_0(i)}\neq x_{\sigma_0(j)}} \frac{x_{\sigma_0(i)}}{x_{\sigma_0(i)}-x_{\sigma_0(j)}}}{\prod_{i<j, x_{\sigma(i)}\neq x_{\sigma(j)}} \frac{x_{\sigma(i)}}{x_{\sigma(i)}-x_{\sigma(j)}}}\right| = \left|\prod_{x_i\neq x_j, \sigma_0^{-1}(i)>\sigma_0^{-1}(j), \sigma^{-1}(i)<\sigma^{-1}(j)} \frac{x_j}{x_i}\right| \geq 1$$

Proof. Note that

$$\left|\frac{\prod_{i<j, x_{\sigma_0(i)}\neq x_{\sigma_0(j)}} \frac{x_{\sigma_0(i)}}{x_{\sigma_0(i)}-x_{\sigma_0(j)}}}{\prod_{i<j, x_{\sigma(i)}\neq x_{\sigma(j)}} \frac{x_{\sigma(i)}}{x_{\sigma(i)}-x_{\sigma(j)}}}\right| = \left|\frac{\prod_{x_i\neq x_j, \sigma_0^{-1}(i)<\sigma_0^{-1}(j)} \left(1-\frac{x_j}{x_i}\right)^{-1}}{\prod_{x_i\neq x_j, \sigma^{-1}(i)<\sigma^{-1}(j)} \left(1-\frac{x_j}{x_i}\right)^{-1}}\right|$$

$$= \left|\frac{\prod_{x_i\neq x_j, \sigma_0^{-1}(i)>\sigma_0^{-1}(j), \sigma^{-1}(i)<\sigma^{-1}(j)} \left(1-\frac{x_j}{x_i}\right)}{\prod_{x_i\neq x_j, \sigma_0^{-1}(i)<\sigma_0^{-1}(j), \sigma^{-1}(i)>\sigma^{-1}(j)} \left(1-\frac{x_j}{x_i}\right)}\right| = \left|\prod_{x_i\neq x_j, \sigma_0^{-1}(i)>\sigma_0^{-1}(j), \sigma^{-1}(i)<\sigma^{-1}(j)} \frac{x_j}{x_i}\right|$$

$$\geq 1$$

where the last inequality holds because $\sigma_0^{-1}(i) > \sigma_0^{-1}(j)$, by (2.6), we obtain $x_j \geq x_i$ □

Now we consider

$$\left|\frac{\prod_{i=1}^{n}\prod_{1\leq j<k\leq |T_i|} \frac{\phi_j^{(i,\sigma_0)}-\phi_k^{(i,\sigma_0)}+k-j}{k-j}}{\prod_{i=1}^{n}\prod_{1\leq j<k\leq |T_i|} \frac{\phi_j^{(i,\sigma)}-\phi_k^{(i,\sigma)}+k-j}{k-j}}\right|$$

It is straightforward to see that

$$\left|\frac{\prod_{i=1}^{n}\prod_{1\leq j<k\leq |T_i|} \frac{\phi_j^{(i,\sigma_0)}-\phi_k^{(i,\sigma_0)}+k-j}{k-j}}{\prod_{i=1}^{n}\prod_{1\leq j<k\leq |T_i|} \frac{\phi_j^{(i,\sigma)}-\phi_k^{(i,\sigma)}+k-j}{k-j}}\right| \geq \prod_{i=1}^{n}\prod_{1\leq j<k\leq |T_i|} \frac{k-j}{\phi_j^{(i,\sigma)}-\phi_k^{(i,\sigma)}+k-j} \tag{4.18}$$

Before giving a lower bound to the right hand side of (4.18), we first recall the following estimates for factorials.

Lemma 4.4. *Let k be an arbitrary positive integer. Then*

$$\sqrt{2\pi k}\left(\frac{k}{e}\right)^k e^{\frac{1}{12k+1}} \leq k! \leq \sqrt{2\pi k}\left(\frac{k}{e}\right)^k e^{\frac{1}{12k}}$$

Proof. See Exercise 3.1.9 of [13]. □

We have the following lemma

Lemma 4.5. *Under Assumptions 2.5 and 2.6, we have*

$$\prod_{i=1}^{n}\prod_{1\leq j<k\leq |T_i|} \frac{k-j}{\phi_j^{(i,\sigma)}-\phi_k^{(i,\sigma)}+k-j} \geq e^{-C_3N^2} \tag{4.19}$$

where $C_3 > 0$ is a constant independent of N.

Proof. For any $1 \leq j < k \leq |T_i|$, there exists $j', k' \in [N] \cap \sigma^{-1}(T_i)$, $j' < k'$, such that

$$0 \leq \phi_j^{(i,\sigma)} - \phi_k^{(i,\sigma)} = \lambda_{j'} - \lambda_{k'} + \eta_{j'} - \eta_{k'}$$

Under Assumption 2.6, we have

$$0 \leq \lambda_{j'} - \lambda_{k'} \leq C_2 N.$$

Moreover

$$0 \leq \eta_{j'} - \eta_{k'} \leq N - |T_i|$$

and

$$0 \leq k - j \leq |T_i|$$

Therefore we have

$$\begin{aligned}\prod_{i=1}^{n} \prod_{1\leq j<k\leq|T_i|} \frac{k-j}{\phi_j^{(i,\sigma)} - \phi_k^{(i,\sigma)} + k - j} &\geq \prod_{i=1}^{n} \frac{\prod_{j=1}^{|T_i|-1}(j!)}{\{(C_2+1)N\}^{\frac{|T_i|(|T_i|-1)}{2}}} \\ &\geq \prod_{i=1}^{n}\left(\frac{|T_i|!}{\{(C_2+1)N\}^{|T_i|}}\right)^{|T_i|}\end{aligned}$$

By Lemma 4.4 and Assumption 2.5 we have

$$\frac{|T_i|!}{\{(C_2+1)N\}^{|T_i|}} \geq \left(\frac{|T_i|}{eN(C_2+1)}\right)^{|T_i|} = \left(\frac{1}{en(C_2+1)}\right)^{\frac{N}{n}}$$

Then

$$\prod_{i=1}^{n} \prod_{1\leq j<k\leq|T_i|} \frac{k-j}{\phi_j^{(i,\sigma)} - \phi_k^{(i,\sigma)} + k - j} \geq \left(\frac{1}{en(C_2+1)}\right)^{\frac{N^2}{n}}$$

Choose $C_3 = \frac{1}{n}(1 + \log n + \log(C_2+1))$, then the lemma follows. □

Remark 4.6. *Lemma 4.5 can also be obtained as follows. Note that the left hand side of (4.19) is exactly the reciprocal of $\prod_{i=1}^n s_{\phi^{(i,\sigma)}}(1,\ldots,1)$. The Schur polynomial $s_{\phi^{(i,\sigma)}}(1,\ldots,1)$ counts the number of perfect matchings on a contracting hexagon lattice with boundary partition given by $\phi^{(i,\sigma)}$. Under Assumption 2.6, all the boundaries of the contracting hexagon lattice grow linearly in N, and therefore the number of vertices in the contraction hexagon lattice is $O(N^2)$. Hence the total number of perfect matchings is bounded above by $e^{O(N^2)}$.*

Proposition 4.7. *Suppose Assumptions 2.8 and 2.6 hold, and let α be given as in Assumption 2.8. For each given $\{a_i, b_i\}_{i=1}^n$, when α is sufficiently large, for any $\sigma \notin \overline{\sigma}_0$ we have*

$$\left|\frac{\left(\prod_{i=1}^n x_{i,N}^{|\phi^{(i,\sigma_0)}(N)|}\right)\left(\prod_{i=1}^n s_{\phi^{(i,\sigma_0)}(N)}(1,\ldots,1)\right)}{\left(\prod_{i=1}^n x_{i,N}^{|\phi^{(i,\sigma)}(N)|}\right)\left(\prod_{i=1}^n s_{\phi^{(i,\sigma)}(N)}(1,\ldots,1)\right)}\right| \times \left|\frac{\left(\prod_{i<j, x_{\sigma_0(i),N}\neq x_{\sigma_0(j),N}} \frac{1}{x_{\sigma_0(i),N} - x_{\sigma_0(j),N}}\right)}{\left(\prod_{i<j, x_{\sigma(i),N}\neq x_{\sigma(j),N}} \frac{1}{x_{\sigma(i),N} - x_{\sigma(j),N}}\right)}\right| \geq e^{CN^2} \tag{4.20}$$

where $C > 0$ is a constant independent of N and σ, and increases as α increases. Indeed, we have

$$\lim_{\alpha\to\infty} C = \infty.$$

Proof. Let $\mathcal{R}$ denote the left hand side of (4.20). By Lemma 4.3, we have

$$\mathcal{R} \geq \frac{\left(\prod_{i=1}^{N} x_{\sigma_0(i),N}^{\lambda_i}\right)\left(\prod_{i=1}^{n} s_{\phi^{(i,\sigma_0)}(N)}(1,\ldots,1)\right)}{\left(\prod_{i=1}^{N} x_{\sigma(i),N}^{\lambda_i}\right)\left(\prod_{i=1}^{n} s_{\phi^{(i,\sigma)}(N)}(1,\ldots,1)\right)} \tag{4.21}$$

By Lemma 4.5, we have

$$\frac{\prod_{i=1}^{n} s_{\phi^{(i,\sigma_0)}(N)}(1,\ldots,1)}{\prod_{i=1}^{n} s_{\phi^{(i,\sigma)}(N)}(1,\ldots,1)} \geq e^{-C_3 N^2}$$

where $C_3 > 0$ is a constant independent of N. Under Assumption 2.8 and Assumption 2.6, we have

$$\left|\frac{x_{\sigma_0(1),N}^{\lambda_1}\cdots x_{\sigma_0(N),N}^{\lambda_N}}{x_{\sigma(1),N}^{\lambda_1}\cdots x_{\sigma(N),N}^{\lambda_N}}\right| \geq \left(\min_{1\leq i<j\leq n} \frac{x_{i,N}}{x_{j,N}}\right)^{\frac{C_1 N}{2}\left(\sum_{1\leq i\leq n}\sum_{1\leq j\leq n, j\neq i}|I_{ij}^{\sigma}|\right)} \tag{4.22}$$

If α in Assumption 2.8 satisfies

$$\alpha > \frac{2C_3}{C_1},$$

we obtain

$$\mathcal{R} \geq e^{-\left(\frac{C_1\alpha}{2}-C_3\right)N^2}$$

Choose $C = \frac{C_1\alpha}{2} - C_3$, then the lemma follows. □

Lemma 4.8. *For $i \in [n]$, let J_i be defined as in (2.8). Under Assumptions 2.5 and 2.6, assume that*

$$J_i = \begin{cases} \{d_i, d_i+1, \ldots, d_{i+1}-1\} & \text{if } 1 \leq i \leq n-1 \\ \{d_n, d_n+1, \ldots, s\} & \text{if } i = n \end{cases}$$

where $d_1, \ldots, d_n$ are positive integers satisfying

$$1 = d_1 < d_2 < \ldots < d_n \leq s$$

Let

$$d_{n+1} := s+1.$$

For $j \in [s]$, let a_j, b_j be given by (2.15).

- *If $i \in [n]$, for $0 \leq k \leq d_{i+1} - d_i - 1$, let*

$$\begin{aligned}
\beta_{i,k} &= n\left(a_1 + \sum_{l=2}^{s-d_i-k+1}(a_l - b_{l-1})\right) + n - i + 1 - n\left(\sum_{l=s-d_i-k+1}^{s-d_i+1}(b_l - a_l)\right)\\
\gamma_{i,k} &= n\left(a_1 + \sum_{l=2}^{s-d_i-k+1}(a_l - b_{l-1})\right) + n - i + 1 - n\left(\sum_{l=s-d_i-k+2}^{s-d_i+1}(b_l - a_l)\right).
\end{aligned}$$

Then the counting measures of $\phi^{(i,\sigma_0)}(N)$ converge weakly to a limit measure $\mathbf{m}_i$ as $N \to \infty$. Moreover,

- *if $i \in [n]$, for $0 \leq k \leq d_{i+1} - d_i - 1$, the limit counting measure $\mathbf{m}_i$ is a probability measure on $[\beta_{i,1}, \gamma_{i,d_{i+1}-d_i-1}]$ with density given by*

$$\frac{\mathbf{dm}_i}{dx} = \begin{cases} 1, & \text{if } \beta_{i,k} < x < \gamma_{i,k}; \\ 0, & \text{if } \gamma_{i,k} \leq x \leq \beta_{i,k+1}. \end{cases}$$

Proof. First of all, it is straightforward to check that Under Assumptions 2.5 and 2.6, for $i \in [n]$, the limiting measure $\mathbf{m}_i$ has constant densities of 0's and 1's on finitely many alternating intervals. It suffices to determine the endpoints of these intervals on which $\mathbf{m}_i$ has constant densities.

Note that under Assumption 2.5, we have

$$x_{\sigma_0\left(\frac{lN}{n}+1\right)} = \ldots = x_{\sigma_0\left(\frac{(l+1)N}{n}\right)} = x_{l+1}, \ \forall l \in \{0, \ldots n-1\}$$

Then for $i \in [n]$, $j \in [N]$, such that

$$j = (i-1)\frac{N}{n} + p \tag{4.23}$$

for some $p \in \left[\frac{N}{n}\right]$ we have

$$\eta_j^{\sigma_0}(N) = \frac{(n-i)N}{n}.$$

Then

$$\lambda_j = \mu_t, \ t \in \{d_i, d_i+1, \ldots, d_{i+1}-1\} \tag{4.24}$$

Under Assumption 2.6, we obtain

$$\phi_p^{(i,\sigma_0)}(N) = \frac{(n-i)N}{n} + \mu_t.$$

Let $k = t - d_i$, then $k \in \{0, 1, \ldots, d_{i+1} - d_i - 1\}$. From (2.15), we obtain

$$\lim_{N\to\infty} \frac{\mu_t}{N} = a_1 + \sum_{l=2}^{s-t+1} (a_l - b_{l-1})$$

Moreover,

$$1 - n\left(\sum_{l=s-t+2}^{s-d_i+1} (b_l - a_l)\right) \geq \lim_{N\to\infty} \frac{\frac{N}{n} - p}{\frac{N}{n}} \geq 1 - n\left(\sum_{l=s-t+1}^{s-d_i+1} (b_l - a_l)\right)$$

Hence for each fixed $t \in J_i$, for all $p \in \left[\frac{N}{n}\right]$ satisfying (4.24) and (4.23), we have

$$\lim_{N\to\infty} \frac{\phi_p^{(i,\sigma_0)}(N) + \frac{N}{n} - p}{\frac{N}{n}} \in [\beta_{i,k}, \gamma_{i,k}]$$

One can check that $\mathbf{m}_i$ has density 1 in $(\beta_{i,k}, \gamma_{i,k})$ for $k \in \{0, 1, \ldots, d_{i+1} - d_i - 1\}$, and density 0 everywhere else. Then the lemma follows. □

Lemma 4.9. *Let σ_0 satisfy (2.6), and let $\overline{\sigma}_0 \in [\Sigma_N/\Sigma_N^X]^r$. For $1 \leq i \leq k$, assume $\frac{u_i}{x_i}$ is in an open complex neighborhood of 1. For any $\sigma \in \Sigma_N$, let*

$$\begin{aligned} G_\sigma \quad = \quad & \left(\prod_{i=1}^n x_i^{|\phi^{(i,\sigma)}(N)|}\right)\left(\prod_{i=1}^r s_{\phi^{(i,\sigma)}(N)}\left(\frac{u_i}{x_i}, \frac{u_{n+i}}{x_i} \ldots, \frac{u_{qn+i}}{x_i}, 1, \ldots, 1\right)\right) \\ & \times \left(\prod_{i=r+1}^n s_{\phi^{(i,\sigma)}(N)}\left(\frac{u_i}{x_i}, \frac{u_{n+i}}{x_i} \ldots, \frac{u_{(q-1)n+i}}{x_i}, 1, \ldots, 1\right)\right) \\ & \times \left(\prod_{i<j, x_{\sigma(i)} \neq x_{\sigma(j)}} \frac{1}{w_{\sigma(i)} - w_{\sigma(j)}}\right) \end{aligned}$$

Suppose that Assumption 2.8 holds. When α in Assumption 2.8 is sufficiently large, we have

$$\left|\frac{G_{\sigma_0}}{G_\sigma}\right| \geq e^{CN^2}$$

where $C > 0$ is a constant independent of σ, N and $(u_1, \ldots, u_k)$.

