Strichartz estimates for the Schrödinger flow on compact symmetric spaces

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CHAPTER 1

Introduction

We start with a complete Riemannian manifold (M, g) of dimension d, associated to which are the Laplace-Beltrami operator Δ_g and the volume form measure μ_g. Then it is well known that Δ_g is essentially self-adjoint on $L^2(M) := L^2(M, d\mu_g)$ (see [**Str83**] for a proof). This gives the functional calculus of Δ_g, and in particular gives the one-parameter unitary operators $e^{it\Delta_g}$ which provides the solution to the linear Schrödinger equation on (M, g). We refer to $e^{it\Delta_g}$ as the *Schrödinger flow*. The functional calculus of Δ_g also gives the definition of the Bessel potentials thus the definition of the Sobolev space

$$H^s(M) := \{u \in L^2(M) : \|u\|_{H^s(M)} := \|(I - \Delta)^{s/2} u\|_{L^2(M)} < \infty\}.$$

We are interested in obtaining estimates of the form

$$\|e^{it\Delta_g} f\|_{L^p L^r(I \times M)} \leq C \|f\|_{H^s(M)} \tag{0.1}$$

where $I \subset \mathbb{R}$ is a fixed time interval, $L^p L^q(I \times M)$ is the space of L^p functions on I with values in $L^q(M)$, and C is a constant that does not depend on f. Such estimates are often called *Strichartz estimates* (for the Schrödinger flow), in honor of Robert Strichartz who first derived such estimates for the wave flow on Euclidean spaces (see [**Str77**]).

The significance of the Strichartz estimates is evident in many ways. The Strichartz estimates have important applications in the field of nonlinear dispersive equations, in the sense that many perturbative results often require a good control on the linear solution which is exactly provided by the Strichartz estimates. The Strichartz estimates can also be interpreted as Fourier restriction estimates, which play a fundamental rule in the field of classical harmonic analysis and have deep connections to arithmetic combinatorics (see [**Łab08**]). Furthermore, the relevance of the distribution of eigenvalues and the norm of eigenfunctions of Δ_g in deriving the estimates makes the Strichartz estimates also a subject in the field of semiclassical analysis and spectral geometry.

Many cases of Strichartz estimates for the Schrödinger flow are known in the literature. For noncompact manifolds, first we have the sharp Strichartz estimates on the Euclidean spaces obtained in [**GV95, KT98**]:

$$\|e^{it\Delta} f\|_{L^p L^q(\mathbb{R} \times \mathbb{R}^d)} \leq C \|f\|_{L^2(\mathbb{R}^d)} \tag{0.2}$$

where $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$, $p, q \geq 2$, $(p, q, d) \neq (2, \infty, 2)$. Such pairs (p, q) are called *admissible*. This implies by Sobolev embedding that

$$\|e^{it\Delta} f\|_{L^p L^r(\mathbb{R} \times \mathbb{R}^d)} \leq C \|f\|_{H^s(\mathbb{R}^d)} \tag{0.3}$$

where

$$s = \frac{d}{2} - \frac{2}{p} - \frac{d}{r} \geq 0, \tag{0.4}$$

$p, r \geq 2$, $(p, r, d) \neq (2, \infty, 2)$. Note that the equality in (0.4) can be derived from a standard scaling argument, and we call exponent triples (p, r, s) that satisfy (0.4) as well as the corresponding Strichartz estimates *scaling* *critical*. An essential ingredient in the derivation of (0.3) is the dispersive estimates

$$\|e^{it\Delta} f\|_{L^\infty(\mathbb{R}^d)} \leq C|t|^{-\frac{d}{2}} \|f\|_{L^1(\mathbb{R}^d)}. \tag{0.5}$$

Similar dispersive estimates hold on many noncompact manifolds, which are essential in the derivation of Strichartz estimates. For example, see [**AP09, Ban07, IS09, Pie06**] for Strichartz estimates on the real hyperbolic spaces, [**APV11, Pie08, BD07**] for Damek-Ricci spaces which include all rank one symmetric spaces of noncompact type, [**Bou11**] for asymptotically hyperbolic manifolds, [**HTW06**] for asymptotically conic manifolds, [**BT08, ST02**] for some perturbed Schrödinger equations on Euclidean spaces, and [**FMM15**] for symmetric spaces G/K where G is complex and K is a maximal compact subgroup of G.