Proof. First of all note that

$$\left|\frac{\prod_{i<j, x_{\sigma(i)} \neq x_{\sigma(j)}} \frac{1}{w_{\sigma(i)} - w_{\sigma(j)}}}{\prod_{i<j, x_{\sigma_0(i)} \neq x_{\sigma_0(j)}} \frac{1}{w_{\sigma_0(i)} - w_{\sigma_0(j)}}}\right| = 1$$

We can express the quotient of two Schur polynomials as an HCIZ integral as follows

$$\begin{aligned}
&\frac{s_{\phi^{(i,\sigma)}(N)}\left(\frac{u_i}{x_i}, \frac{u_{n+i}}{x_i} \ldots, \frac{u_{qn+i}}{x_i}, 1, \ldots, 1\right)}{s_{\phi^{(i,\sigma)}(N)}(1, \ldots, 1)} \\
= &\left[\prod_{1 \leq j < l \leq q+1} \frac{\log\left(\frac{u_{(j-1)n+i}}{x_i}\right) - \log\left(\frac{u_{(l-1)n+i}}{x_i}\right)}{\frac{u_{(j-1)n+i}}{x_i} - \frac{u_{(l-1)n+i}}{x_i}}\right]\left[\prod_{1 \leq t \leq q+1} \prod_{q+2 \leq j \leq N} \frac{\log\left(\frac{u_{(t-1)n+i}}{x_i}\right)}{\frac{u_{(t-1)n+i}}{x_i} - 1}\right] \\
&\times \int_{U(N)} e^{\mathrm{tr}(U^* A_N U B_N)} dU
\end{aligned}$$

where

$$\begin{aligned}
A_N &= \mathrm{diag}\left[\log\left(\frac{u_i}{x_i}\right), \log\left(\frac{u_{n+i}}{x_i}\right), \ldots, \log\left(\frac{u_{qn+i}}{x_i}\right), 0, \ldots, 0\right] \\
B_N &= \mathrm{diag}\left[\phi_1^{(i,\sigma)}(N) + N - 1, \phi_2^{(i,\sigma)}(N) + N - 2, \ldots, \phi_N^{(i,\sigma)}(N)\right]
\end{aligned}$$

Under Assumptions 2.5 and 2.6, we have

$$\begin{aligned}
&|\phi_j^{(i,\sigma)}(N) + N - j| \leq CN \\
&\left|\log\left(\frac{u_{sn+i}}{x_i}\right)\right| \leq C, \text{ for } 0 \leq s \leq q
\end{aligned}$$

for all $j \in [N]$ and $i \in [n]$, where $C > 0$ is a constant independent of i, j and N. We obtain

$$\begin{aligned}
\left|\int_{U(N)} e^{\mathrm{tr}(U^* A_N U B_N)} dU\right| &= \left|\int_{U(N)} e^{\sum_{1 \leq i \leq q+1, 1 \leq j \leq N} A_N(i,i) U(i,j) B_N(j,j) \overline{U(i,j)}} dU\right| \\
&\leq e^{(q+1)C^2 N}
\end{aligned}$$

Then there exists a constant $C > 0$ (which might be different from the C above, we abuse the notation here, and similar below), such that

$$\left|\frac{s_{\phi^{(i,\sigma)}(N)}\left(\frac{u_i}{x_i}, \frac{u_{n+i}}{x_i} \ldots, \frac{u_{qn+i}}{x_i}, 1, \ldots, 1\right)}{s_{\phi^{(i,\sigma)}(N)}(1, \ldots, 1)}\right| \leq e^{CN}. \tag{4.25}$$

Note also that for each $i \in [N]$, as $N \to \infty$, by Lemma 4.8 the counting measure for $\phi^{(i,\sigma_0)}(N)$ converges to a measure $\mathbf{m}_i$. By Theorem 4.2 of [7], we have

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \log \frac{s_{\phi^{(i,\sigma_0)}(N)}\left(\frac{u_i}{x_i}, \frac{u_{n+i}}{x_i} \ldots, \frac{u_{qn+i}}{x_i}, 1, \ldots, 1\right)}{s_{\phi^{(i,\sigma_0)}(N)}(1, \ldots, 1)} \\
=& \frac{1}{n}\left[H_{\mathbf{m}_i}\left(\frac{u_i}{x_i}\right) + H_{\mathbf{m}_i}\left(\frac{u_{n+i}}{x_i}\right) + \ldots + H_{\mathbf{m}_i}\left(\frac{u_{qn+i}}{x_i}\right)\right]
\end{aligned}$$

where $H_{\mathbf{m}_i}$ is a function defined by (2.10). In particular, $H_{\mathbf{m}_i}(u)$ is an holomorphic function of u when u is in an open complex neighborhood of 1. Therefore there exists constant $C > 0$, such that

$$e^{-CN} \leq \left| \frac{s_{\phi^{(i,\sigma_0)}(N)}\left(\frac{u_i}{x_i}, \frac{u_{n+i}}{x_i} \ldots, \frac{u_{qn+i}}{x_i}, 1, \ldots, 1\right)}{s_{\phi^{(i,\sigma_0)}(N)}(1, \ldots, 1)} \right| \leq e^{CN} \tag{4.26}$$

when $\frac{u_i}{x_i}, \frac{u_{n+i}}{x_i}, \ldots, \frac{u_{qn+i}}{x_i}$ are in an open complex neighborhood of 1 and when N is sufficiently large. Then

$$\left| \frac{s_{\phi^{(i,\sigma_0)}(N)}\left(\frac{u_i}{x_i}, \frac{u_{n+i}}{x_i} \ldots, \frac{u_{qn+i}}{x_i}, 1, \ldots, 1\right)}{s_{\phi^{(i,\sigma)}(N)}\left(\frac{u_i}{x_i}, \frac{u_{n+i}}{x_i} \ldots, \frac{u_{qn+i}}{x_i}, 1, \ldots, 1\right)} \right| = D \cdot E \cdot F$$

where

$$\begin{aligned}
D &= \left| \frac{s_{\phi^{(i,\sigma_0)}(N)}\left(\frac{u_i}{x_i}, \frac{u_{n+i}}{x_i} \ldots, \frac{u_{qn+i}}{x_i}, 1, \ldots, 1\right)}{s_{\phi^{(i,\sigma_0)}(N)}(1, \ldots, 1)} \right| \\
E &= \left| \frac{s_{\phi^{(i,\sigma_0)}(N)}(1, \ldots, 1)}{s_{\phi^{(i,\sigma)}(N)}(1, \ldots, 1)} \right| \\
F &= \left| \frac{s_{\phi^{(i,\sigma)}(N)}(1, \ldots, 1)}{s_{\phi^{(i,\sigma)}(N)}\left(\frac{u_i}{x_i}, \frac{u_{n+i}}{x_i} \ldots, \frac{u_{qn+i}}{x_i}, 1, \ldots, 1\right)} \right|
\end{aligned}$$

For some constant $C > 0$, we have $D \geq e^{-CN}$ by (4.26), $F \geq e^{-CN}$ by (4.25). Then the lemma follows from Proposition 4.7.

□

Recall that J_i was defined as in (2.8).

Lemma 4.10. *For $i \in [n]$, by Lemma 4.8 let $\mathbf{m}_i$ is the limit of counting measures for the partition $\phi^{(i,\sigma_0)}$ as $N \to \infty$, where the counting measure for a partition is defined by (2.2). Let $H_{\mathbf{m}_i}$ be defined by (2.10). Under Assumptions 2.5, 2.8 and 2.6, the moment generating function for $\mathbf{m}_i$, as defined by (2.11), is given by*

$$S_{\mathbf{m}_i}(z) = \log \left[\prod_{j=0}^{d_{i+1}-d_i-1} \frac{1 - \beta_{i,j} z}{1 - \gamma_{i,j} z} \right].$$

Proof. Under Assumption 2.5, 2.8 and 2.6, the components in $\phi^{(i,\sigma_0)}$ takes finitely many values for all N. Note that if $\sigma_0(j)\in T_i$, then

$$\eta_j^{\sigma_0}=N-\frac{iN}{n}.$$

Then

- if $\sigma_0(j)\in T_i$, $i\in[n]$, then there exists an integer a, such that $0\leq a\leq d_{i+1}-d_i-1$ and

$$\lambda_j=\mu_{d_i+a}=\sum_{l=2}^{s-d_i-a+1}(A_l-B_{l-1}-1)$$

By Lemma 4.8, for each $i\in[n]$ the k-th moment $M_k(\mathbf{m}_i)$ of $\mathbf{m}_i$ is

$$M_k(\mathbf{m}_i)=\sum_{j=0}^{d_{i+1}-d_i-1}\frac{\gamma_{i,j}^{k+1}-\beta_{i,j}^{k+1}}{k+1}$$

Then the Stieljes transformation for $\mathbf{m}_i$ is

$$\begin{aligned}\mathrm{St}_{\mathbf{m}_i}(t) &= \frac{1}{t}+\frac{M_1(\mathbf{m}_i)}{t^2}+\frac{M_2(\mathbf{m}_i)}{t^3}+\dots\\ &= \sum_{j=0}^{d_{i+1}-d_i-1}\log\frac{t-\beta_{i,j}}{t-\gamma_{i,j}}.\end{aligned}$$

Then the lemma follows from the fact that $S_{\mathbf{m}_i}(z)=\mathrm{St}_{\mathbf{m}_i}\left(\frac{1}{z}\right)$. □

By (2.10), we can also compute

$$H'_{\mathbf{m}_i}(u)=\frac{1}{uS_{\mathbf{m}_i}^{(-1)}(\log u)}-\frac{1}{u-1}$$

By Lemma 4.10, we obtain

$$H'_{\mathbf{m}_i}(u)=\frac{1}{uz}-\frac{1}{u-1};$$

where z and u satisfy the following condition:

$$u=\prod_{j=1}^{d_{i+1}-d_i-1}\frac{1-\beta_{i,j}z}{1-\gamma_{i,j}z},\qquad\forall i\in[n].$$

Proof of Theorem 2.9. By Proposition 2.4, both $s_{\lambda(N)}(u_1x_{1,N},\dots,u_kx_{1,N},x_{k+1,N},\dots,x_{N,N})$ and $s_{\lambda(N)}(x_{1,N},\dots,x_{N,N})$ can be expressed as a sum of $\left|[\Sigma_N/\Sigma_N^X]^r\right|$ terms. Under Assumption 2.8, we have

$$\left|[\Sigma_N/\Sigma_N^X]^r\right|=\frac{N!}{\left[\left(\frac{N}{n}\right)!\right]^n}$$

By Stirling's formula (see also Lemma 4.4) we obtain

$$\lim_{N\to\infty}\left|[\Sigma_N/\Sigma_N^X]^r\right|^{\frac{1}{N}}=n. \tag{4.27}$$

By Proposition 4.7 and (4.27), when α in Assumption 2.8 is sufficiently large,

$$\begin{aligned} & s_{\lambda(N)}(x_1,\ldots,x_N) \\ = & \left(\prod_{i=1}^{n} x_i^{|\phi^{(i,\sigma_0)}(N)|}\right)\left(\prod_{i=1}^{n} s_{\phi^{(i,\sigma_0)}(N)}(1,\ldots,1)\right)\left(\prod_{i<j,x_{\sigma_0(i)}\neq x_{\sigma_0(j)}} \frac{1}{x_{\sigma_0(i)}-x_{\sigma_0(j)}}\right)\left(1+e^{-CN^2}\right) \end{aligned}$$

For $1\leq i\leq N$, let

$$w_{i,N}=\begin{cases} u_i x_{i,N} & \text{for } 1\leq i\leq k \\ x_{i,N} & \text{for } k+1\leq i\leq N \end{cases}$$

When each one of $u_1,\ldots,u_k$ is in an open complex neighborhood of 1, respectively, by Proposition 3.4 and Lemma 4.9, we have

$$\begin{aligned} & s_{\lambda(N)}(w_{1,N},\ldots,w_{N,N}) \\ = & \left(\prod_{i=1}^{n} x_{i,N}^{|\phi^{(i,\sigma_0)}(N)|}\right)\left(\prod_{i=1}^{r} s_{\phi^{(i,\sigma_0)}(N)}\left(u_i,u_{n+i},\ldots,u_{qn+i},1,\ldots,1\right)\right) \\ & \times\left(\prod_{i=r+1}^{n} s_{\phi^{(i,\sigma_0)}(N)}\left(u_i,u_{n+i},\ldots,u_{(q-1)n+i},1,\ldots,1\right)\right) \\ & \times\left(\prod_{i<j,x_{\sigma_0(i)}\neq x_{\sigma_0(j)}} \frac{1}{w_{\sigma_0(i),N}-w_{\sigma_0(j),N}}\right)\left(1+e^{-CN^2}\right) \end{aligned}$$

where $w_1,\ldots,w_N$ is defined as in Lemma 4.9, for some constant $C>0$ independent of N.

Therefore,

$$\begin{aligned} & \lim_{N\to\infty}\frac{1}{N}\log\frac{s_{\lambda(N)}(u_1x_{1,N},\ldots,u_kx_{k,N},x_{k+1,N},\ldots,x_{N,N})}{s_{\lambda(N)}(x_{1,N},\ldots,x_{N,N})} \\ = & \lim_{N\to\infty}\frac{1}{N}\log\left[\frac{\prod_{i=1}^{r} s_{\phi^{(i,\sigma_0)}(N)}\left(u_i,u_{n+i}\ldots,u_{qn+i},1,\ldots,1\right)}{\prod_{i=1}^{n} s_{\phi^{(i,\sigma_0)}(N)}(1,\ldots,1)}\right. \\ & \left.\frac{\prod_{i=r+1}^{n} s_{\phi^{(i,\sigma_0)}(N)}\left(u_i,u_{n+i}\ldots,u_{(q-1)n+i},1,\ldots,1\right)\left(\prod_{i<j,x_{\sigma_0(i),N}\neq x_{\sigma_0(j),N}}\frac{1}{w_{\sigma_0(i),N}-w_{\sigma_0(j),N}}\right)}{\left(\prod_{i<j,x_{\sigma_0(i),N}\neq x_{\sigma_0(j),N}}\frac{1}{x_{\sigma_0(i),N}-x_{\sigma_0(j),N}}\right)}\right] \\ = & S_1+S_2, \end{aligned}$$

where

$$\begin{aligned} S_1 = & \lim_{N\to\infty}\frac{1}{N}\log \\ & \left[\frac{\prod_{i=1}^{r} s_{\phi^{(i,\sigma_0)}(N)}\left(u_i,u_{n+i}\ldots,u_{qn+i},1,\ldots,1\right)\prod_{i=r+1}^{n} s_{\phi^{(i,\sigma_0)}(N)}\left(u_i,u_{n+i}\ldots,u_{(q-1)n+i},1,\ldots,1\right)}{\prod_{i=1}^{n} s_{\phi^{(i,\sigma_0)}(N)}(1,\ldots,1)}\right] \\ S_2 = & \lim_{N\to\infty}\frac{1}{N}\log\left[\prod_{i<j,x_{\sigma_0(i),N}\neq x_{\sigma_0(j),N}}\frac{x_{\sigma_0(i),N}-x_{\sigma_0(j),N}}{w_{\sigma_0(i),N}-w_{\sigma_0(j),N}}\right] \end{aligned}$$

Note that for each $1 \leq i \leq n$, $\phi^{(i,\sigma_0)}(N) \in \mathbb{GT}^+_{\frac{N}{n}}$. By Theorem 4.2 of [7], for each $1 \leq i \leq r$, we obtain

$$\lim_{N\to\infty} \frac{1}{N} \log \frac{s_{\phi^{(i,\sigma_0)}(N)}\left(u_i, u_{n+i} \ldots, u_{qn+i}, 1, \ldots, 1\right)}{s_{\phi^{(i,\sigma_0)}(N)}(1, \ldots, 1)} = \frac{1}{n}\left[\sum_{t=0}^{q} H_{\mathbf{m}_i}\left(u_{i+tn}\right)\right]$$

For each $r+1 \leq i \leq n$, we obtain

$$\lim_{N\to\infty} \frac{1}{N} \log \frac{s_{\phi^{(i,\sigma_0)}(N)}\left(u_i, u_{n+i} \ldots, u_{(q-1)n+i}, 1, \ldots, 1\right)}{s_{\phi^{(i,\sigma_0)}(N)}(1, \ldots, 1)} = \frac{1}{n}\left[\sum_{t=0}^{q-1} H_{\mathbf{m}_i}\left(u_{i+tn}\right)\right]$$

Therefore we have

$$S_1 = \frac{1}{n}\left[\sum_{1\leq i\leq k, [i \mod n]\neq 0} H_{\mathbf{m}_{[i \mod n]}}(u_i) + \sum_{1\leq i\leq k, [i \mod n]=0} H_{\mathbf{m}_n}(u_i)\right]$$

Moreover,

$$\begin{aligned}
S_2 &= \lim_{N\to\infty} \frac{1}{N} \log\left[\left(\prod_{i=1}^{r}\prod_{t=0}^{q}\prod_{j=i+1}^{n} \frac{x_{i,N}-x_{j,N}}{u_{i+tn}x_{i,N}-x_{j,N}}\right)\left(\prod_{i=r+1}^{n-1}\prod_{t=0}^{q-1}\prod_{j=i+1}^{n} \frac{x_{i,N}-x_{j,N}}{u_{i+tn}x_{i,N}-x_{j,N}}\right)\right]^{\frac{N}{n}} \\
&= \lim_{N\to\infty} \frac{1}{n} \sum_{\{1\leq i\leq k, [i \mod n]\neq 0.\}} \sum_{\{j=[i \mod n]+1\}}^{n} \log \frac{x_{[i \mod n],N}-x_{j,N}}{u_i x_{[i \mod n],N}-x_{j,N}} \\
&= -\frac{1}{n} \sum_{\{1\leq i\leq k, [i \mod n]\neq 0.\}} \sum_{\{j=[i \mod n]+1\}}^{n} \log(u_i)
\end{aligned}$$

where the last identity is obtained from Assumption 2.8. Then the theorem follows. □

5. Periodic dimer model on contracting square-hexagon lattice with piecewise boundary conditions: limit of the moments of the counting measure

In this section, we study the periodic dimer model on contracting square-hexagon lattice with edge-weight period $1 \times n$ and piecewise boundary conditions by analyzing the Schur function at a general point using the formula in Theorem 2.4 and Corollary 3.4. The main goal is to prove Theorem 2.18. The idea is to define a Schur generating function (see Definition 5.2), such that the moments of the counting measure can be computed by the derivatives of the Schur generating function. The Schur generating function for the uniform perfect matchings on the hexagon lattice was defined and analyzed in [6]; for the periodic perfect matchings on the square-hexagon lattice with periodic boundary conditions was defined and analyzed in [5]. Here we consider the case that the edge weights are periodic and the boundary condition is piecewise. We first recall a few lemmas proved in [5].

Recall that to the boundary row $\Omega = (\Omega_1 < \cdots < \Omega_N)$ of a contracting square-hexagon lattice is naturally associated a partition $\omega \in \mathbb{GT}^+_N$ of length N by:

$$\omega = (\Omega_N - N, \ldots, \Omega_1 - 1).$$

Proposition 5.1. *([5])Let $\mathcal{R}(\Omega, \check{a})$ be a contracting square-hexagon lattice, such that for each $1 \leq i \leq N$*

- *each NE-SW edge joining the $(2i)$th row to the $(2i+1)$th row has weight x_i; and*

- *each NE-SW edge joining the* $(2i-1)$*th row to the* $2i$*th row has weight* y_i*, if such an edge exists;*
- *All the other edges have weight 1.*

Then the partition function for perfect matchings on $\mathcal{R}(\Omega, \check{a})$ *is given by*

$$Z = \left[\prod_{i \in I_2} \Gamma_i\right] s_\omega(x_1, \ldots, x_N)$$

where $\omega \in \mathbb{GT}_N$ *describes the bottom boundary condition of* $\mathcal{R}(\Omega, \check{a})$*, and for* $i \in I_2$*,* Γ_i *is defined by*

$$\Gamma_i = \prod_{t=i+1}^{N} (1 + y_i x_t)\,. \tag{5.1}$$

.