For compact manifolds, however, dispersive estimates that are global in time such as (0.5) are expected to fail and so are Strichartz estimates such as (0.2) (see [**AM12**] which shows (0.2) fails for any p, q with $p = q$). The Sobolev exponent s in (0.1) are expected to be positive for (0.1) to possibly hold. And we also expect sharp Strichartz estimates that fail to be scaling critical and thus are *scaling subcritical*, in the sense that the exponents (p, r, s) in (0.1) satisfy

$$s > \frac{d}{2} - \frac{2}{p} - \frac{d}{r}.$$

For example, from the results in [**BGT04**], we know that on a general compact Riemannian manifold (M, g) it holds that for any finite interval I,

$$\|e^{it\Delta_g} f\|_{L^p L^r(I\times M)} \leq C\|f\|_{H^{1/p}(M)} \tag{0.6}$$

for all admissible pairs (p, r). These estimates are scaling subcritical, and the special case of which when $(p, r, s) = (2, \frac{2d}{d-2}, 1/2)$ can be shown to be sharp on spheres of dimension $d \geq 3$ equipped with canonical Riemannian metrics. The proof of (0.6) in [**BGT04**] hinges on a semiclassical analogue of the dispersive estimate (0.5): given any bump function φ on $\mathbb{R}$, there exists $\alpha > 0$ such that

$$\|e^{it\Delta_g} \varphi(h^2\Delta_g) f\|_{L^\infty(M)} \leq C|t|^{-\frac{d}{2}} \|f\|_{L^1(M)} \tag{0.7}$$

for every $t \in (-\alpha h, \alpha h)$.

On the other hand, scaling critical Strichartz estimates have also been obtained on compact manifolds. On spheres and more generally Zoll manifolds, it holds that

$$\|e^{it\Delta_g} f\|_{L^p(I\times M)} \leq C\|f\|_{H^{\frac{d}{2}-\frac{d+2}{p}}(M)} \tag{0.8}$$

for $p > 4$ when $d \geq 3$ and $p \geq 6$ when $d = 2$ (see [**BGT04, BGT05, Her13**]). We also have that on a d-dimensional torus $\mathbb{T}^d$ equipped with a *rectangular* metric $g = \otimes_{i=1}^d \alpha_i dt_i^2$ where the α_i's are positive numbers and the dt_i^2's are the canonical metrics on the circle components of $\mathbb{T}^d$, Strichartz estimates of the form (0.8) hold for all $p > \frac{2(d+2)}{d}$ (see [**Bou93, Bou13, BD15, GOW14, KV16**]). In [**Bou93**], the author was able to obtain (0.8) for $p \geq \frac{2(d+4)}{d}$ on tori that are *square* in the sense that the underlying metric is a constant multiple of $\otimes_{i=1}^d dt_i^2$, by interpolating a distributional Strichartz estimate

$$\lambda \cdot \mu\{(t, x) \in I \times \mathbb{T}^d : |e^{it\Delta_g} \varphi(N^{-2}\Delta_g) f(x)| > \lambda\}^{1/p} \leq CN^{\frac{d}{2}-\frac{d+2}{p}} \|f\|_{L^2(\mathbb{T}^d)}. \tag{0.9}$$

for $\lambda > N^{d/4}$, $p > \frac{2(d+2)}{d}$, $N \geq 1$, with the trivial subcritical Strichartz estimate

$$\|e^{it\Delta_g} f\|_{L^2(I\times\mathbb{T}^d)} \leq C\|f\|_{L^2(\mathbb{T}^d)}. \tag{0.10}$$