Definition 5.2. *Let*

$$X = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N; \tag{5.2}$$

and

$$(u_1, \ldots, u_N) \in \mathbb{C}^N.$$

Let ρ_N *be a probability measure on* $\mathbb{GT}_N$*. The the* Schur generating function *with respect to* ρ_N*,* X *is given by*

$$S_{\rho_N, X}(u_1, \ldots, u_N) = \sum_{\lambda \in \mathbb{GT}_N} \rho_N(\lambda) \frac{s_\lambda(u_1, \ldots, u_N)}{s_\lambda(x_1, \ldots, x_N)}$$

For a positive integer s, write $\overline{s} = s \mod n$.

Lemma 5.3. *Suppose that Assumption 2.5 holds. Let*

$$\begin{aligned} X^{(N-t)} &= (x_{\overline{t+1}}, \ldots, x_{\overline{N}}), \\ Y^{(t)} &= (x_{\overline{1}}, \ldots, x_{\overline{t}}) \end{aligned}$$

for each integer t *satisfying* $0 \leq t \leq N-1$*. Let* $k \in \{2t+1, 2t+2\}$*. Let* ω *be the partition corresponding to the configuration on the boundary row, let* ρ^k *be the probability measure on* $\mathbb{GT}^+_{N-t}$ *which is the distribution of partitions corresponding to the dimer configuration on the* k*th row of vertices of* $\mathcal{R}(\Omega, \check{a})$*, counting from the bottom. Then the Schur generating function, as defined in Definition 5.2, can be computed by*

(1) If $t = 2k+1$*, then*

$$S_{\rho^k, X^{(N-t)}}(u_1, \ldots, u_{N-t}) = \frac{s_\omega\left(u_1, \ldots, u_{N-t}, Y^{(t)}\right)}{s_\omega(X^{(N)})} \prod_{i \in \{1, \ldots, t\} \cap I_2} \prod_{j=1}^{N-t} \left(\frac{1 + y_{\overline{i}} u_j}{1 + y_{\overline{i}} x_{\overline{t+j}}}\right).$$

(2) If $t = 2k+2$*, then*

$$S_{\rho^k, X^{(N-t)}}(u_1, \ldots, u_{N-t}) = \frac{s_\omega\left(u_1, \ldots, u_{N-t}, Y^{(t)}\right)}{s_\omega(X^{(N)})} \prod_{i \in \{1, \ldots, t+1\} \cap I_2} \prod_{j=1}^{N-t} \left(\frac{1 + y_{\overline{i}} u_j}{1 + y_{\overline{i}} x_{\overline{t+j}}}\right),$$

for $k = 2t+2,\ t = 0, 1, \ldots, N-1.$

where I_2 *is defined in Definition 2.12.*

Proof. See Lemma 3.17 of [5]. □

Let

$$\begin{aligned} X_N^{(N-t)} &= (x_{t+1,N},\ldots,x_{N,N}) \\ Y_N^{(t)} &= (x_{1,N},\ldots,x_{t,N}). \end{aligned}$$

By Lemma 5.3, we have

$$\begin{aligned} &S_{\rho^k, X_N^{(N-t)}}(u_1 x_{t+1,N},\ldots,u_{N-t}x_{N,N}) \\ = &\frac{s_{\lambda(N)}(u_1x_{t+1,N},\ldots,u_{N-t}x_{N,N},x_{1,N},\ldots,x_{t,N})}{s_{\lambda(N)}(x_{1,N},\ldots,x_{N,N})} \prod_{i\in\{1,2,\ldots,t/t+1\}\cap I_2}\prod_{j=1}^{N-t}\left(\frac{1+y_iu_jx_{t+j,N}}{1+y_ix_{t+j,N}}\right). \end{aligned}$$

for $k=2t+1$ or $k=2t+2$. Let

$$j = \begin{cases} (i+t) \mod n & \text{if } 1\leq [(i+t) \mod n]\leq n-1 \\ n & \text{if } [(i+t) \mod n]=0 \end{cases}. \tag{5.3}$$

When the edge weights are assigned periodically and satisfy Assumptions 2.8, letting $N\to\infty$, $\frac{t}{N}\to\kappa\in[0,1)$, by Theorem 2.9 we have

$$\begin{aligned} \lim_{(1-\kappa)N\to\infty}\frac{1}{(1-\kappa)N}\log S_{\rho^k,X_N^{(N-t)}}(u_1x_{t+1,N},\ldots,u_lx_{t+l,N},x_{t+1+l,N},\ldots,x_{N-t,N}) \\ = \sum_{1\leq i\leq l}\left[Q_{j,\kappa}(u_i)\right], \end{aligned}$$

where j and s are given by (5.3), respectively; and

$$Q_{j,\kappa}(u) = \begin{cases} \frac{1}{1-\kappa}\left[Q_j(u)+\frac{\kappa}{n}\sum_{r\in\{1,2,\ldots,n\}\cap I_2}\log\frac{1+y_rux_1}{1+y_rx_1}\right] & \text{if } [j \mod n]=1 \\ \frac{Q_j(u)}{1-\kappa} & \text{otherwise} \end{cases}$$

Let $p\geq 1$ be a positive integer. Let $\rho_{\lfloor(1-\kappa)N\rfloor} := \rho^{2(N-\lfloor(1-\kappa)N)\rfloor)+1}$ be a probability measure on $\mathbb{GT}^+_{\lfloor(1-\kappa)N\rfloor}$(Indeed, we will obtain exactly the same result in the limit as $N\to\infty$ if we define $\rho_{\lfloor(1-\kappa)N\rfloor} := \rho^{2(N-\lfloor(1-\kappa)N)\rfloor)+2}$), and let $\mathbf{m}_{\rho_{\lfloor(1-\kappa)N\rfloor}}$ be the corresponding random counting measure. Let $\mathcal{N}=\lfloor(1-\kappa)N\rfloor$. Let

$$\begin{aligned} U &= (u_1,\ldots,u_N); \\ X_N &= (x_{1,N},\ldots,x_{N,N}); \\ U_{X,N} &= (u_1x_{1,N},\ldots,u_Nx_{N,N}). \\ U_{X,N}^{(N-t)} &= (u_1x_{t+1,N},\ldots,u_{N-t}x_{N,N}). \end{aligned}$$

Suppose that X_N satisfies Assumption 2.8.

For a positive integer p, define an operator

$$\mathcal{D}_{p,N} = \frac{1}{\prod_{1\leq i<j\leq N}(u_ix_{i,N}-u_jx_{j,N})}\circ\left(\sum_{i=1}^N\left(u_i\frac{\partial}{\partial u_i}\right)^p\right)\circ\prod_{1\leq i<j\leq N}(u_ix_{i,N}-u_jx_{j,N})$$

where $\circ$ denotes composition of operators; the left operator and the right operator above are multiplication operators, and the middle operator above is a differential operator.

Let $\lambda \in \mathbb{GT}_N^+$ be a length-N partition. Explicit computations show that

$$\mathcal{D}_{p,N} s_\lambda(U_{X,N}) = \sum_{i=1}^{N} (\lambda_i + N - i)^p s_\lambda(U_{X,N}) \tag{5.4}$$

We write the Schur generating function as defined by Definition 5.2 as

$$S_{\rho_{\lfloor(1-\kappa)N\rfloor}, X_N^{(N-t)}}\left(U_{X,N}^{(N-t)}\right) = \exp\left(\sum_{i=1}^{N} \mathcal{N} Q_{j,\kappa}(u_i)\right) T_{N,\kappa}\left(U_{X,N}^{(N-t)}\right)$$

Since $S_{\rho_{\lfloor(1-\kappa)N\rfloor}, X_N^{(N-t)}}\left(X_N^{(N-t)}\right) = 1$, the definition of $Q_{j,\kappa}$ implies that $Q_{j,\kappa}(1) = 0$ for all $j \in [N]$ and $0 < \kappa < 1$. Therefore $T_{N,\kappa}\left(X_N^{(N-t)}\right) = 1$ and

$$\lim_{N\to\infty} \frac{1}{N} \log\left[T_{N,\kappa}\left(u_1 x_{t+1,N}, \ldots, u_l x_{t+l,N}, x_{t+l+1,N}, \ldots, x_{N,N}\right)\right] = 0.$$

for any fixed positive integer l. Moreover, the convergence is uniform when $(u_1, \ldots, u_l)$ is in an open complex neighborhood of $(1, \ldots, 1)$. Therefore any partial derivative of $T_{N,\kappa}\left(u_1 x_{t+1,N}, \ldots, u_l x_{t+l,N}, x_{t+l+1,N}, \ldots, x_{N,N}\right)$ with respect to $(u_1, \ldots, u_l)$, divided by $N T_{N,\kappa}\left(u_1 x_{t+1,N}, \ldots, u_l x_{t+l,N}, x_{t+l+1,N}, \ldots, x_{N,N}\right)$ tends to 0 uniformly when $(u_1, \ldots, u_l)$ is in a certain complex neighborhood of $(1, \ldots, 1)$.

We write

$$\begin{aligned} &\mathcal{D}_{p,\lfloor(1-\kappa)N\rfloor} \\ =\ & \frac{1}{\prod_{1\le i<j\le N-t}(u_i x_{i+t,N} - u_j x_{j+t,N})} \circ \left(\sum_{i=1}^{N-t}\left(u_i \frac{\partial}{\partial u_i}\right)^p\right) \circ \prod_{1\le i<j\le N-t}(u_i x_{i+t,N} - u_j x_{j+t,N}) \end{aligned}$$

By (5.4) we have

$$\begin{aligned} &\mathbb{E}\left(\int_{\mathbb{R}} x^p \mathbf{m}_{\rho_{\lfloor(1-\kappa)N\rfloor}}(dx)\right)^m \\ &= \frac{1}{\mathcal{N}^{m(p+1)}} \lim_{(u_1,\ldots,u_N)\to(1,\ldots,1)} \left(\mathcal{D}_{p,\lfloor(1-\kappa)N\rfloor}\right)^m S_{\rho_{\lfloor(1-\kappa)N\rfloor}, X_N^{(N-t)}}\left(U_{X,N}^{(N-t)}\right) \end{aligned} \tag{5.5}$$

Using the Leibnitz rule to expand $\left(\mathcal{D}_{p,\lfloor(1-\kappa)N\rfloor}\right)^m S_{\rho_{\lfloor(1-\kappa)N\rfloor}, X_N^{(N-t)}}\left(U_{X,N}^{(N-t)}\right)$, we obtain a linear combination of terms of the following form

$$\begin{aligned} &(u_{g_1}\cdots u_{g_\gamma}) \cdot \left(\frac{\frac{\partial}{\partial u_{t_1}} \cdots \frac{\partial}{\partial u_{t_\tau}} \prod_{1\le i<j\le N-t}(u_i x_{i+t,N} - u_j x_{j+t,N})}{\prod_{1\le i<j\le N-t}(u_i x_{i+t,N} - u_j x_{j+t,N})}\right) \\ &\times \left(\frac{\partial}{\partial u_{a_1}} \cdots \frac{\partial}{\partial u_{a_\alpha}} \exp\left(\sum_{i=1}^{N-t} \mathcal{N} Q_{j,\kappa}(u_i)\right)\right) \cdot \left(\frac{\partial}{\partial u_{b_1}} \cdots \frac{\partial}{\partial u_{b_\beta}} T_{N,\kappa}\left(U_{X,N}^{(N-t)}\right)\right), \end{aligned} \tag{5.6}$$

where $\gamma \le mp$ and $\alpha + \beta + \tau \le mp$.

Further expanding the second term in (5.6) we obtain a linear combination of terms with the following form

$$(u_{g_1}\cdots u_{g_\gamma})(x_{t_1+t}\cdots x_{t_\tau+t})\cdot\left(\prod_{(a,b)\in P}\frac{1}{u_a x_{a+t}-u_b x_{b+t}}\right) \tag{5.7}$$
$$\times\left(\frac{\partial}{\partial u_{a_1}}\cdots\frac{\partial}{\partial u_{a_\alpha}}\exp\left(\sum_{i=1}^{N}\mathcal{N}Q_{j,\kappa}(u_i)\right)\right)\cdot\left(\frac{\partial}{\partial u_{b_1}}\cdots\frac{\partial}{\partial u_{b_\beta}}T_{N,\kappa}\left(U_{X,N}^{(N-t)}\right)\right),$$

where $P\subset\{(a,b)|1\leq a<b\leq N-t\}$.

Note that in each derivation of the exponent in (5.7), a multiple of N appears. Also recall that for any $\beta\geq 1$,

$$\frac{\partial}{\partial u_{b_1}}\cdots\frac{\partial}{\partial u_{b_\beta}}T_{N,\kappa}\left(U_{X,N}^{(N-t)}\right)=o(N).$$

Therefore when $m=1$ and N large, the leading term for

$$\mathcal{N}^{p+1}\mathbb{E}\int_{\mathbb{R}}x^p\mathbf{m}_{\rho_{\lfloor(1-\kappa)N\rfloor}}(dx)$$

is the same as that of

$$\left(\frac{1}{\prod_{1\leq i<r\leq\mathcal{N}}(u_i x_{i+t,N}-u_r x_{r+t,N})}\right)$$
$$\times\left(\sum_{a=1}^{\mathcal{N}}u_a^p\frac{\partial^p}{\partial u_a^p}\left[\exp\left(\mathcal{N}\left(\sum_{i=1}^{\mathcal{N}}Q_{j,\kappa}(u_i)\right)\right)\prod_{b<r}(u_b x_{b+t,N}-u_r x_{r+t,N})\right]\right)\Bigg|_{(u_1,\ldots,u_N)=(1,\ldots,1)}.$$

The latter has the same leading term as

$$\begin{aligned}\mathcal{M}_{p,\mathcal{N}} &:= \sum_{l=0}^{p}\sum_{i=1}^{\mathcal{N}}\binom{p}{l}\mathcal{N}^{p-l}u_i^p\frac{\frac{\partial^l}{\partial u_i^l}\prod_{1\leq b<r\leq\mathcal{N}}(u_b x_{b+t,N}-u_r x_{r+t,N})}{\prod_{1\leq b<r\leq\mathcal{N}}(u_b x_{b+t,N}-u_r x_{r+t,N})}Q'_{j,\kappa}(u_i)^{p-l}\Bigg|_{(u_1,\ldots,u_N)=(1,\ldots,1)}\\
&= \sum_{l=0}^{p}\sum_{i=1}^{\mathcal{N}}\binom{p}{l}\mathcal{N}^{p-l}u_i^p\\
&\quad\times\left[\sum_{1\leq j_1<\ldots<j_l\leq\mathcal{N},j_s\neq i(1\leq s\leq l)}\frac{x_{i+t,N}^l}{\prod_{s=1}^{l}(u_i x_{i+t,N}-u_{j_s}x_{j_s+t,N})}\right]Q'_{j,\kappa}(u_i)^{p-l}\Bigg|_{(u_1,\ldots,u_N)=(1,\ldots,1)}\\
&\approx \sum_{i=1}^{\mathcal{N}}u_i^p\left(\mathcal{N}Q'_{j,\kappa}(u_i)+\sum_{r\in\{1,2,\ldots,\mathcal{N}\},r\neq i}\frac{x_{j,N}}{u_i x_{j,N}-u_r x_{r+t,N}}\right)^p\Bigg|_{(u_1,\ldots,u_N)=(1,\ldots,1)}.\end{aligned}$$

where $A\approx B$ means that A and B have the same leading term as $\mathcal{N}\to\infty$. For $i\in[n]$, let

$$S_{\mathcal{N}}(i)=\{j\in[\mathcal{N}]\}:((j-i)\mod n)=0\}=\{an+i;0\leq a\leq\lfloor\mathcal{N}/n\rfloor\}.$$

Then

$$\begin{aligned}\mathcal{M}_{p,\mathcal{N}} &\approx \lim_{(u_1,\ldots,u_N)\to(1,\ldots,1)} \sum_{i=1}^{\mathcal{N}}\sum_{l=0}^{p}\mathcal{N}^{p-l}\binom{p}{l}u_i^p[Q'_{j,\kappa}(u_i)]^{p-l}\left[\sum_{k=0}^{l}\binom{l}{k}\right.\\ &\quad\times\left.\left(\sum_{r\in\{1,2,\ldots,\mathcal{N}\}\setminus S_{\mathcal{N}}(i)}\frac{x_{j,N}}{u_ix_{j,N}-u_rx_{r+t,N}}\right)^{l-k}\left(\sum_{r\in S_{\mathcal{N}}(i)\setminus\{i\}}\frac{x_{j,N}}{u_ix_{j,N}-u_rx_{r+t,N}}\right)^{k}\right]\\ &= \lim_{(u_1,\ldots,u_N)\to(1,\ldots,1)}\sum_{i=1}^{\mathcal{N}}\sum_{k=0}^{p}\sum_{l=k}^{p}\frac{p!}{k!(l-k)!(p-l)!}\\ &\quad\times\left[\sum_{j_1,\ldots,j_k\in S_{\mathcal{N}}(i)\setminus\{i\}}\frac{u_i^p\left[\mathcal{N}Q'_{j,\kappa}(u_i)\right]^{p-l}\left(\sum_{r\in\{1,2,\ldots,\mathcal{N}\}\setminus S_{\mathcal{N}}(i)}\frac{x_{j,N}}{u_ix_{j,N}-u_rx_{r+t,N}}\right)^{l-k}x_{j,N}^k}{\prod_{r=1}^{k}(u_ix_{j,N}-u_{j_r}x_{j_r+t,N})}\right]\\ &= \lim_{(u_1,\ldots,u_N)\to(1,\ldots,1)}\sum_{i=1}^{\mathcal{N}}\sum_{k=0}^{p}\frac{p!}{k!(p-k)!}\\ &\quad\times\left[\sum_{j_1,\ldots,j_k\in S_{\mathcal{N}}(i)\setminus\{i\}}\frac{u_i^p\left[\mathcal{N}Q'_{j,\kappa}(u_i)+\sum_{r\in\{1,2,\ldots,\mathcal{N}\}\setminus S_{\mathcal{N}}(i)}\frac{x_{j,N}}{u_ix_{j,N}-u_rx_{r+t,N}}\right]^{p-k}x_{j,N}^k}{\prod_{s=1}^{k}(u_ix_{j,N}-u_{j_s}x_{j_s+t,N})}\right]\end{aligned}$$

We then apply the following lemma slightly adapted from [7] after a change of variables, to compute the limit as $(u_1,\ldots,u_N)$ approaches $(1,\ldots,1)$:

Lemma 5.4 ([7], Lemma 5.5)**.** *Let $\xi\in\mathbb{C}\setminus\{0\}$ be a nonzero complex number; and $n>0$ be a positive integer. Assume $g(z)$ is analytic in a neighborhood of ξ. Then*

$$\lim_{\forall\ i,z_i\to\xi}\sum_{j=1}^{n}\frac{g(z_j)}{\prod_{i\neq j}(z_j-z_i)}=\frac{\partial^{n-1}}{\partial z^{n-1}}\left(\frac{g(z)}{(n-1)!}\right)\Big|_{z=\xi}.$$

Given $x_{i \mod n}=x_i$, by Lemma 5.4 and Assumption 2.8 we have:

$$\begin{aligned}\lim_{N\to\infty}\frac{\mathcal{M}_{p,\mathcal{N}}}{\mathcal{N}^{p+1}} &= \lim_{N\to\infty}\lim_{(u_1,\ldots,u_N)\to(1,\ldots,1)}\frac{1}{n}\sum_{i=1}^{n}\sum_{k=0}^{p}\frac{p!}{k!(p-k)!}\frac{1}{\mathcal{N}^k(k)!}\\ &\quad\times\left.\frac{\partial^k\left[u_i^p\left(Q'_{j,\kappa}(u_i)+\frac{1}{n}\sum_{1\leq r\leq n,r\neq i}\frac{x_{j,N}}{u_ix_{j,N}-u_rx_{r+t,N}}\right)^{p-k}\right]}{\partial^k u_i}\right|_{u_i=1}\\ &= \lim_{(u_1,\ldots,u_N)\to(1,\ldots,1)}\frac{1}{n}\sum_{i=1}^{n}\left[u_i^p\left(Q'_{j,\kappa}(u_i)+\frac{n-j}{nu_i}\right)^p\right]\Big|_{u_i=1}\end{aligned}$$

Using residue we obtain

$$\mathbb{E}\left[\int_{\mathbb{R}}x^p\mathbf{m}^\kappa(dx)\right]=\frac{1}{2(p+1)\pi\mathbf{i}}\sum_{i=1}^{n}\oint_{C_1}\frac{dz}{z}\left(zQ'_{i,\kappa}(z)+\frac{n-i}{n}+\frac{z}{n(z-1)}\right)^{p+1},$$

where C_1 is a small counterclockwise contour enclosing 1 and no other singularities of the integrand.