(0.9) is a consequence of an arithmetic version of dispersive estimates:

$$\|e^{it\Delta_g}\varphi(N^{-2}\Delta_g)f\|_{L^\infty(\mathbb{T}^d)} \leq C\Big(\frac{N}{\sqrt{q}(1+N\|\frac{t}{T}-\frac{a}{q}\|^{1/2})}\Big)^d\|f\|_{L^1(\mathbb{T}^d)} \tag{0.11}$$

for $\|\frac{t}{T}-\frac{a}{q}\| < \frac{1}{qN}$, where $\|\cdot\|$ stands for the distance from 0 on the standard circle with length 1, a, q are nonnegative integers with $a < q$ and $(a, q) = 1$, $q < N$. Here T is the *period* for the *Schrödinger flow* $e^{it\Delta_g}$. Then in [**Bou13**], the author improved (0.10) into a stronger subcritical Strichartz estimate

$$\|e^{it\Delta_g}f\|_{L^{\frac{2(d+1)}{d}}(I\times\mathbb{T}^d)} \leq C\|f\|_{L^2(\mathbb{T}^d)} \tag{0.12}$$

which yields (0.8) for $p \geq \frac{2(d+3)}{d}$, which is further upgraded to the full range $p > \frac{2(d+2)}{d}$ in [**BD15**]. Then authors in [**GOW14, KV16**] extend the results to all rectangular tori.

The understanding of Strichartz estimates on compact manifolds is far from complete. The sublime goal is to understand how the exponents (p, r, s) in the sharp Strichartz estimates are related to the geometry and topology of the underlying manifold. This thesis picks up a modest goal, that is to explore scaling critical Strichartz estimates on the special case of compact Lie groups and more generally compact Riemannian globally symmetric spaces. By the previous discussion, for such spaces, the cases already solved in the literature are

1. Euclidean type, i.e. tori (but only for rectangular ones);
2. Symmetric space of compact type of rank one, which are Zoll manifolds, i.e. manifolds such that the geodesics are all closed and have the same length (see Proposition 10.2 of Ch. VII in [**Hel01**]).

Symmetric spaces are equipped with rich tools of harmonic analysis, which provide a possible general approach to Strichartz estimates. In this thesis, scaling critical (with an ε-loss, respectively) Strichartz estimates will be proved for general compact Riemannian globally symmetric spaces with canonical *rational metrics*, conditional on a conjectured scaling critical (with an ε-loss, respectively) dispersive estimate associated to the spherical functions. This scaling critical dispersive estimate will be proved for the special case of connected compact Lie groups. More generally, for products of connected compact Lie groups and spheres of odd dimension, the dispersive estimate will be proved with an ε-loss.

1. Statement of the Main Theorem

1.1. Rational Metric and Rank. Throughout the thesis, a *compact symmetric space* always means a compact Riemannian globally symmetric space. Let M be a compact symmetric space. It can be shown that M is finitely covered by $\tilde{M} = \mathbb{T}^n \times N$ where $\mathbb{T}^n$ is the n-dimensional torus and N is a simply connected Riemannian globally symmetric space of compact type[1]. As a simply connected Riemannian globally symmetric space of compact type, N is a direct product $U_1/K_1 \times U_2/K_2 \times \cdots \times U_m/K_m$ of irreducible simply connected Riemannian globally symmetric space of compact type (see Proposition 5.5 in Ch. VIII in [**Hel01**]).

DEFINITION 1. *We call such* $\tilde{M} = \mathbb{T} \times U_1/K_1 \times U_2/K_2 \times \cdots \times U_m/K_m$ *a universal covering compact symmetric space, and say that* M *is universally covered by* $\tilde{M}$.