Now we need to show that the moments of random measures $\mathbf{m}^\kappa$ become deterministic as $N\to\infty$. It suffices to show that

$$\lim_{N\to\infty}\mathbb{E}\left[\left(\int_{\mathbb{R}}x^p\mathbf{m}^\kappa(dx)\right)^2\right]=\lim_{N\to\infty}\left[\mathbb{E}\left(\int_{\mathbb{R}}x^p\mathbf{m}^\kappa(dx)\right)\right]^2 \tag{5.8}$$

Let $2^{[\mathcal{N}]}$ be the collection of all the subsets of $[\mathcal{N}]$. For $1\leq i,i'\leq\mathcal{N}$, define sets $A_{i,i'},C_{i,i'}\in 2^{[\mathcal{N}]}\times 2^{[\mathcal{N}]}$ as follows. If $i\neq i'$,then

$$\begin{aligned}A_{i,i'}:&=\Big\{(M,M')\in 2^{[\mathcal{N}]}\times 2^{[\mathcal{N}]}:|M|=l,|M'|=l',i\notin M,i'\notin M';\\&\qquad\text{if }i\in M',\text{then }i'\notin M;\text{if }i'\in M,\text{then }i\notin M'\Big\}.\end{aligned}$$

If $i=i'$, then

$$A_{i,i}=\{(M,M')\in 2^{[\mathcal{N}]}\times 2^{[\mathcal{N}]}:|M|=l,|M'|=l',i\notin M,i\notin M';M\cap M'=\emptyset\}.$$

and

$$C_{i,i'}:=\{(M,M')\in 2^{[\mathcal{N}]}\times 2^{[\mathcal{N}]}:|M|=l,|M'|=l',i\notin M,i'\notin M'\}.$$

Let

$$j'=\begin{cases}(i'+t)\mod n & \text{if } 1\leq[(i'+t)\mod n]\leq n-1\\ n & \text{if } [(i'+t)\mod n]=0\end{cases}.$$

Note that the right hand side of (5.8) is

$$\left\{\sum_{l=0}^{p}\sum_{i=1}^{\mathcal{N}}\binom{p}{l}\mathcal{N}^{-l-1}u_i^p\times\left[\sum_{1\leq j_1<\ldots<j_l\leq\mathcal{N},j_s\neq i(1\leq s\leq l)}\frac{x_{i+t,N}^l}{\prod_{s=1}^l(u_ix_{i+t,N}-u_{j_s}x_{j_s+t,N})}\right]Q'_{j,\kappa}(u_i)^{p-l}\Big|_{(u_1,\ldots,u_N)=(1,\ldots,1)}\right\}^2; \tag{5.9}$$

Expanding (5.9), we obtain

$$\sum_{l=0}^{p}\sum_{l'=0}^{p}\sum_{i=1}^{\mathcal{N}}\sum_{i'=1}^{\mathcal{N}}\binom{p}{l}\binom{p}{l'}\mathcal{N}^{-l-l'-2}u_i^pu_{i'}^p\times\left[\sum_{C_{i,i'}}\frac{x_{i+t,N}^l}{\prod_{r\in M}(u_ix_{i+t,N}-u_rx_{r+t,N})}\frac{x_{i'+t,N}^{l'}}{\prod_{r'\in M'}(u_{i'}x_{i'+t,N}-u_{r'}x_{r'+t,N})}\right]Q'_{j,\kappa}(u_i)^{p-l}Q'_{j',\kappa}(u_{i'})^{p-l'}\Big|_{(u_1,\ldots,u_N)=(1,\ldots,1)}. \tag{5.10}$$

Let $m=2$ in (5.5), one can compute that the leading term on the left hand side of (5.8) is

$$\sum_{l=0}^{p}\sum_{l'=0}^{p}\sum_{i=1}^{\mathcal{N}}\sum_{i'=1}^{\mathcal{N}}\binom{p}{l}\binom{p}{l'}\mathcal{N}^{-l-l'-2}u_i^pu_{i'}^p\times\left[\sum_{A_{i,i'}}\frac{x_{j,N}^l}{\prod_{r\in M}(u_ix_{j,N}-u_rx_{r+t,N})}\frac{x_{j',N}^{l'}}{\prod_{r'\in M'}(u_{i'}x_{j',N}-u_{r'}x_{r'+t,N})}\right]Q'_{j,\kappa}(u_i)^{p-l}Q'_{j',\kappa}(u_{i'})^{p-l'}\Big|_{(u_1,\ldots,u_N)=(1,\ldots,1)} \tag{5.11}$$

It is not hard to see that

$$\sum_{i=1}^{\mathcal{N}}\sum_{i'=1}^{\mathcal{N}}\left|A_{i,i'}\triangle C_{i,i'}\right|=o(N^{l+l'+2})$$

Therefore the leading term for (5.10) and (5.11) when N is large are the same. Then (5.8) follows.

6. Periodic dimer model on contracting square-hexagon lattice with piecewise boundary conditions: limit shape of the height function

In this section, we prove the limit shape of the height function defined in Section 2 from the moment formulas of the limit counting measure proved in Section 4.

Let

$$r = n - |I_2 \cap [n]|.$$

In other words, r is the number of rows of white vertices in a fundamental domain such that each white vertex in the row is adjacent to exactly one black vertex in the row above.

Let V_j^* (resp. V_i) be the collection of all the vertices in $\mathcal{R}^*(\Omega, \check{a})$ (resp. $\mathcal{R}(\Omega, \check{a})$) with y-coordinate j (resp. i).

Lemma 6.1. *Define the height function h_M associated to a perfect matching M of the contracting square-hexagon lattice $\mathcal{R}(\Omega, \check{a})$ as in Definition 2.21. Assume that $j \in \mathbb{R}^+$ is such that $V_j^* \neq \emptyset$. Then the value of h_M at the leftmost vertex of V_j^* lies in the following interval*

$$\left[\left(\left\lfloor \frac{j}{n} \right\rfloor - 1\right)(2n-r), \left(\left\lfloor \frac{j}{n} \right\rfloor + 1\right)(2n-r)\right]$$

Proof. By Definition 2.21, the value of h_M at the lexicographic smallest vertex of $\mathcal{R}^*(\Omega, \check{a})$ is 0. Starting from the lexicographic smallest vertex of $\mathcal{R}^*(\Omega, \check{a})$ and moving up along the left boundary of $\mathcal{R}^*(\Omega, \check{a})$, each time we move from a vertex (p, q) to an above vertex (p', q') of $\mathcal{R}^*(\Omega, \check{a})$ on the left boundary, the following cases might occur.

- If we cross an absent NE-SW edge e_1 of $\mathrm{SH}(\check{a})$ and an absent NW-SE edge e_2 of $\mathrm{SH}(\check{a})$ with the write vertices of both e_1 and e_2 on the left, and e_1 and e_2 are incident to a common black vertex of $\mathcal{R}(\Omega, \check{a})$ which has degree 4 in $\mathrm{SH}(\check{a})$, then
$$h_M(p', q') - h_M(p, q) = 2.$$
- if we cross one absent NW-SE edge e of $\mathrm{SH}(\check{a})$ with the write vertex on the left, and e is incident to a black vertex of $\mathcal{R}(\Omega, \check{a})$ which has degree 3 in $\mathrm{SH}(\check{a})$, then
$$h_M(p', q') - h_M(p, q) = 1.$$

When the edge weights of $\mathrm{SH}(\check{a})$ are assigned periodically as in Assumption 2.5, the length of each fundamental domain is n. Moving up by n units in a fundamental domain, there are r rows of black vertices with degree 3, and $n - r$ rows of black vertices with degree 4. Hence moving up by n units along the left boundary of $\mathcal{R}^*(\Omega, \check{a})$, the height increases by $(2n - r)$. Moreover, moving from the lexicographic smallest vertex of $\mathcal{R}^*(\Omega, \check{a})$ to a vertex of $\mathcal{R}^*(\Omega, \check{a})$ with y-coordinate j, we move across at least $\left(\left\lfloor \frac{j}{n} \right\rfloor - 1\right)$ fundamental domains, and at most $\left(\left\lfloor \frac{j}{n} \right\rfloor + 1\right)$ fundamental domains. Then the lemma follows. □

Lemma 6.2. *Let (i, j) be a vertex of $\mathcal{R}^*(\Omega, \check{a})$.*

- *If (i, j) is the center of a hexagon face of $\mathrm{SH}(\check{a})$, then the number of vertices in $V_{j+\frac{1}{4}}$ to the left of (i, j) lies in the following interval*
$$\left[i - \left(\left\lfloor \frac{j}{n} \right\rfloor + 1\right) \cdot \frac{r}{2} - 1, i - \left(\left\lfloor \frac{j}{n} \right\rfloor - 1\right) \cdot \frac{r}{2} + 1\right]$$

- *If (i,j) is the center of a square face of* $\mathrm{SH}(\check{a})$, *then the number of vertices in* V_j *to the left of* (i,j) *lies in the following interval*

$$\left[i-\left(\left\lfloor\frac{j}{n}\right\rfloor+1\right)\cdot\frac{r}{2}-1, i-\left(\left\lfloor\frac{j}{n}\right\rfloor-1\right)\cdot\frac{r}{2}+1\right]$$

Proof. First of all, if the leftmost vertex of V_j^* is the lexicographic smallest vertex of $\mathcal{R}^*(\Omega,\check{a})$, then the number of vertices with y-coordinate j, on the right of the leftmost vertex in V_j^* (including the leftmost vertex of V_j^*) and to the left of (i,j) lies in the interval

$$[i-1, i+1].$$

and $j\in\left\{\frac{1}{2},\frac{3}{4}\right\}$, depending on whether $a_1=1$ or $a_1=0$.

Moving along the left boundary of $\mathcal{R}^*(\Omega,\check{a})$ from a vertex (p,q) of $\mathrm{SH}^*(\Omega,\check{a})$ to an above vertex (p',q') of $\mathrm{SH}^*(\Omega,\check{a})$, the following cases might occur.

- If we cross an absent NE-SW edge e_1 of $\mathrm{SH}(\check{a})$ and an absent NW-SE edge e_2 of $\mathrm{SH}(\check{a})$, and e_1 and e_2 are incident to a common black vertex of $\mathcal{R}(\Omega,\check{a})$ which has degree 4 in $\mathrm{SH}(\check{a})$ then $p'=p$.
- If we cross one absent NW-SE edge e of $\mathrm{SH}(\check{a})$, and e as adjacent to a black vertex of $\mathcal{R}(\Omega,\check{a})$ which has degree 3 in $\mathrm{SH}(\check{a})$, then $p'=p+\frac{1}{2}$.

Moving up by n units in a fundamental domain, there are r rows of black vertices with degree 3, and $n-r$ rows of black vertices with degree 4. Hence moving up by n units along the left boundary of $\mathcal{R}^*(\Omega,\check{a})$, the leftmost vertex along the row of $\mathcal{R}^*(\Omega,\check{a})$ moves to the right by $\frac{r}{2}$ units. Then the lemma follows. □

Lemma 6.3. *Let $(i,j)\in V_j$ be a vertex of $\mathcal{R}(\Omega(N),\check{a})$. Let $L(i)$ be the number of vertices in V_j to the left of (i,j). Then for each perfect matching $M\in\mathcal{M}(\Omega(N),\check{a})$, the number of V-vertices in V_j not to the right of (i,j) (including (i,j) if (i,j) itself is a V-vertex) is*

$$(N+1-\lceil j\rceil)\int \mathbf{1}_{\left[0,\frac{L(i)}{N+1-\lceil j\rceil}\right]}dm\left[\mu^{M,j}\right], \tag{6.1}$$

where $\mu^{M,j}=\left(\mu_1^{M,j}\geq\mu_2^{M,j}\geq\ldots\geq\mu_{N+1-\lceil j\rceil}^{M,j}\right)$ is the signature corresponding to the dimer configuration M restricted on the row of vertices with y-coordinates j, and $m\left[\mu^{M,j}\right]$ is the counting measure for $\mu^{M,j}$.

Proof. The vertex (i,j) of $\mathcal{R}(\Omega(N),\check{a})$ is a vertex on the $(2j)$th row of vertices of $\mathcal{R}(\Omega(N),\check{a})$ by construction. The total number of V-vertices on the $(2j)$th row is $N+1-\lceil j\rceil$.

We define

$$\Delta_M^N(i,j)=|\{1\leq s\leq N+1-\lceil j\rceil : \mu_s^{M,j}+(N+1-\lceil j\rceil)-s\leq L(i)\}|$$

Note that $\mu_s^{M,j}$ is the number of Λ-vertices to the left of the sth V-vertex in the dimer configuration M, where the V-vertices are counted from the right; $(N+1-\lceil j\rceil)-s$ is the number of V-vertices to the left of the sth V-vertex. Hence $\mu_s^{M,j}+(N+1-\lceil j\rceil)-s$ is the total number of vertices to the left of s-th V-vertex in the $(2j)$th row of vertices of $\mathcal{R}(\Omega(N),\check{a})$. Therefore $\Delta_M^N(i,j)$ is exactly the total number of V-vertices in V_j not to the right of (i,j), which contains (i,j) if (i,j) itself is a V-vertex. From the definition of the counting measure, it is straightforward to check that expression (6.1) is equal to $\Delta_M^N(i,j)$. □

Lemma 6.4. *Let $M \in \mathcal{M}(\Omega(N), \check{a})$ be a perfect matching of the contracting square-hexagon lattice $\mathcal{R}(\Omega, \check{a})$. Let (i, j) be a vertex of $\mathcal{R}^*(\Omega, \check{a})$. Then the value of the height function $h_M((i, j))$ lies in the following interval*

$$
\begin{aligned}
(6.2)\qquad & h_M((i,j)) \in \\
& \Bigg[4(N - \lceil j \rceil) \int 1_{\left[0, \frac{L_1(i,j)}{N+1-\lceil j \rceil}\right]} dm \left[\mu^{M,j}\right] - 2L_2(i,j) + \left(\left\lfloor \frac{j}{n} \right\rfloor - 1\right)(2n - r), \\
& 4(N + 1 - \lceil j \rceil) \int 1_{\left[0, \frac{L_2(i,j)}{N+1-\lceil j \rceil}\right]} dm \left[\mu^{M,j}\right] - 2L_1(i,j) + \left(\left\lfloor \frac{j}{n} \right\rfloor + 1\right)(2n - r)\Bigg]
\end{aligned}
$$

where

$$
\begin{aligned}
L_1(i,j) &= i - \left(\left\lfloor \frac{j}{n} \right\rfloor + 1\right) \cdot \frac{r}{2} - 1; \\
L_2(i,j) &= i - \left(\left\lfloor \frac{j}{n} \right\rfloor - 1\right) \cdot \frac{r}{2} + 1.
\end{aligned}
$$

Proof. The vertex (i, j) of $\mathcal{R}^*(\Omega, \check{a})$ may be either the center of a hexagon face of $\mathrm{SH}(\check{a})$, or the center of a square face of $\mathrm{SH}(\check{a})$. Consider a path consisting of edges of $\mathcal{R}^*(\Omega, \check{a})$, passing through every vertex in V_j^* to the left of (i, j) and joining the leftmost vertex in V_j^* and (i, j), constructed as follows. It is allowed to go from the vertex (i, j) to the vertex $(i + 1, j)$ along one or two edges of $\mathcal{R}^*(\Omega, \check{a})$ whose dual edges are not present in M.

By Definition 2.21, the height function along this path changes by 2 when moving from (i, j) to $(i + 1, j)$. When (i, j) is the center of a hexagon face (resp. square face) of $\mathrm{SH}(\check{a})$, the sign of $h_M((i + 1, j)) - h_M((i, j))$ depends on whether the vertex $(i + \frac{1}{2}, j + \frac{1}{4})$ (resp. $(i + \frac{1}{2}, j)$ of $\mathrm{SH}(\check{a})$ is a V-vertex or a Λ-vertex. More precisely,

- if the vertex $\left(i + \frac{1}{2}, j + \frac{1}{4}\right)$ (resp. $\left(i + \frac{1}{2}, j\right)$) is a V-vertex, then $h_M((i + 1, j)) - h_M((i, j)) = 2$;
- if the vertex $\left(i + \frac{1}{2}, j + \frac{1}{4}\right)$ (resp. $\left(i + \frac{1}{2}, j\right)$) is a Λ-vertex, then $h_M((i + 1, j)) - h_M((i, j)) = -2$.