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[1] This fact can be proved as follows. Let $M = U/K$ be a compact symmetric space and $\mathfrak{u}, \mathfrak{k}$ be respectively the Lie algebras of U, K. Then $\mathfrak{u} = \mathfrak{c} + \mathfrak{u}'$ where $\mathfrak{c}$ is the center of $\mathfrak{u}$ and $\mathfrak{u}'$ is the semisimple part of $\mathfrak{u}$. Let $\mathfrak{u} = \mathfrak{k} + \mathfrak{m}$ be the Cartan decomposition. Then $\mathfrak{k} = \mathfrak{c}_\mathfrak{k} + \mathfrak{k}'$ for $\mathfrak{c}_\mathfrak{k} = \mathfrak{c} \cap \mathfrak{k}$, $\mathfrak{k}' = \mathfrak{u}' \cap \mathfrak{k}$, and $\mathfrak{m} = \mathfrak{c}_\mathfrak{m} + \mathfrak{m}'$ for $\mathfrak{c}_\mathfrak{m} = \mathfrak{c} \cap \mathfrak{m}$, $\mathfrak{m}' = \mathfrak{u}' \cap \mathfrak{m}$. Let U', K' be the subgroups of U associated to $\mathfrak{u}', \mathfrak{k}'$ respectively. Then U'/K' is a symmetric space of compact type and let $\tilde{U}'/\tilde{K}'$ be its universal cover, the covering map induced from the universal covering $\pi : \tilde{U}' \to U'$. Let $C_\mathfrak{m}$ be the toric subgroup of U associated to $\mathfrak{c}_\mathfrak{m}$. Then the map $C_\mathfrak{m} \times \tilde{U}'/\tilde{K}' \to U/K$, $(c, uK') \to c\pi(u)K$ is a finite covering map.

Now let U/K be a simply connected Riemannian globally symmetric space of compact type. We consider the dual symmetric space G/K with G and U analytic subgroups of the simply connected group $G^{\mathbb{C}}$ whose Lie algebra is the complexification $\mathfrak{g}^{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$ of G. Let $\mathfrak{u}, \mathfrak{k}$ be respectively the Lie algebra of U and K. Then we have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \tag{1.1}$$

$$\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}. \tag{1.2}$$

The negative of the *Cartan-Killing form* $-\langle \ , \ \rangle$ defined on $\mathfrak{u}$ (as well as on $\mathfrak{g}$ and $\mathfrak{g}^{\mathbb{C}}$) restricts to $i\mathfrak{p}$ as a positive definite bilinear form invariant under the adjoint action of U, which induces a Riemannian metric on U/K invariant under the left action of U.

We equip each irreducible factor U_i/K_i with such a metric g_i defined above. We then equip the torus factor $\mathbb{T}^n$ a flat metric g_0 inherited from its representation as the quotient $\mathbb{R}^n/2\pi\Gamma$ and require that there exists some $D \in \mathbb{N}$ such that $\langle \lambda, \mu \rangle \in D^{-1}\mathbb{Z}$ for all $\lambda, \mu \in \Gamma$ (cf. Definition 27). Then we equip $\tilde{M} \cong \mathbb{T}^n \times U_1/K_1 \times \cdots \times U_m/K_m$ the metric

$$\tilde{g} = \otimes_{j=0}^{m} \beta_j g_j, \tag{1.3}$$

$\beta_j > 0$, $j = 0, \ldots, m$. Then $\tilde{g}$ induces a metric g on M.

Definition 2. *Let g be the metric induced from $\tilde{g}$ in* (1.3) *as described above. We call g a rational metric provided the numbers $\beta_0, \cdots, \beta_m$ are rational multiples of each other.*

Provided the numbers $\beta_0, \cdots, \beta_m$ are rational multiples of each other, the periods of the Schrödinger flow $e^{it\Delta_{\tilde{g}}}$ on each factor of $\tilde{M}$ are rational multiples of each other, which implies that the Schrödinger flow on $\tilde{M}$ as well as on M is still periodic (see Proposition 10 and Section 1).

Next, we define the *rank* of a Riemannian symmetric space U/K of compact type as the dimension of any maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. In general, let M be a compact symmetric space with a universal covering compact symmetric space $\tilde{M} = \mathbb{T}^n \times U_1/K_1 \times \cdots \times U_m/K_m$. We define the rank of M as well as $\tilde{M}$ to be $n + r_1 + \cdots + r_m$, where r_j is the rank of U_j/K_j, $j = 1, \cdots, m$.

Example 3. *Any compact connected Lie group M is a compact symmetric space. M is covered by a universal covering compact Lie group $\tilde{M} = \mathbb{T}^n \times M_1 \times \cdots \times M_m$, where the M_i's are compact simply connected simple Lie groups (see Theorem 4, Section 7.2, Chapter 10 in* [**Pro07**]*). Suppose M is a compact simply connected simple Lie group with Lie algebra $\mathfrak{m}$. Then $M \cong U/K$ where $U = M \times M$ and $K = \{(x,x) : x \in M\}$, of which the Lie algebras are $\mathfrak{u} = \mathfrak{m} \times \mathfrak{m}$ and $\mathfrak{k} = \{(X,X) : X \in \mathfrak{m}\}$ respectively, and the complement of $\mathfrak{k}$ in the Cartan decomposition* (1.2) *is $i\mathfrak{p} = \{(X,-X) : X \in \mathfrak{m}\}$. We have the identifications*

$$\begin{aligned} U/K \cong M,&\ (x,y)K \mapsto xy^{-1}, \\ i\mathfrak{p} \cong \mathfrak{m},&\ (X,-X) \mapsto 2X. \end{aligned} \tag{1.4}$$

Under the above identification, the Cartan-Killing form on $i\mathfrak{p}$ is half the value of the Cartan-Killing form on $\mathfrak{m}$, and any Cartan subalgebra (i.e. maximal abelian subspace) $i\mathfrak{a}$ of $\mathfrak{m}$ corresponds to a maximal abelian subspace of $i\mathfrak{p}$.

1.2. Main Conjecture and Main Theorem. Inspired by the result of Strichartz estimates on tori and Zoll manifolds, we have the following conjecture.

Conjecture 4. *Let M be a compact symmetric space equipped with a rational metric g. Let d be the dimension of M and r the rank of M. Let $I \subset \mathbb{R}$ be a finite time interval. Then the following scaling critical Strichartz estimates*

$$\|e^{it\Delta_g} f\|_{L^p(I\times M)} \leq C\|f\|_{H^{\frac{d}{2}-\frac{d+2}{p}}(M)} \tag{1.5}$$

hold for all $p > 2 + \frac{4}{r}$.

This thesis proves some special cases of this conjecture.

Theorem 5. *Let M be a compact symmetric space universally covered by $\tilde{M} = \mathbb{T}^n \times U_1/K_1 \times \cdots \times U_m/K_m$. Equip M with a rational metric g and let d, r be respectively the dimension and rank of M. Let $I \subset \mathbb{R}$ be a finite time interval.*

Case 1. [**Zha**] *If each U_j/K_j is a compact simply connected simple Lie group, in other words, by Example 3, if M itself is a connected compact Lie group, then the following scaling critical Strichartz estimates*

$$\|e^{it\Delta_g} f\|_{L^p(I\times M)} \leq C\|f\|_{H^{\frac{d}{2}-\frac{d+2}{p}}(M)} \tag{1.6}$$

hold for all $p \geq 2 + \frac{8}{r}$.

Case 2. *If each U_j/K_j is either a compact simply connected simple Lie group or a sphere of odd dimension ≥ 5, then*

$$\|e^{it\Delta_g} f\|_{L^p(I\times M)} \leq C_\varepsilon\|f\|_{H^{\frac{d}{2}-\frac{d+2}{p}+\varepsilon}(M)} \tag{1.7}$$

hold for all $p \geq 2 + \frac{8}{r}$, $\varepsilon > 0$.