We shall use / to denote "or". From the leftmost vertex v_0 of V_j^* to (i, j), the total increment of height function is equal to

$$
\begin{aligned}
& 2\left(\#\left\{V - \text{vertices in } V_{j+\frac{1}{4}/j} \text{ to the left of } \left(i, j + \frac{1}{4}/j\right)\right.\right. \\
& \quad \left.\left. -\#\{\Lambda - \text{vertices in } V_{j+\frac{1}{4}/j} \text{ to the left of } \left(i, j + \frac{1}{4}/j\right)\right\}\right) \\
= \; & 2\left(2\#\left\{V - \text{vertices in } V_{j+\frac{1}{4}/j} \text{ to the left of } \left(i, j + \frac{1}{4}/j\right)\right.\right. \\
& \quad \left.\left. -\#\{\text{vertices in } V_{j+\frac{1}{4}/j} \text{ to the left of } \left(i, j + \frac{1}{4}/j\right)\right\}\right)
\end{aligned}
$$

Note that $h_M((i, j)) = h(v_0) + (h_M(i, j) - h(v_0))$. Then the lemma follows from Lemmas 6.1, 6.2 and 6.3. □

Proof of Theorem 2.22. Assume $i = [\chi N]$ and $j = [\kappa N]$. Dividing both the left hand side of (6.2) and the right hand side of (6.2) by N and letting $N \to \infty$, we obtain the theorem. □

7. Frozen Boundary

In this section, we prove Theorem 2.20. This is obtained by analyzing the density of the limit measure obtained in Section 4, and find explicitly the region where the density is 0 or 1. This turns out to be related to the real and complex roots of a sequence of equations. In the special case when $I_2 \cap [n] = \emptyset$, for which the graph is a hexagonal lattice, or when $|I_2 \cap [n]| = 1$, we explicitly write down the equation of the frozen boundary, and showed that the frozen boundary is a union of n disjoint cloud curves, where n is the size of a period. Similar approaches were used in [8] to study the frozen region of uniform perfect matchings on the square grid, and in [5] to study the frozen region of periodic perfect matchings on the square-hexagon lattice with periodic boundary conditions. Here we shall study the periodic perfect matchings on the square-hexagon lattice with piecewise boundary conditions and prove the surprising result that when the edge weights satisfy certain conditions, the liquid region becomes disconnected.

For $i \in [n]$ and $\kappa \in (0,1)$, let

$$F_{i,\kappa}(z) = zQ'_{i,\kappa}(z) + \frac{n-i}{n} + \frac{z}{n(z-1)}.$$

We can compute the Stieltjes transform of the measure $\mathbf{m}^{\kappa}$ when x is in a neighborhood of infinity, by Theorem 2.18 we obtain

$$\begin{aligned}
\mathrm{St}_{\mathbf{m}^{\kappa}}(x) &= \sum_{j=0}^{\infty} \frac{\int_{\mathbb{R}} y^j \mathbf{m}^{\kappa}(dy)}{x^{j+1}} \\
&= \sum_{i=1}^{n}\sum_{j=0}^{\infty} \frac{1}{2(j+1)\pi\mathbf{i}} \oint_{C_1} \left(\frac{F_{i,\kappa}(z)}{x}\right)^{j+1} \frac{dz}{z} \\
&= -\frac{1}{2\pi\mathbf{i}} \sum_{i=1}^{n} \oint_{C_1} \log\left(1 - \frac{F_{i,\kappa}(z)}{x}\right)\frac{dz}{z}
\end{aligned}$$

Integration by parts we have

$$\mathrm{St}_{\mathbf{m}^{\kappa}}(x) = \frac{1}{2\pi\mathbf{i}}\left(\sum_{i=1}^{n}\left(\oint_{C_1} \log z \frac{\frac{d}{dz}\left(1-\frac{F_{i,\kappa}(z)}{x}\right)}{1-\frac{F_{i,\kappa}(z)}{x}} dz\right) - \oint_{C_{x_1,\ldots,x_n}} d\left(\log z \log\left(1-\frac{F_{i,\kappa}(z)}{x}\right)\right)\right)$$

We claim that when $|x|$ is sufficiently large, $F_{i,\kappa}(z) = x$ has exactly one root in a neighborhood of 1 for each $i \in [n]$. Indeed, $F_{i,\kappa}(z)$ has a Laurent series expansion in a neighborhood of 1 given by

$$F_{i,\kappa}(z) = \frac{1}{n(z-1)} + \sum_{k=0}^{\infty} \alpha_k (z-1)^k.$$

We can find a unique composite inverse Laurent series of $F_{i,\kappa}(z)$ given by

$$G_{i,\kappa}(w) = 1 + \sum_{k=1}^{\infty} \frac{\beta_k}{w^k},$$

such that $F_{i,\kappa}(G_{i,\kappa}(w)) = w$ when w is in a neighborhood of infinity. Then

$$z_i(x) = G_{i,\kappa}(x) \tag{7.1}$$

is the unique root of $F_{i,\kappa}(z) = x$ in a neighborhood of 1.

Since $1-\frac{F_{i,\kappa}(z)}{x}$ has exactly one zero $z_i(x)$ and one pole 1 in a neighborhood of 1 when $|x|$ is sufficiently large, we have

$$\oint_{C_1} d\left(\log z \log\left(1-\frac{F_{i,\kappa}(z)}{x}\right)\right)=0;$$

and therefore

$$\mathrm{St}_{\mathbf{m}^{\kappa}}(x)=\sum_{i=1}^{n}\log(z_i(x)) \tag{7.2}$$

when x is in a neighborhood of infinity. By the complex analyticity of both sides of (7.2), we infer that (7.2) holds whenever x is outside the support of $\mathbf{m}^{\kappa}$.

Recall that if a measure μ has a continuous density f with respect to the Lebesgue measure, then

$$f(x)=-\lim_{\epsilon\to 0+}\frac{1}{\pi}\Im\left[\mathrm{St}_{\mu}(x+i\epsilon)\right]; \tag{7.3}$$

see Lemma 4.2 of [8].

Let (χ,κ) be the continuous coordinates in the limit of rescaled square-hexagon lattice $\frac{1}{N}\mathcal{R}(\Omega,\check{a})$ as $N\to\infty$. The frozen region is the region consisting of all points (χ,κ) where the density of the counting measure $\mathbf{m}^{\kappa}$ is 0 or 1.

Proposition 7.1. *If the equation*

$$F_{i,\kappa}(z)=\frac{\chi}{1-\kappa} \tag{7.4}$$

only has real roots for all $i\in[n]$; then (χ,κ) is in the frozen region.

Proof. The proposition follows directly from (7.2), (7.3) and the definition of the frozen region. □

Proposition 7.2. *For any $x>0$, and $i\in[n]$ the equation $F_{i,\kappa}(z)=x$ has at most one pair of complex conjugate roots.*

Proof. For $2\leq i\leq n$, we can write down the equation $F_{i,\kappa}(z)=x$ explicitly as follows

$$\frac{zH'_{\mathbf{m}_i}(z)}{n}-\frac{\kappa(n-i)}{n}+\frac{(1-\kappa)z}{n(z-1)}=x(1-\kappa);$$

and we have

$$\frac{zH'_{\mathbf{m}_1}(z)}{n}+\frac{\kappa z}{n}\sum_{l\in([n]\cap I_2)}\frac{y_l x_1}{1+y_l x_1 z}-\frac{\kappa(n-1)}{n}+\frac{(1-\kappa)z}{n(z-1)}=x(1-\kappa).$$

where $\mathbf{m}_i$ is a probability measure on an interval which is divided into finitely many sub-intervals, and each sub-interval has probability density 0 or 1. For $i\in[n]$

$$H'_{\mathbf{m}_i}(z)=\frac{\mathrm{St}_{\mathbf{m}_i}^{(-1)}(\log(z))}{z}-\frac{1}{z-1}.$$

By introducing additional variables t_i such that $\mathrm{St}_{\mathbf{m}_i}(t_i)=\log(z)$ for $1\leq i\leq n$, one can then write for $\kappa\in(0,1)$, $2\leq i\leq n$:

$$F_{i,\kappa}(z,t_i):=\frac{z}{(1-\kappa)n}\left(\frac{t_i}{z}-\frac{1}{z-1}-\frac{n-i}{z}\right)+\frac{z}{n(z-1)}+\frac{n-i}{n}.$$

and

$$F_{1,\kappa}(z,t_1):=\frac{z}{(1-\kappa)n}\left(\frac{t_1}{z}-\frac{1}{z-1}-\frac{n-1}{z}+\kappa\sum_{r\in I_2\cap\{1,2,\ldots,n\}}\frac{y_rx_1}{1+y_rx_1z}\right)+\frac{z}{n(z-1)}+\frac{n-1}{n}.$$

As a consequence, injecting the expression of the moments of the limiting measure into the definition of the Stieltjes transform, one gets implicit equations to be solved: for any $x\in\mathbb{C}$, finding $(z,t_i)\in(\mathbb{C}\setminus\mathbb{R}_-)\times\mathbb{C}\setminus\mathrm{Support}(\mathbf{m}_i)$ such that

$$\begin{cases}F_{i,\kappa}(z,t_i)=x,\\ \mathrm{St}_{\mathbf{m}_i}(t_i)=\log(z)\end{cases}. \tag{7.5}$$

Let $x\mapsto z_i^\kappa(x)$ be the composite inverse of $u:z\mapsto F_{i,\kappa}\left(z,\mathrm{St}_{\mathbf{m}_i}^{(-1)}(\log z)\right)$ as given by (7.1). Note that $z_i^\kappa(x)$ is a uniformly convergent Laurent series in x when x is in a neighborhood of infinity, and

$$z_i^\kappa\left(F_{i,\kappa}\left(z,\mathrm{St}_{\mathbf{m}_i}^{(-1)}(\log z)\right)\right)=z;$$

See Section 4.1 of [8].

By (7.2) The following identity holds when x is in a neighborhood of infinity

$$\mathrm{St}_{\mathbf{m}^\kappa}(x)=\sum_{i=1}^n\log(z_i^\kappa(x)),$$

The first equation of the system (7.5) is linear in t for given x and z, which gives with $c_r=\frac{1}{y_rx_1}$:

$$t_1(z,\kappa,x)=nx(1-\kappa)+\frac{\kappa z}{z-1}+\kappa(n-1)-\kappa z\sum_{r\in(I_2\cap[n])}\frac{1}{z+c_r}$$

and for $2\leq i\leq n$

$$t_i(z,\kappa,x)=nx(1-\kappa)+\kappa(n-i)+\frac{\kappa z}{z-1}.$$

For a given value $y\in\mathbb{R}$, and fixed x (and κ), we investigate properties of the complex numbers z such that $t(z,\kappa,x)=y$. In particular, we have the following:

Lemma 7.3. *Let $c_r>0$, for $r\in I_2\cap[n]$. Let $\kappa\in(0,1)$, and $x,y\in\mathbb{R}$. Then*

- *the the following equation in z*

$$t_1(z,\kappa,x)=y \tag{7.6}$$

  has $m+1$ roots on the Riemann sphere $\mathbb{C}\cup\{\infty\}$, where m is the number of distinct values of c_r, and all these roots are real and simple.
- *For $2\leq i\leq n$, the equation $t_i(z,\kappa,x)=y$ has exactly one root on the Riemann sphere $\mathbb{C}\cup\{\infty\}$.*

Proof. Let $0<\gamma_1<\cdots<\gamma_m$ be all the possible distinct values for the c_i, and $n_1,\ldots,n_m$ be their respective multiplicities among the c_i's. Let

$$l=|I_2\cap[n]|.$$

For $2 \leq i \leq n$, let

$$K_i = nx(1-\kappa) + \kappa(n-i+1) - y$$

and let

$$K_1 = nx(1-\kappa) + \kappa(n-l) - y$$

Define

$$H_1(z;x,y) = t(z,\kappa,x) - y = K_1 + \left(\frac{\kappa}{z-1} + \kappa\sum_{j=1}^{m}\frac{n_j\gamma_j}{z+\gamma_j}\right) \tag{7.7}$$

and for $2 \leq i \leq n$, define

$$H_i(z;x,y) = t(z,\kappa,x) - y = K_i + \frac{\kappa}{z-1} \tag{7.8}$$

When writing $H_1(z;x,y)$ (resp. $H_i(z;x,y)$ for $2 \leq i \leq n$) as a single rational fraction by bringing all the terms onto the same polynomial denominator (of degree $m+1$), the polynomial on the numerator has degree at most $m+1$ (resp. 1). So there are at most $m+1$ (resp. 1) roots in $\mathbb{C}$ (and exactly $m+1$ (resp. 1) if we add roots at infinity). Notice that the denominator does not depend on x and y, but just on the γ_j's.

Moreover, each factor of the form $\frac{1}{z-b}$ with $b = -\gamma_j$ or 1 is a decreasing function of z on any interval where it is defined. As a consequence, on each of the intervals $(-\gamma_{j+1}, -\gamma_j)$, $j = 1, \ldots, m-1$ and $(-\gamma_1, 1)$, H_1 realizes a bijection with $\mathbb{R}$. In particular, the equation $H_1(z;x,y)$ has a unique solution in every such interval. It is also decreasing on $(\infty, -\gamma_m)$ and $(1, +\infty)$. Since the limits of $H_1(z;x,y)$ when z goes to $\pm\infty$ coincide, and

$$\lim_{z\to 1+} H_1(z;x,y) = +\infty, \qquad \lim_{z\to -\gamma_m^-} H_1(z;x,y) = -\infty,$$

the equation $H_1(z;x,y) = 0$ has a unique solution in $(-\infty, -\gamma_m) \cup (1, +\infty) \cup \{\infty\}$, which is in fact infinite if and only if the limits of H_1 at infinity is zero, that is, when $K_1 = 0$. This gives thus $m+1$ real roots (with possibly one at infinity). Similar arguments hold for H_i when $2 \leq i \leq n$. □

Remark 7.4. *Let*

$$L_1 = nx(1-\kappa) + \kappa(n-l). \tag{7.9}$$

For $2 \leq i \leq n$, let

$$L_i = nx(1-\kappa) + \kappa(n-i+1). \tag{7.10}$$

Increasing the value of y translates downward the graph of the function $z \in \mathbb{R} \mapsto H_1(z;x,y)$. Since $H_1(z;x,y)$ is decreasing with respect to z in any interval of definition, the roots present in the bounded intervals decrease. The one in $(-\infty, -\gamma_m) \cup (1, +\infty) \cup \{\infty\}$ moves also to the left, and if it started in $\mathbb{R}_-$, when it reaches $-\infty$, it jumps to the right part of $(1, +\infty)$ and then continues to decrease. In particular, it means that if $y < y'$, the respective roots $z_1 < \cdots < z_{m+1}$ and $z'_1 < \cdots < z'_{m+1}$ are interlaced:

- *if $y < y' < L_1$,*

$$z'_1 < z_1 < -\gamma_m < z'_2 < z_2 < -\gamma_{m-1} < \cdots < -\gamma_1 < z'_{m+1} < z_{m+1} < 1,$$

- *if $y < L_1 < y'$,*

$$z_1 < -\gamma_m < z'_1 < z_2 < -\gamma_{m-1} < \cdots < -\gamma_1 < z'_m < z_{m+1} < 1 < z'_{m+1},$$

- *if* $L_1 < y < y'$,

$$-\gamma_m < z_1' < z_1 < -\gamma_{m-1} < z_2' < z_2 < -\gamma_{m-1} < \cdots < 1 < z_{m+1}' < z_{m+1},$$

The limiting case when y *or* y' *is equal to* $x(1-\kappa) + \frac{\kappa r}{n}$ *is obtained by sending the corresponding root in* $(-\infty, -\gamma_m) \cup (1, +\infty)$ *to* ∞.

For $2 \leq i \leq n$, *we have*

- *if* $y < y' < L_i$, $z_1' < z_1 < 1$.
- *if* $y < L_i < y'$, $z_1 < 1 < z_1'$.
- *if* $L_i < y < y'$, $1 < z_1' < z_1$.

Rational fractions where zeros of the numerator and denominator interlace have interesting monotonicity properties, already used for example in [29], which are straightforwardly checked by induction using the decomposition of $R(z)$ into the sum of simple fractions:

Lemma 7.5. *(1) Let*

$$R(z) = \frac{(z-u_1)(z-u_2)\cdots(z-u_h)}{(z-v_1)(z-v_2)\cdots(z-v_h)},$$

where $\{u_i\}$ *and* $\{v_i\}$ *are two sets of real numbers, and* h *is a positive integer.*

- *If* $\{u_i\}$ *and* $\{v_i\}$ *satisfy*

$$v_1 < u_1 < v_2 < u_2 < \cdots < v_h < u_h.$$

Then $R(z)$ *is monotone increasing in each one of the following intervals*

$$(-\infty, v_1), (v_1, v_2), \ldots, (v_{h-1}, v_h), (v_h, \infty).$$

- *If* $\{u_i\}$ *and* $\{v_i\}$ *satisfy*

$$u_1 < v_1 < u_2 < v_2 < \cdots < u_h < v_h.$$

Then $R(z)$ *is monotone decreasing in each one of the following intervals*

$$(-\infty, v_1), (v_1, v_2), \ldots, (v_{h-1}, v_h), (v_h, \infty).$$

(2) Let

$$R(z) = \frac{(z-u_1)\cdots(z-u_{h-1})}{(z-v_1)\cdots(z-v_h)} \text{ with } v_1 < u_1 < \cdots < u_{h-1} < v_h$$

Then $R(z)$ *is monotone decreasing in each one of the following intervals*

$$(-\infty, v_1), (v_1, v_2), \ldots, (v_{h-1}, v_h), (v_h, \infty).$$

(3) Let

$$R(z) = \frac{(z-u_1)\cdots(z-u_{h+1})}{(z-v_1)\cdots(z-v_h)} \text{ with } u_1 < v_1 < \cdots < v_h < u_{h+1}.$$

Then $R(z)$ *is monotone increasing in each one of the following intervals*

$$(-\infty, v_1), (v_1, v_2), \ldots, (v_{h-1}, v_h), (v_h, \infty).$$

This is helpful to determine the number of solutions of Equation (7.4), as shown in the following lemma:

Lemma 7.6. *Let d_i, $\beta_{i,k}$, $\gamma_{i,k}$ be defined as in Lemma 4.10. Let*

$$D_i = d_{i+1} - d_i - 1 \quad \text{if } i \in [n]$$

Then $\mathbf{m}_i$ is a measure with a density with respect to the Lebesgue measure equal to the indicator of a union of intervals $\bigcup_{r=0}^{D_i}[\beta_{i,k}, \gamma_{i,k}]$, with

$$\beta_{i,0} < \gamma_{i,0} < \beta_{i,1} < \gamma_{i,1} < \cdots < \beta_{i,D_i} < \gamma_{i,D_i} \quad \text{and} \quad \sum_{k=0}^{D_i}(\gamma_{i,k} - \beta_{i,k}) = 1;$$

Then the system of equations (7.4) has at most one pair of complex conjugate solutions. Moreover, let m is the number of distinct c_j's, for $j \in I_2 \cap [n]$.