Case 3. *If $\tilde{M} = \mathbb{T}^d$, then* (1.6) *hold for all $p > 2 + \frac{4}{d}$.*

The framework of the proof of Theorem 5 will be based on [**Bou93**], in which the author proves it for the special case of square tori. Note that Case 3 in the above theorem provides full Strichartz estimates for non-rectangular tori of the type $\mathbb{R}^n/2\pi\Gamma$, Γ being a *rational lattice* (see Definition 27).

2. Organization and Notation Conventions

The organization of the thesis is as follows. In Chapter 2, several reductions will be made to reduce the conjectured Strichartz estimate (1.5) into a spectrally localized form posed on a universal covering compact symmetric space. This reduction in particular dissolves the issue of convergence of the Schrödinger kernel. In Chapter 3, basic facts of harmonic analysis on compact symmetric spaces that are crucial in the sequel, including spherical Fourier series, reduced root systems, and functional calculus of the Laplace-Beltrami operator, will be reviewed, which are used to give the explicit formula of the Schrödinger kernel. In Chapter 4, a conjectured dispersive estimate will be posed on a general compact symmetric space, and we will show that it implies the Strichartz estimates, by the method of Stein-Tomas type interpolation. In Chapter 5, the conjectured spectrally localized dispersive estimate will be proved on a general symmetric space of compact type for a neighborhood of diameter $\lesssim N^{-1}$ of any corner in the space. Special approaches to this result for the case of compact Lie groups will also be given. Chapter 5 ends with proving with an ε-loss the dispersive estimate on spheres of odd dimension and remarking on the difficulty for the general case. In Chapter 6, the dispersive estimate for connected compact Lie groups will be proved. We will first make a crucial observation that the Schrödinger kernel can be rewritten as an exponential sum over the whole weight lattice instead of just a Weyl chamber of the lattice, which is unique among symmetric spaces of compact type. We will decompose the maximal torus into regions according to the distance from the cell walls, and prove the dispersive estimate for each region. The most difficult case is when the variable in the maximal torus stays away from some cell walls but close to the other cell walls. These other walls will be identified as those of a root subsystem, which induces a decomposition of Schrödinger kernel that makes the proof work.

Throughout the paper:

- $A \lesssim B$ means $A \leq CB$ for some constant C.
- $A \lesssim_{a,b,\dots} B$ means $A \leq CB$ for some constant C that depends on $a, b, \dots$.
- Δ, μ are short for the Laplace-Beltrami operator Δ_g and the associated normalized volume form measure μ_g respectively when the underlying Riemannian metric g is clear from context.
- $L_x^p, H_x^s, L_t^p, L_t^p L_x^q, L_{t,x}^p$ are short for $L^p(M), H^s(M), L^p(I), L^p L^q(I \times M), L^p(I \times M)$ respectively when the underlying manifold M and time interval I are clear from context.
- Let $\mathbb{T} = \mathbb{R}/T\mathbb{Z}$. For $f \in L^1(\mathbb{T})$, let $\widehat{f}$ denote the Fourier transform of f such that $\widehat{f}(n) = \frac{1}{T}\int_0^T f(t)e^{-int}\,dt$, $n \in \frac{2\pi}{T}\mathbb{Z}$.
- p' denotes the number such that $\frac{1}{p} + \frac{1}{p'} = 1$.

CHAPTER 2

First Reductions

1. Littlewood-Paley Theory

Let (M,g) be a compact Riemannian manifold and Δ be the Laplace-Beltrami operator. Let φ be a bump function on $\mathbb{R}$. Then for $N \geq 1$, $P_N := \varphi(N^{-2}\Delta)$ defines a bounded operator on $L^2(M)$ through the functional calculus of Δ. These operators P_N are often called the *Littlewood-Paley projections*. We reduce the problem of obtaining Strichartz estimates for $e^{it\Delta}$ to those for $P_N e^{it\Delta}$.