- *when $i = 1$, if for any $0 \le k \le D_1$, $\gamma_{1,k} \neq L_1$, then for each fixed $x \in \mathbb{R}$, (7.4) has at least $(m+1)(D_1+1) - 1$ distinct real roots;*
- *when $i = 1$, if $\gamma_{1,k} = L_1$ for some $k \in \{0, 1, \dots, D_1\}$, then for each fixed $x \in \mathbb{R}$, (7.4) has at least $(m+1)(D_1+1) - 2$ distinct real roots.*
- *when $2 \le i \le n$, if for any $0 \le k \le D_i$, $\gamma_{i,k} \neq L_i$, then for each fixed $x \in \mathbb{R}$, (7.4) has at least D_i distinct real roots;*
- *when $2 \le i \le n$, if $\gamma_{i,k} = L_i$ for some $k \in \{0, 1, \dots, D_i\}$, then for each fixed $x \in \mathbb{R}$, (7.4) has at least $(D_i - 1)$ distinct real roots;*

Proof. The Stieltjes transform can be computed explicitly from the definition:

$$\mathrm{St}_{\mathbf{m}_i}(t_i) = \log \prod_{k=0}^{D_i} \frac{t_i - \beta_{i,k}}{t_i - \gamma_{i,k}}. \tag{7.11}$$

We use the second expression from (7.5) and (7.11), we obtain

$$z = \prod_{k=0}^{D_i} \frac{t_i - \beta_{i,k}}{t_i - \gamma_{i,k}}$$

By (7.7) and (7.8) get (after exponentiation)

$$z = \prod_{k=0}^{D_i} \frac{H_i(z; x, \beta_{i,k})}{H_i(z; x, \gamma_{i,k})}. \tag{7.12}$$

Let us suppose that

$$\{\beta_{i,0}, \dots, \beta_{i,D_i}, \gamma_{i,0}, \dots, \gamma_{i,D_i}\} \cap \{L_i\} = \emptyset;$$

where L_i's, for $1 \le i \le n$, are defined by (7.9) and (7.10).

The rational fractions $\prod_{k=0}^{D_i} H_i(z; x, \beta_{i,k})$ and $\prod_{k=0}^{D_i} H_i(z; x, \gamma_{i,k})$ have the same poles $m+1$ poles (of same order s) when $i = 1$; and they have the same pole 1 of order s when $2 \le i \le n$. According to Lemma 7.3 they have $s(m+1)$ distinct real roots when $i = 1$; and s distinct real roots when $2 \le i \le n$. These roots interlace. Therefore, the ratio:

$$\prod_{r=0}^{D_i} \frac{H_i(z; x, \beta_{i,k})}{H_i(z; x, \gamma_{i,k})}$$

is a rational fraction of the form described in the hypotheses of Lemma 7.5; with $h = m+1$ for $i = 1$ and $h = 1$ for $2 \le i \le n$. Therefore, when $i = 1$ on each bounded interval between two consecutive poles, by monotonicity, the graph of the rational fraction will cross the first diagonal exactly once and there are $(m+1)s - 1$ such intervals.

If (no $\gamma_{1,k}$, and exactly) one $\beta_{1,k}$ is equal to L_1, the same argument is applicable. The only difference is that the rational fraction on the right hand side of Equation 7.12 has only $(s-1)(m+1)+m = s(m+1)-1$ zeros, but still $s(m+1)$ poles. Therefore we still get the same number $s(m+1)-1$ of intersection with the first diagonal, one on each finite interval between two consecutive poles.

If (no $\beta_{1,k}$ and exactly) one $\gamma_{1,k}$ is equal to L_1, then this time the rational fraction has $s(m+1)-1$ finite real poles. Therefore, there is only $s(m+1)-2$ roots found by this approach between two successive poles.

Similar arguments apply when $2 \leq i \leq n$. □

Note that when rewriting Equation 7.12 as a polynomial equation in z, it has degree

- When $i = 1$,

$$\begin{cases} s(m+1)+1 & \text{when no } b_r \text{ equals } L_1, \\ s(m+1) & \text{when a } b_r \text{ is equal to } L_1 \end{cases}$$

- When $2 \leq i \leq n$,

$$\begin{cases} s+1 & \text{when no } b_r \text{ equals } L_1, \\ s & \text{when a } b_r \text{ is equal to } L_1 \end{cases}$$

Indeed, in all the cases, the leading coefficients of the numerator and denominator of the rational fraction are distinct, thus there is no cancellation of the monomials of higher degree when multiplying both sides by the denominator. In both case, it is exactly the number of real roots we found plus 2. Which means that Equation (7.12), and thus Equation (7.4) has at most a pair of complex conjugated roots. □

Proposition 7.7. *For each $i \in [n]$, the boundary of the region such that (7.4) has only real roots and the region such that (7.4) has a pair of complex conjugate roots is a rational algebraic curve C_i with an explicit parametrization $(\chi_i(t_i), \kappa_i(t_i))$ defined as follows:*

$$\chi_i(t_i) = \frac{1}{n}\left[t_i - \frac{J_i(t_i)}{J_i'(t_i)}\right], \quad \kappa_i(t_i) = \frac{1}{J_i'(t_i)},$$

where

$$J_1(t_1) = \frac{1}{\Psi_1(t_1) - 1} + (n-l) + \sum_{j=1}^{m} \frac{n_j \gamma_j}{\Psi_1(t_1) + \gamma_j}; \tag{7.13}$$

for $2 \leq i \leq n$,

$$J_i(t_i) = (n-i+1) + \frac{1}{\Psi_i(t_i) - 1}; \tag{7.14}$$

and

$$\Psi_i(t_i) = \frac{(t_i - \beta_{i,0})(t_i - \beta_{i,1}) \cdots (t_i - \beta_{i,D_i})}{(t_i - \gamma_{i,0})(t_i - \gamma_{i,1}) \cdots (t_i - \gamma_{i,D_i})}. \tag{7.15}$$

Proof. According to Proposition 7.2, the boundary of the region such that (7.4) has only real roots and the region such that (7.4) has a pair of complex conjugate is given by the condition that

$$z = \prod_{k=0}^{D_i} \frac{H_i(z; \frac{\chi}{1-\kappa}, \beta_{i,k})}{H_i(z; \frac{\chi}{1-\kappa}, \gamma_{i,k})}$$

has double roots; where $H_i(z;x,y)$ is defined by Equation (7.8). We can also rewrite the system of equations (7.5) as follows

- if $2 \leq i \leq n$,

$$\begin{cases}\Psi_i(t_i) = z;\\ n(1-\kappa)F_{i,\kappa}(z) = t_i - \kappa\left[(n-i+1) + \frac{1}{z-1}\right] = n\chi.\end{cases}$$

- if $i = 1$,

$$\begin{cases}\Psi_1(t_1) = z;\\ n(1-\kappa)F_{i,\kappa}(z) = t_1 - \kappa\left[\frac{1}{z-1} + (n-l) + \sum_{j=1}^{m}\frac{n_j\gamma_j}{z+\gamma_j}\right] = n\chi.\end{cases}$$

In each one of the two system of equations above, we plug the expression of z from the first equation into the second equation; and for $1 \leq i \leq n$, let $J_i(t_i)$ be defined as in (7.13) and (7.14). Note that the condition that the resulting equation has a double root is equivalent to the following system of equations (where $1 \leq i \leq n$)

$$\begin{cases}\chi_i = \frac{t_i - \kappa J_i(t_i)}{n},\\ 1 = \kappa J_i'(t_i).\end{cases}$$

Then the parametrization of the curve separating the region with a pair of complex conjugate roots and the region with only real roots follows. □

Proposition 7.8. *The curve C_1 (resp. C_i, for $2 \leq i \leq n$) is a cloud curve of class $(m+1)(D_1+1)$ (resp. (D_i+1)), where (D_i+1) is the number of segments in the measure $\mathbf{m}_i$ for $1 \leq i \leq n$, and m is the number of distinct values of $c_r = \frac{1}{y_r x_1}$ for $r \in |I_2 \cap [n]|$ in one period. Moreover, the curve C_i has the following properties*

(1) it is tangent to the line $\kappa = 1$ with a unique tangent point for $i \in [n]$.
(2) it is tangent to the line $\kappa = 0$ with $(m+1)(D_1+1) - 1$ points of tangency when $i = 1$, and with D_i points of tangency when $2 \leq i \leq n$.

Proof. We recall that the class of a curve is the degree of its dual curve. So we need to show that the dual curve $C_1^\vee$ (resp. $C_i^\vee$, for $2 \leq i \leq n$) has degree $(m+1)(D_1+1)$ (resp. (D_i+1)) and is winding.

We apply the classical formula to obtain from a parametrization $(x(t), y(t))$ of the curve C_i for the frozen boundary one for its dual $C_i^\vee$, $(x^\vee(t), y^\vee(t))$:

$$x^\vee = \frac{y'}{yx' - xy'}, \quad y^\vee = -\frac{x'}{yx' - xy'}.$$

and obtain that the dual curve $C_i^\vee$ given in the following parametric form

$$C_i^\vee = \left\{\left(-\frac{n}{t_i}, -\frac{J_i(t_i)}{t_i}\right)\ ;\ t \in \mathbb{C} \cup \{\infty\}\right\}. \tag{7.16}$$

from which we can read that its degree is $(m+1)(D_1+1)$ for $i = 1$ and (D_i+1) for $2 \leq i \leq n$. To show that $C_i^\vee$ is winding, we need to look at real intersections with straight lines.

First, from Equation (7.16), one sees that the first coordinate x of the dual curve $C^\vee$ and the parameter t are linked by the simple relation $xt_i = -1$.

Using this relation to eliminate t from the expression of the second coordinate, we obtain that the points (x,t) on the dual curve satisfy the following implicit equation:

$$y = \frac{x}{n} J_i\left(-\frac{n}{x}\right).$$

The points of intersection $(x(t_i), y(t_i))$ of the dual curve with a straight line of the form $y = cx + d$ have a parameter t_i satisfying:

$$cn - dt_i = J_i(t_i). \tag{7.17}$$

the exact same argument as in Lemma 7.6 (but with the role of s and $(m+1)$ exchanged) shows that the (7.17) has at least $(m+1)(D_1+1)-2$, if $i=1$, (resp. D_i-1, if $2\leq i\leq n$) distinct real solutions, yielding $(m+1)(D_1+1)-2$, when $i=1$, (resp. D_i-1, if $2\leq i\leq n$) points of intersections for the dual curve and the line $y=cx+d$. Moreover, if t_0 doesn't lie in a compact interval containing all the zeros of J_i, then any non vertical straight line passing through $(t_0,0)$ will have $(m+1)(D_1+1)-1$, when $i=1$, (resp. D_i, when $2\leq i\leq n$) intersections with the graph of J_i. This means that x_0 in some closed interval, there are at least $(m+1)(D_1+1)-1$, when $i=1$, (resp. D_i, when $2\leq i\leq n$) real intersections of the dual curve with a line $y=cx+d$ passing through (x_0,y), thus exactly $(m+1)(D_1+1)$, when $i=1$, (resp. $(m+1)(D_i+1)$, when $2\leq i\leq n$) real intersections, since there cannot be a single complex one. Such points (x_0,y) are candidates to be the center of the dual curve.

To consider the vertical lines $x=d$, we rewrite the equations in homogeneous coordinate $[x:y:z]$ and get that the line $x=dz$ intersects the curve at the point $[0:1:0]$ with multiplicity $(m+1)(D_1+1)-1$ when $i=1$ (resp. D_i when $2\leq i\leq n$) so again, by the same argument as above, $(m+1)(D_1+1)$, when $i=1$, (resp. D_i+1, when $2\leq i\leq n$,) real intersections. The case of the line $z=0$ is similar.

Recall that each point on the dual curve $C_i^\vee$ corresponds to a tangent line of C_i. For $1\leq i\leq n$, let

$$\begin{aligned} U_i &= (t_i-\beta_{i,0})(t_i-\beta_{i,1})\cdots(t_i-\beta_{i,D_i}) \\ V_i &= (t_i-\gamma_{i,0})(t_i-\gamma_{i,1})\cdots(t_i-\gamma_{i,D_i}). \end{aligned}$$

When $t_i=\infty$, we have

$$\lim_{t_i\rightarrow\infty}\frac{J_i(t_i)}{t_i} = \lim_{t_i\rightarrow\infty}\frac{V_i}{t_i(U_i-V_i)}.$$

The leading term in V_i is $t_i^{D_i+1}$, while the leading terms for $t_i(U_i-V_i)$ is $\left[\sum_{k=0}^{D_i}(\gamma_{i,k}-\beta_{i,k})\right]t_i^{D_i+1}$, therefore we have $\lim_{t_i\rightarrow\infty}\frac{V_i}{t_i(U_i-V_i)}=1$. Therefore we have $(0,-1)\in C_i^\vee$, which corresponds to the tangent line $\kappa=1$ of $C_i^\vee$. The unique tangent point is given by $\lim_{t_i\rightarrow\infty}(\chi_i(t_i),\kappa_i(t_i))$.

Those t_i such that $J_i(t_i)=\infty$ corresponds to tangent points with the tangent line $\kappa=0$. When $i=1$, the tangent points with the tangent line $\kappa=0$ are solutions of

$$\left[\Psi_1\left(-\frac{1}{x}\right)-1\right]\prod_{j=1}^{m}\left[\Psi_1\left(-\frac{1}{x}\right)-1\right]=0$$

There are $(m+1)(D_1+1)-1$ such points. When $2\leq i\leq n$, the tangent points with the tangent line $\kappa=0$ are solutions of

$$\Psi_1\left(-\frac{1}{x}\right)=1$$

and there are D_i such points. □

7.1. **Hexagonal lattice.** When $I_2 = \emptyset$, the square-hexagon lattice we constructed is actually a hexagonal lattice. In this case we shall show that when $I_2 = \emptyset$, for each $i \in [n]$, if a pair of complex conjugate roots exist for (7.5), then the root $z_i(x)$ as used to compute the density of the limit counting measure, can not be real. This follows from an adaptation of Lemma 4.5 in [8], in which the uniform perfect matching on a square grid is considered.

When $I_2 = \emptyset$, for each $i \in [n]$ we can write (7.5) as follows

$$\begin{cases} \frac{t_i}{1-\kappa} - \frac{\kappa}{1-\kappa}\frac{z}{z-1} = nx + \frac{\kappa(n-i)}{1-\kappa}. \\ \mathrm{St}_{\mathbf{m}_i}(t_i) = \log(z). \end{cases} \tag{7.18}$$

Then we have

$$z = \frac{\prod_{k=0}^{D_i}\left[\frac{\kappa z}{z-1} + nx(1-\kappa) + \kappa(n-i) - \beta_{i,k}\right]}{\prod_{k=0}^{D_i}\left[\frac{\kappa z}{z-1} + nx(1-\kappa) + \kappa(n-i) - \gamma_{i,k}\right]} := G_i(z,x) \tag{7.19}$$

Lemma 7.9. *Assume $x_0 > 0$ is such that equation (7.19) has a pair of complex conjugate roots. Let $s_i(x)$ be a real root of (7.19). Then*

$$\left.\frac{\partial s_i(x)}{\partial x}\right|_{x=x_0} \geq 0.$$

It is equal to 0 if and only if $s_i(x_0) = 1$.

Proof. The derivative $s_i'(x)$ can be computed explicitly from (7.19) as follows

$$s_i'(x) = \frac{\frac{\partial G_i(z,x)}{\partial x}}{1 - \frac{\partial G_i(z,x)}{\partial z}}$$

First we claim that $\frac{\partial G_i(z,x)}{\partial x} \leq 0$. Note that

$$G_i(z,x) = \frac{\prod_{k=0}^{D_i}\left[x - \left(\frac{\beta_{i,k}}{n(1-\kappa)} - \frac{\kappa z}{n(z-1)(1-\kappa)} - \frac{\kappa(n-i)}{n(1-\kappa)}\right)\right]}{\prod_{k=0}^{D_i}\left[x - \left(\frac{\gamma_{i,k}}{n(1-\kappa)} - \frac{\kappa z}{n(z-1)(1-\kappa)} - \frac{\kappa(n-i)}{n(1-\kappa)}\right)\right]}$$

where

$$\frac{\beta_{i,k}}{n(1-\kappa)} - \frac{\kappa z}{n(z-1)(1-\kappa)} - \frac{\kappa(n-i)}{n(1-\kappa)} < \frac{\gamma_{i,k}}{n(1-\kappa)} - \frac{\kappa z}{n(z-1)(1-\kappa)} - \frac{\kappa(n-i)}{n(1-\kappa)}$$
$$< \frac{\beta_{i,k+1}}{n(1-\kappa)} - \frac{\kappa z}{n(z-1)(1-\kappa)} - \frac{\kappa(n-i)}{n(1-\kappa)}.$$

By Lemma 7.5 that for each fixed $z \in \mathbb{R} \setminus \{1\}$, $G_i(z,x)$ is strictly decreasing in x whenever it is defined. Hence $\frac{\partial G_i(z,x)}{\partial x} < 0$ whenever $z \neq 1$, and $\frac{\partial G_i(z,x)}{\partial x} = 0$ if $z = 1$.

Now we show that $\frac{\partial G_i(z,x)}{\partial z} > 1$. Let

$$p_{i,k} = nx(1-\kappa) + \kappa(n-i) - \beta_{i,k}$$
$$q_{i,k} = nx(1-\kappa) + \kappa(n-i) - \gamma_{i,k}$$

We have

$$G_i(z,x) = \left(\prod_{k=0}^{D_i}\frac{\kappa + p_{i,k}}{\kappa + q_{i,k}}\right)\prod_{k=0}^{D_i}\frac{z - \frac{p_{i,k}}{\kappa+p_{i,k}}}{z - \frac{q_{i,k}}{\kappa+q_{i,k}}}$$

Let $f(t)=\frac{t}{\kappa+t}$, then $f'(t)=\frac{\kappa}{(\kappa+t)^2}>0$, so $p_{i,k}>q_{i,k}>p_{i+1,k}$ implies that

$$\frac{p_{i,k}}{\kappa+p_{i,k}}>\frac{q_{i,k}}{\kappa+q_{i,k}}>\frac{p_{i+1,k}}{\kappa+p_{i+1,k}};$$

when $p_{i+1,k}+\kappa$ and $p_{i,k}+\kappa$ have the same sign.

The following cases might occur

(1) $\prod_{k=0}^{D_i}\frac{\kappa+p_{i,k}}{\kappa+q_{i,k}}>0$.
(2) $\prod_{k=0}^{D_i}\frac{\kappa+p_{i,k}}{\kappa+q_{i,k}}<0$.
(3) there exists $0\leq k\leq D_i$, such that $\kappa+p_{i,k}=0$.
(4) there exists $0\leq k\leq D_i$, such that $\kappa+q_{i,k}=0$.

First we consider case (1). There exists some $0\leq k_0\leq D_i-1$, such that

$$\kappa+p_{i,k_0}>\kappa+q_{i,k_0}>0>\kappa+p_{i,k_0+1}>\kappa+q_{i,k_0+1}$$

Then we have

$$f(p_{i,k_0+1})>f(q_{i,k_0+1})>\ldots>f(p_{i,D_i})>f(q_{i,D_i})>f(p_{1,1})>f(q_{1,1})>\ldots>f(p_{i,k_0})>f(q_{i,k_0})$$

By Lemma 7.5, for each fixed $x\in\mathbb{R}$, $G_i(z,x)$ is strictly increasing in z whenever it is defined. On each bounded interval $z\in\left(\frac{p_{i,k+1}}{\kappa+p_{i,k+1}},\frac{p_{i,k}}{\kappa+p_{i,k}}\right)$ for $k\neq k_0$ or $z\in\left(\frac{p_{i,1}}{\kappa+p_{i,1}},\frac{p_{i,D_i}}{\kappa+p_{i,D_i}}\right)$, $G_i(z,x)$ increases from $-\infty$ to ∞, hence the equation $z=G_i(z,x)$ has at least one root on each such bounded interval. There are D_i such bounded intervals, therefore $z=G_i(z,x)$ has at least D_i distinct real roots. Moreover, the roots $z=G_i(z,x)$ are those of a polynomial of degree at most D_i+2, therefore when it has a pair of complex conjugate roots, each bounded interval $z\in\left(\frac{p_{i+1,k}}{\kappa+p_{i+1,k}},\frac{p_{i,k}}{\kappa+p_{i,k}}\right)$ has exactly one real root, counting multiplicities. At the real root we have $\frac{\partial G_i(z,x)}{\partial z}>1$.

All the other cases can be proved using similar arguments. □

Now let us consider a contracting hexagon lattice with boundary partition given by $\phi^{(i,\sigma_0)}(N)\in\mathbb{GT}^+_{\frac{N}{n}}$. Let $\kappa\in(0,1)$, and $\mathbf{m}_i^\kappa$ be the limit counting measure for the partitions on the $\left\lfloor\frac{2\kappa N}{n}\right\rfloor$th row, counting from the bottom. Then using the same arguments as before, we obtain that

$$\mathrm{St}_{\mathbf{m}_i^\kappa}\left(nx+\frac{\kappa(n-i)}{1-\kappa}\right)=\log(z_i^\kappa(x))$$

Hence we have

$$z_i^\kappa(x)=\exp\left(\int_{\mathbb{R}}\frac{\mathbf{m}_i^\kappa[ds]}{nx+\frac{\kappa(n-i)}{1-\kappa}-s}\right);$$

and

$$z_i^\kappa(x+\mathbf{i}\epsilon)=\exp\left(\int_{\mathbb{R}}\frac{\left(nx+\frac{\kappa(n-i)}{1-\kappa}-s-\mathbf{i}\epsilon\right)\mathbf{m}_i^\kappa[ds]}{\left(nx+\frac{\kappa(n-i)}{1-\kappa}-s\right)^2+\epsilon^2}\right)$$

Therefore $\Im[z_i^\kappa(x+\mathbf{i}\epsilon)]<0$ when ϵ is a small positive number. However, when complex roots exist for (7.19), for real root $s_i(x)$, Lemma 7.9 implies that $\Im[s_i(x+\mathbf{i}\epsilon)]>0$ when ϵ is a small positive number. This implies that when complex roots exist for (7.19), $z_i^\kappa(x+\mathbf{i}\epsilon)$ cannot be real. Then we have the following theorem

Theorem 7.10. *Assume $I_2 = \emptyset$. For the contracting hexagon lattice, (χ, κ) is in the frozen region if and only if (7.4) only has real roots for all $i \in [n]$. The frozen boundary consists of n disjoint cloud curve $C_1, \ldots, C_n$, where for $i \in [n]$, C_i is a cloud curve of class $D_i + 1$ with an explicit parametrization given by*

$$\chi_i(t_i) = \frac{1}{n}\left[t_i - \frac{J_i(t_i)}{J_i'(t_i)}\right], \quad \kappa_i(t_i) = \frac{1}{J_i'(t_i)},$$

where

$$J_i(t_i) = (n - i + 1) + \frac{1}{\Psi_i(t_i) - 1};$$

and Ψ_i is given by (7.15). Moreover, each C_i is tangent to $\kappa = 0$ with D_i tangent points, and is tangent to $\kappa = 1$ with a unique tangent point. The curve C_n is tangent to $\chi = 0$, and the curve C_1 is tangent to $\chi - r_{d_1} + \kappa - 1 = 0$.

Proof. For each $i \in [n]$, given the parametrization of C_i, the fact that C_i is a cloud curve of class $D_i + 1$ follows from Proposition 7.8. We need to show that $C_1, \ldots, C_n$ are disjoint. Note that C_i is characterized by the condition that the system (7.18) of equations have double roots (note that $x = \frac{\chi}{1-\kappa}$). We make a change of variables in (7.18)

$$\begin{cases} \tilde{\chi}_i = n\chi + \kappa(n - i) \\ \tilde{\kappa}_i = \kappa \end{cases},$$

and let $\tilde{C}_i$ be the corresponding curve in the new coordinate system $\tilde{\chi}_i, \tilde{\kappa}_i$. Then $\tilde{C}_i$ is the frozen boundary of a uniform dimer model on contracting hexagon lattice with boundary condition given by $\mathbf{m}_i$. For $1 \leq j \leq s$, let $r_j = \lim_{N \to \infty} \frac{\mu_j(N)}{N}$.

By the results in [10, 7, 8, 5], $\tilde{C}_i$ satisfies the following conditions

(1) It is tangent to $\tilde{\kappa}_i = 0$ with D_i tangent points;
(2) It is tangent to $\tilde{\kappa}_i = 1$ with a unique tangent point;
(3) It is tangent to $\tilde{\chi}_i = nr_{d_{i+1}-1} + (n - i)$;
(4) It is tangent to $\tilde{\kappa}_i = -\tilde{\chi}_i + nr_{d_i} + n - i + 1$;
(5) It is in the bounded region bounded by the curves $\tilde{\kappa}_i = 0$, $\tilde{\kappa}_i = 1$, $\tilde{\chi}_i = 0$, $\tilde{\kappa}_i = -\tilde{\chi}_i + nr_{d_i} + n - i + 1$.

Then C_i satisfies the following conditions

(1) It is tangent to $\kappa_i = 0$ with D_i tangent points;
(2) It is tangent to $\kappa_i = 1$ with a unique tangent point;
(3) It is tangent to $n(\chi - r_{d_{i+1}-1}) + (\kappa - 1)(n - i) = 0$;
(4) It is tangent to $n(\chi - r_{d_i}) + (n - i + 1)(\kappa - 1) = 0$;
(5) It is in the bounded region R_i bounded by the curves $\kappa_i = 0$, $\kappa_i = 1$, $n(\chi - r_{d_{i+1}-1}) + (\kappa - 1)(n - i) = 0$, $n(\chi - r_{d_i}) + (n - i + 1)(\kappa - 1) = 0$.

Under the assumption that $r_1 > r_2 > \ldots > r_s$, it is straightforward to check that $R_i \cap R_j = \emptyset$. Then the theorem follows. □

To illustrate Theorem 7.10, let us see the following example.

Example 7.11. *Consider a contracting hexagon lattice with period* 1×2*. Let* $x_1=1$*, and* $\frac{x_2}{x_1}\leq e^{-\alpha N}$*. Assume* N *is an integer multiple of* 6*.*

$$\begin{aligned}
&\lambda_1(N)=\lambda_2(N)=\ldots=\lambda_{\frac{N}{4}}(N)=\mu_1(N)\\
&\lambda_{\frac{N}{4}+1}(N)=\lambda_{\frac{N}{4}+2}(N)=\ldots=\lambda_{\frac{N}{2}}(N)=\mu_2(N)\\
&\lambda_{\frac{N}{2}+1}(N)=\lambda_{\frac{N}{2}+2}(N)=\ldots=\lambda_{\frac{2N}{3}}(N)=\mu_3(N)\\
&\lambda_{\frac{2N}{3}+1}(N)=\lambda_{\frac{N}{2}+2}(N)=\ldots=\lambda_{\frac{5N}{6}}(N)=\mu_4(N)\\
&\lambda_{\frac{5N}{6}+1}(N)=\lambda_{\frac{N}{2}+2}(N)=\ldots=\lambda_{N}(N)=\mu_5(N)=0.
\end{aligned}$$

For $1\leq j\leq 5$*, let*

$$r_j=\lim_{N\rightarrow\infty}\frac{\mu_j(N)}{N}.$$

Note that $r_5=0$*. Then we have* $\phi^{(1,\sigma_0)}(N)\in\mathbb{GT}^+_{\lfloor\frac{N}{2}\rfloor}$ *is given by*

$$\phi_i^{(1,\sigma_0)}(N)=\begin{cases}\mu_1(N)+\frac{N}{2}, \text{ if } 1\leq i\leq \frac{N}{4}\\ \mu_2(N)+\frac{N}{2}, \text{ if } \frac{N}{4}+1\leq i\leq\frac{N}{2}\end{cases}.$$

and $\phi^{(2,\sigma_0)}(N)\in\mathbb{GT}^+_{\lfloor\frac{N}{2}\rfloor}$ *is given by*

$$\phi_i^{(2,\sigma_0)}(N)=\begin{cases}\mu_3(N), \text{ if } 1\leq i\leq \frac{N}{6}\\ \mu_4(N), \text{ if } \frac{N}{6}+1\leq i\leq\frac{N}{3}\\ 0, \text{ if } \frac{N}{3}+1\leq i\leq\frac{N}{2}\end{cases}.$$

Hence $\mathbf{m}_1$ *is the uniform measure on* $\left[2r_2+1,2r_2+\frac{3}{2}\right]\cup\left[2r_1+\frac{3}{2},2r_1+2\right]$*; and* $\mathbf{m}_2$ *is the uniform measure on* $\left[0,\frac{1}{3}\right]\cup\left[2r_4+\frac{1}{3},2r_4+\frac{2}{3}\right]\cup\left[2r_3+\frac{2}{3},2r_3+1\right]$*. Then we have the following two systems of linear equations*

$$\begin{cases}t_1-\frac{\kappa z}{z-1}=2\chi+\kappa\\ z=\frac{\left(t_1-2r_1-\frac{3}{2}\right)(t_1-2r_2-1)}{(t_1-2r_1-2)\left(t_1-2r_2-\frac{3}{2}\right)}\end{cases}$$

and

$$\begin{cases}t_2-\frac{\kappa z}{z-1}=2\chi\\ z=\frac{t_2\left(t_2-2r_4-\frac{1}{3}\right)\left(t_2-2r_3-\frac{2}{3}\right)}{\left(t_2-\frac{1}{3}\right)\left(t_2-2r_4-\frac{2}{3}\right)(t_2-2r_3-1)}\end{cases}$$

Then

$$\Psi_1(t_1)=\frac{\left(t_1-2r_1-\frac{3}{2}\right)(t_1-2r_2-1)}{(t_1-2r_1-2)\left(t_1-2r_2-\frac{3}{2}\right)};\qquad \Psi_2(t_2)=\frac{t_2\left(t_2-2r_4-\frac{1}{3}\right)\left(t_2-2r_3-\frac{2}{3}\right)}{\left(t_2-\frac{1}{3}\right)\left(t_2-2r_4-\frac{2}{3}\right)(t_2-2r_3-1)}.$$

$$J_1(t_1)=\frac{1}{\Psi_1(t_1)-1}+2;\qquad J_2(t_2)=\frac{1}{\Psi_2(t_2)-1}+1.$$

The boundary separating the region where the first system has only real roots and the first system has a pair of complex conjugate roots is given by

$$\begin{cases}\chi_1(t_1)=\frac{1}{2}\left[t_1-\frac{J_1(t_1)}{J_1'(t_1)}\right]\\ \kappa_1(t_1)=\frac{1}{J_1'(t_1)}\end{cases}$$

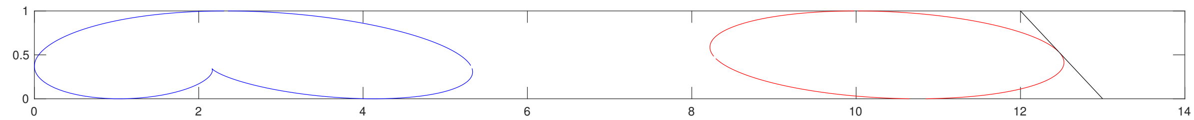


FIGURE 7.1. Frozen boundary for a contracting hexagonal lattice when $n = 2$, $(r_1, r_2, r_3, r_4) = (12, 8, 5, 2)$, represented by the union of the red curve and the blue curve.

The boundary separating the region where the second system has only real roots and the second system has a pair of complex conjugate roots is given by

$$\begin{cases} \chi_2(t_2) = \frac{1}{2}\left[t_2 - \frac{J_2(t_2)}{J_2'(t_2)}\right] \\ \kappa_2(t_2) = \frac{1}{J_2'(t_2)} \end{cases}$$

For $(r_1, r_2, r_3, r_4) = (12, 8, 5, 2)$*; see Figure 7.1 for a picture of the frozen boundary.*

7.2. **Square-hexagonal lattice with** $|I_2 \cap [n]| = 1$**.** Now we consider the case when

$$|I_2 \cap [n]| = \{r\},$$

where r is a positive integer satisfying $1 \leq r \leq n$. Heuristically, there is exactly one row with the structure of a square grid in each period; and all the other rows in the period has the structure of a hexagon lattice. In this case, for each $2 \leq i \leq n$ we can write (7.5) as follows

$$\begin{cases} \frac{t_i}{1-\kappa} - \frac{\kappa}{1-\kappa}\frac{z}{z-1} = nx + \frac{\kappa(n-i)}{1-\kappa}. \\ \mathrm{St}_{\mathbf{m}_i}(t_i) = \log(z). \end{cases}$$

When $i = 1$, (7.5) can be written as

$$\begin{cases} \frac{t_1}{1-\kappa} - \frac{\kappa}{1-\kappa}\frac{z}{z-1} + \frac{\kappa}{1-\kappa}\frac{z}{z+c_r} = nx + \frac{\kappa(n-1)}{1-\kappa}. \\ \mathrm{St}_{\mathbf{m}_i}(t_i) = \log(z). \end{cases}$$

where $c_r = \frac{1}{y_r x_1}$.

Then we have for $2 \leq i \leq n$,

$$z = \frac{\prod_{k=0}^{D_i}\left[\frac{\kappa z}{z-1} + nx(1-\kappa) + \kappa(n-i) - \beta_{i,k}\right]}{\prod_{k=0}^{D_i}\left[\frac{\kappa z}{z-1} + nx(1-\kappa) + \kappa(n-i) - \gamma_{i,k}\right]} := G_i(z, x) \tag{7.20}$$

and for $i = 1$,

$$z = \frac{\prod_{k=0}^{D_i}\left[\frac{\kappa z}{z-1} - \frac{\kappa z}{z+c_r} + nx(1-\kappa) + \kappa(n-1) - \beta_{i,k}\right]}{\prod_{k=0}^{D_i}\left[\frac{\kappa z}{z-1} - \frac{\kappa z}{z+c_r} + nx(1-\kappa) + \kappa(n-i) - \gamma_{i,k}\right]} := G_i(z, x) \tag{7.21}$$

Lemma 7.12. *Assume* $x_0 > 0$ *is such that equation (7.21) (resp. (7.20)) has a pair of complex conjugate roots when* $i = 1$ *(resp.* $2 \leq i \leq n$*). Let* $s_i(x)$ *be a real root of (7.19). Then*

$$\left.\frac{\partial s_i(x)}{\partial x}\right|_{x=x_0} \geq 0.$$

It is equal to 0 if and only if $s_i(x_0) = 1$.

Proof. When $2 \leq i \leq n$, the lemma follows from lemma 7.9. When $i = 1$, (7.21) is the same as the equation for rectangular Aztec diamond with period 1×1 and parameter $q = c_r$, boundary condition given by $\mathbf{m}_i$; see equation (8.6) of [8]. Then the lemma follows from the same argument as the proof of Lemma 4.5 in [8]. □

Now let us consider a rectangular Aztec diamond (resp. contracting hexagon lattice) with boundary partition given by $\phi^{(i,\sigma_0)}(N) \in \mathbb{GT}^+_{\frac{N}{n}}$ when $i = 1$ (resp. $2 \leq i \leq n$). Let $\kappa \in (0,1)$, and $\mathbf{m}_i^\kappa$ be the limit counting measure for the partitions on the $\lfloor \frac{2\kappa N}{n} \rfloor$th row, counting from the bottom. Then using the same arguments as before, we obtain that

$$\mathrm{St}_{\mathbf{m}_i^\kappa}\left(nx + \frac{\kappa(n-i)}{1-\kappa}\right) = \log(z_i^\kappa(x))$$

Hence we have

$$z_i^\kappa(x) = \exp\left(\int_{\mathbb{R}} \frac{\mathbf{m}_i^\kappa[ds]}{nx + \frac{\kappa(n-i)}{1-\kappa} - s}\right);$$

and

$$z_i^\kappa(x + \mathbf{i}\epsilon) = \exp\left(\int_{\mathbb{R}} \frac{\left(nx + \frac{\kappa(n-i)}{1-\kappa} - s - \mathbf{i}\epsilon\right)\mathbf{m}_i^\kappa[ds]}{\left(nx + \frac{\kappa(n-i)}{1-\kappa} - s\right)^2 + \epsilon^2}\right)$$

Therefore $\Im[z_i^\kappa(x + \mathbf{i}\epsilon)] < 0$ when ϵ is a small positive number. However, when complex roots exist for (7.19), for real root $s_i(x)$, Lemma 7.9 implies that $\Im[s_i(x+\mathbf{i}\epsilon)] > 0$ when ϵ is a small positive number. This implies that when complex roots exist for (7.19), $z_i^\kappa(x+\mathbf{i}\epsilon)$ cannot be real. Then we have the following theorem

Theorem 7.13. *Assume $I_2 \cap [n] = \{r\}$, where r is a positive integer satisfying $1 \leq r \leq n$. For the contracting square-hexagon lattice, (χ, κ) is in the frozen region if and only if (7.4) only has real roots for all $1 \leq i \leq n$. The frozen boundary consists of n disjoint cloud curve $C_1, \ldots, C_n$, where for $2 \leq i \leq n$, C_i is a cloud curve of class $D_i + 1$; C_1 is a cloud curve of class $2(D_1 + 1)$. The curve C_i has an explicit parametrization given by*

$$\chi_i(t_i) = \frac{1}{n}\left[t_i - \frac{J_i(t_i)}{J_i'(t_i)}\right], \quad \kappa_i(t_i) = \frac{1}{J_i'(t_i)},$$

where for $2 \leq i \leq n$;

$$J_i(t_i) = (n - i + 1) + \frac{1}{\Psi_i(t_i) - 1};$$

for $i = 1$

$$J_1(t_1) = \frac{1}{\Psi_1(t_1) - 1} + n - 1 + \frac{c_r}{\Psi_1(t_1) + c_r}$$

and Ψ_i is given by (7.15). Moreover, for $2 \leq i \leq n$, each C_i is tangent to $\kappa = 0$ with D_i tangent points, and C_1 is tangent to $\kappa = 0$ with $2D_i + 1$ points. For $1 \leq i \leq n$, C_i is tangent to $\kappa = 1$ with a unique tangent point. The curve C_n is tangent to $\chi = 0$, and the curve C_1 is tangent to $\chi - r_{d_1} + \frac{1}{2}(\kappa - 2) = 0$.

To illustrate Theorem 7.13, let us see the following example.

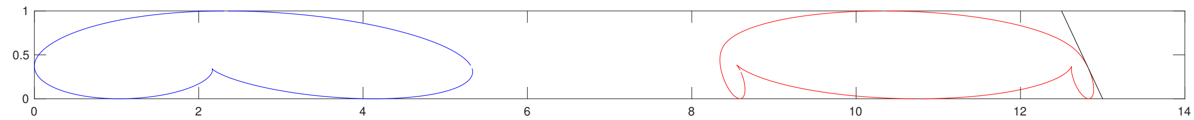


FIGURE 7.2. Frozen boundary for a contracting square hexagon lattice with $n=2$, $|I_2 \cap \{1,2\}|=1$ when $(r_1,r_2,r_3,r_4)=(12,8,5,2)$, $c_r=\frac{1}{2}$, represented by the union of the red curve and the blue curve.

Example 7.14. *Consider a contracting square-hexagon lattice with period* 1×2. *Let* $x_1=1$, *and* $\frac{x_2}{x_1}\leq e^{-\alpha N}$. *Assume* N *is an integer multiple of* 6. *Let* $\lambda(N)$, $\mu(N)$, r_j $(1\leq j\leq 5)$, ϕ^{i,σ_0} $(1\leq i\leq 2)$, $\mathbf{m}_i$ *be given as in Example 7.11. Then we have the following two systems of linear equations*

$$\begin{cases} t_1-\frac{\kappa z}{z-1}=2\chi+\kappa\\ z=\frac{\left(t_1-2r_1-\frac{3}{2}\right)(t_1-2r_2-1)}{(t_1-2r_1-2)\left(t_1-2r_2-\frac{3}{2}\right)}\end{cases}$$

and

$$\begin{cases} t_2-\frac{\kappa z}{z-1}+\frac{\kappa z}{z+c_r}=2\chi\\ z=\frac{t_2\left(t_2-2r_4-\frac{1}{3}\right)\left(t_2-2r_3-\frac{2}{3}\right)}{\left(t_2-\frac{1}{3}\right)\left(t_2-2r_4-\frac{2}{3}\right)(t_2-2r_3-1)}\end{cases}$$

Then

$$\Psi_1(t_1)=\frac{\left(t_1-2r_1-\frac{3}{2}\right)(t_1-2r_2-1)}{(t_1-2r_1-2)\left(t_1-2r_2-\frac{3}{2}\right)};\qquad \Psi_2(t_2)=\frac{t_2\left(t_2-2r_4-\frac{1}{3}\right)\left(t_2-2r_3-\frac{2}{3}\right)}{\left(t_2-\frac{1}{3}\right)\left(t_2-2r_4-\frac{2}{3}\right)(t_2-2r_3-1)}.$$

$$J_1(t_1)=\frac{1}{\Psi_1(t_1)-1}+1+\frac{c_r}{\Psi_1(t_1)+c_r};\qquad J_2(t_2)=\frac{1}{\Psi_2(t_2)-1}+1.$$

The boundary separating the region where the first system has only real roots and the first system has a pair of complex conjugate roots is given by

$$\begin{cases}\chi_1(t_1)=\frac{1}{2}\left[t_1-\frac{J_1(t_1)}{J_1'(t_1)}\right]\\ \kappa_1(t_1)=\frac{1}{J_1'(t_1)}\end{cases}$$

The boundary separating the region where the second system has only real roots and the second system has a pair of complex conjugate roots is given by

$$\begin{cases}\chi_2(t_2)=\frac{1}{2}\left[t_2-\frac{J_2(t_2)}{J_2'(t_2)}\right]\\ \kappa_2(t_2)=\frac{1}{J_2'(t_2)}\end{cases}$$

For $(r_1,r_2,r_3,r_4)=(12,8,5,2)$, $c_r=\frac{1}{2}$; *see Figure 7.2 for a picture of the frozen boundary.*

8. APPENDIX

In this section, we give concrete examples to illustrate the combintorial formula to compute the Schur functions. Example 8.1 is to illustrate Theorem 2.4; and Example 8.2 is to illustrate Corollary 3.4.

Example 8.1. *Let* $N=4$, $\lambda=(3,3,3,1)$, *and* $X=(x_1,x_2,x_1,x_2)$. *Assume* $x_1\neq x_2$ *Then*

$$s_\lambda(x_1,x_2,x_1,x_2)=x_1^4x_2^4(3x_1^2+4x_1x_2+3x_2^2) \tag{8.1}$$

Moreover,

$$|[\Sigma_4/\Sigma_4^X]^r| = \frac{4!}{2!2!} = 6.$$

We find an representative for each right cosets in $[\Sigma_4/\Sigma_4^X]^r$*, as follows:*

$$\sigma_1 = \mathrm{id}; \qquad \sigma_1 = (12); \qquad \sigma_3 = (34);$$
$$\sigma_4 = (23); \qquad \sigma_5 = (14); \qquad \sigma_6 = (12)(34).$$

Then we can compute

$$(\eta_1^{\sigma_k}, \eta_2^{\sigma_k}, \eta_3^{\sigma_k}, \eta_4^{\sigma_k}) = \begin{cases} (2,1,1,0). & \text{if } k = 1,2,3,6 \\ (2,2,0,0) & \text{if } k = 4,5 \end{cases}$$

and

$$\begin{aligned}
\phi^{(1,\sigma_1)} &= (5,4), & \phi^{(2,\sigma_1)} &= (4,1)\\
\phi^{(1,\sigma_2)} &= (4,4), & \phi^{(2,\sigma_2)} &= (5,1)\\
\phi^{(1,\sigma_3)} &= (5,1), & \phi^{(2,\sigma_3)} &= (4,4)\\
\phi^{(1,\sigma_4)} &= (5,5), & \phi^{(2,\sigma_4)} &= (3,1)\\
\phi^{(1,\sigma_5)} &= (3,1), & \phi^{(2,\sigma_5)} &= (5,5)\\
\phi^{(1,\sigma_6)} &= (4,1), & \phi^{(2,\sigma_6)} &= (5,4)
\end{aligned}$$

Computing the right hand side of (3.5), we obtain exactly the right hand side of (8.1).

Example 8.2. *Let* $N = 4$*,* $\lambda = (3,3,3,1)$*, and* $X = (x_1, x_2, x_1, x_2)$*. Assume* $x_1 \neq x_2$ *Then*

$$\begin{aligned}
& s_\lambda(u_1, x_2, x_1, x_2)\\
&= u_1x_1x_2^2(3u_1^2x_1^2x_2^2 + 2u_1^2x_1x_2^3 + u_1^2x_2^4 + 2u_1x_1^2x_2^3 + u_1x_1x_2^4 + x_1^2x_2^4); \qquad (8.2)
\end{aligned}$$

$$\begin{aligned}
& s_\lambda(u_1, u_2, x_1, x_2)\\
&= u_1u_2x_1x_2(u_1^2u_2^2x_1^2 + u_1^2u_2^2x_1x_2 + u_1^2u_2^2x_2^2 + u_1^2u_2x_1^2x_2 + u_1^2u_2x_1x_2^2\\
&+u_1^2x_1^2x_2^2 + u_1u_2^2x_1^2x_2 + u_1u_2^2x_1x_2^2 + u_1u_2x_1^2x_2^2 + u_2^2x_1^2x_2^2); \qquad (8.3)
\end{aligned}$$

$$\begin{aligned}
& s_\lambda(u_1, u_2, u_3, x_2)\\
&= u_1u_2u_3x_2(u_1^2u_2^2u_3^2 + u_1^2u_2^2u_3x_2 + u_1^2u_2^2x_2^2 + u_1^2u_2u_3^2x_2 + u_1^2u_2u_3x_2^2 + u_1^2u_3^2x_2^2\\
&+u_1u_2^2u_3^2x_2 + u_1u_2^2u_3x_2^2 + u_1u_2u_3^2x_2^2 + u_2^2u_3^2x_2^2) \qquad (8.4)
\end{aligned}$$

For $i \in \{1,2\}$ *and* $j \in \{1,2,3,4,5,6\}$*,* $\phi^{(i,\sigma_j)}$ *can be computed as in Example 8.1. We have*

$$\begin{aligned}
s_{\phi^{(1,\sigma_1)}}\left(\frac{u_1}{x_1}, 1\right) &= s_{(5,4)}\left(\frac{u_1}{x_1}, 1\right) = \left(\frac{u_1}{x_1}\right)^4\left(\frac{u_1}{x_1} + 1\right).\\
s_{\phi^{(1,\sigma_1)}}\left(\frac{u_1}{x_1}, \frac{u_3}{x_1}\right) &= s_{(5,4)}\left(\frac{u_1}{x_1}, \frac{u_3}{x_1}\right) = \left(\frac{u_1}{x_1}\right)^4\left(\frac{u_3}{x_1}\right)^4\frac{u_1+u_3}{x_1}.
\end{aligned}$$

$$\begin{aligned}
s_{\phi^{(2,\sigma_1)}}(1,1) &= s_{(4,1)}(1,1) = 4\\
s_{\phi^{(2,\sigma_1)}}\left(\frac{u_2}{x_2}, 1\right) &= s_{(4,1)}\left(\frac{u_2}{x_2}, 1\right) = \left(\frac{u_2}{x_2}\right)\left[\left(\frac{u_2}{x_2}\right)^2 + 1\right]\left[\frac{u_2}{x_2} + 1\right]
\end{aligned}$$

$$\begin{aligned} s_{\phi^{(1,\sigma_2)}}\left(\frac{u_1}{x_1},1\right) &= s_{(4,4)}\left(\frac{u_1}{x_1},1\right)=\left(\frac{u_1}{x_1}\right)^4 \\ s_{\phi^{(1,\sigma_2)}}\left(\frac{u_1}{x_1},\frac{u_3}{x_1}\right) &= s_{(4,4)}\left(\frac{u_1}{x_1},\frac{u_3}{x_1}\right)=\left(\frac{u_1}{x_1}\right)^4\left(\frac{u_3}{x_1}\right)^4 \end{aligned}$$

$$\begin{aligned} s_{\phi^{(2,\sigma_2)}}(1,1) &= s_{(5,1)}(1,1)=5 \\ s_{\phi^{(2,\sigma_2)}}\left(\frac{u_2}{x_2},1\right) &= s_{(5,1)}\left(\frac{u_2}{x_2},1\right)=\left(\frac{u_2}{x_2}\right)\left[\left(\frac{u_2}{x_2}\right)^4+\left(\frac{u_2}{x_2}\right)^3+\left(\frac{u_2}{x_2}\right)^2+\left(\frac{u_2}{x_2}\right)+1\right] \end{aligned}$$

$$\begin{aligned} s_{\phi^{(1,\sigma_3)}}\left(\frac{u_1}{x_1},1\right) &= s_{(5,1)}\left(\frac{u_1}{x_1},1\right)=\left(\frac{u_1}{x_1}\right)\left[\left(\frac{u_1}{x_1}\right)^4+\left(\frac{u_1}{x_1}\right)^3+\left(\frac{u_1}{x_1}\right)^2+\left(\frac{u_1}{x_1}\right)+1\right]. \\ s_{\phi^{(1,\sigma_3)}}\left(\frac{u_1}{x_1},\frac{u_3}{x_1}\right) &= s_{(5,1)}\left(\frac{u_1}{x_1},\frac{u_3}{x_1}\right) \\ &= \left(\frac{u_1}{x_1}\right)\left(\frac{u_3}{x_1}\right)\left[\left(\frac{u_1}{x_1}\right)^4+\left(\frac{u_1}{x_1}\right)^3\left(\frac{u_3}{x_1}\right)+\left(\frac{u_1}{x_1}\right)^2\left(\frac{u_3}{x_1}\right)^2+\left(\frac{u_1}{x_1}\right)\left(\frac{u_3}{x_1}\right)^3+\left(\frac{u_3}{x_1}\right)^4\right] \end{aligned}$$

$$\begin{aligned} s_{\phi^{(2,\sigma_3)}}(1,1) &= s_{(4,4)}(1,1)=1 \\ s_{\phi^{(2,\sigma_3)}}\left(\frac{u_2}{x_2},1\right) &= s_{(4,4)}\left(\frac{u_2}{x_2},1\right)=\left(\frac{u_2}{x_2}\right)^4 \end{aligned}$$

$$\begin{aligned} s_{\phi^{(1,\sigma_4)}}\left(\frac{u_1}{x_1},1\right) &= s_{(5,5)}\left(\frac{u_1}{x_1},1\right)=\left(\frac{u_1}{x_1}\right)^5. \\ s_{\phi^{(1,\sigma_4)}}\left(\frac{u_1}{x_1},\frac{u_3}{x_1}\right) &= s_{(5,5)}\left(\frac{u_1}{x_1},\frac{u_3}{x_1}\right)=\left(\frac{u_1}{x_1}\right)^5\left(\frac{u_3}{x_1}\right)^5. \end{aligned}$$

$$\begin{aligned} s_{\phi^{(2,\sigma_4)}}(1,1) &= s_{(3,1)}(1,1)=3 \\ s_{\phi^{(2,\sigma_4)}}\left(\frac{u_2}{x_2},1\right) &= s_{(3,1)}\left(\frac{u_2}{x_2},1\right)=\left(\frac{u_2}{x_2}\right)\left[\left(\frac{u_2}{x_2}\right)^2+\left(\frac{u_2}{x_2}\right)+1\right] \end{aligned}$$

$$\begin{aligned} s_{\phi^{(1,\sigma_5)}}\left(\frac{u_1}{x_1},1\right) &= s_{(3,1)}\left(\frac{u_1}{x_1},1\right)=\left(\frac{u_1}{x_1}\right)\left[\left(\frac{u_1}{x_1}\right)^2+\left(\frac{u_1}{x_1}\right)+1\right] \\ s_{\phi^{(1,\sigma_5)}}\left(\frac{u_1}{x_1},\frac{u_3}{x_1}\right) &= s_{(3,1)}\left(\frac{u_1}{x_1}\right)\left(\frac{u_3}{x_1}\right)\left[\left(\frac{u_1}{x_1}\right)^2+\left(\frac{u_1}{x_1}\right)\left(\frac{u_3}{x_1}\right)+\left(\frac{u_3}{x_1}\right)^2\right] \end{aligned}$$

$$\begin{aligned} s_{\phi^{(2,\sigma_5)}}(1,1) &= s_{(5,5)}(1,1)=1 \\ s_{\phi^{(2,\sigma_5)}}\left(\frac{u_2}{x_2},1\right) &= s_{(5,5)}\left(\frac{u_2}{x_2},1\right)=\left(\frac{u_2}{x_2}\right)^5 \end{aligned}$$

$$s_{\phi^{(1,\sigma_6)}}\left(\frac{u_1}{x_1},1\right) = s_{(4,1)}\left(\frac{u_1}{x_1},1\right)=\left(\frac{u_1}{x_1}\right)\left[\left(\frac{u_1}{x_1}\right)^2+1\right]\left[\frac{u_1}{x_1}+1\right].$$

$$s_{\phi^{(1,\sigma_6)}}\left(\frac{u_1}{x_1},\frac{u_3}{x_1}\right) = s_{(4,1)}\left(\frac{u_1}{x_1},\frac{u_3}{x_1}\right)$$
$$= \left(\frac{u_1}{x_1}\right)\left(\frac{u_3}{x_1}\right)\left[\left(\frac{u_1}{x_1}\right)^2+\left(\frac{u_3}{x_1}\right)^2\right]\left[\frac{u_1}{x_1}+\frac{u_3}{x_1}\right]$$

$$s_{\phi^{(2,\sigma_6)}}(1,1) = s_{(5,4)}(1,1)=2$$
$$s_{\phi^{(2,\sigma_6)}}\left(\frac{u_2}{x_2},1\right) = s_{(5,4)}\left(\frac{u_2}{x_2},1\right)=\left(\frac{u_2}{x_2}\right)^4\left(\frac{u_2}{x_2}+1\right)$$

Computing the right hand side of (3.5) when $k=1,2,3$, we obtain exactly the right hand side of (8.2)-(8.4).

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