PROPOSITION 6. *Fix $p, q \geq 2$, $s \geq 0$. Then the Strichartz estimate* (0.1) *is equivalent to the following statement: Given any bump function φ,*

$$\|P_N e^{it\Delta} f\|_{L^p L^q(I\times M)} \lesssim N^s \|f\|_{L^2(M)}, \tag{1.1}$$

holds for all $N \in 2^{\mathbb{N}}$. In particular, (1.5) *reduced to*

$$\|P_N e^{it\Delta} f\|_{L^p(I\times M)} \leq N^{\frac{d}{2}-\frac{d+2}{p}} \|f\|_{L^2(M)}. \tag{1.2}$$

We quote the following Littlewood-Paley theory from [**BGT04**].

PROPOSITION 7 (Corollary 2.3 in [**BGT04**]). *Let $\tilde{\varphi} \in C_c^\infty(\mathbb{R})$ and $\varphi \in C_c^\infty(\mathbb{R}^*)$ such that*

$$\tilde{\varphi}(\lambda) + \sum_{N=2^{\mathbb{N}}} \varphi(N^{-2}\lambda) = 1$$

for all $\lambda \in \mathbb{R}$. Then for all $q \geq 2$, we have

$$\|f\|_{L^q(M)} \lesssim_q \|\tilde{\varphi}(\Delta) f\|_{L^q(M)} + (\sum_{N=2^{\mathbb{N}}} \|\varphi(N^{-2}\Delta) f\|^2_{L^q(M)})^{1/2}. \tag{1.3}$$

PROOF OF PROPOSITION 6. The implication of (1.1) from (0.1) is immediate by letting f in (0.1) be $P_N f$, and noting that P_N and $e^{it\Delta}$ commute, and that $\|P_N f\|_{H^s} \lesssim N^s \|f\|_{L^2}$. For the other direction, assume that φ and $\tilde{\varphi}$ is given as in Proposition 6 and define $P_N = \varphi(N^{-2}\Delta)$ and $\tilde{P}_1 = \tilde{\varphi}(\Delta)$. Let $\tilde{\tilde{\varphi}} \in C_c^\infty(\mathbb{R})$ and define $\tilde{\tilde{P}}_N = \tilde{\tilde{\varphi}}(N^{-2}\Delta)$ such that $\tilde{\tilde{\varphi}}\varphi = \varphi$ and thus $\tilde{\tilde{P}}_N P_N = P_N$. By (1.3), we have that

$$\begin{aligned}
\|e^{it\Delta} f\|_{L^p_t L^q_x} &= \big\| \|e^{it\Delta} f\|_{L^q_x} \big\|_{L^p_t} \\
&\lesssim \left\| \|\tilde{P}_1 e^{it\Delta} f\|_{L^q_x} + (\sum_{N=2^{\mathbb{N}}} \|P_N e^{it\Delta} f\|^2_{L^q_x})^{1/2} \right\|_{L^p_t} \\
&\lesssim \|\tilde{P}_1 e^{it\Delta} f\|_{L^p_t L^q_x} + \left\| (\sum_{N=2^{\mathbb{N}}} \|P_N e^{it\Delta} f\|^2_{L^q_x})^{1/2} \right\|_{L^p_t} \\
&\lesssim \|\tilde{P}_1 e^{it\Delta} f\|_{L^p_t L^q_x} + \left\| (\sum_{N=2^{\mathbb{N}}} \|P_N e^{it\Delta} \tilde{\tilde{P}}_N f\|^2_{L^q_x})^{1/2} \right\|_{L^p_t}
\end{aligned}$$

which by the Minkowski inequality and the estimates (1.1) for both P_N and $\tilde{P}_1$ implies

$$\begin{aligned}\|e^{it\Delta}f\|_{L_t^pL_x^q} &\lesssim \|f\|_{L_x^2} + (\sum_{N=2^{\mathbb{N}}}(N^s\|\tilde{\tilde{P}}_Nf\|_{L_x^2})^2)^{1/2}\\ &\lesssim \|f\|_{H_x^s}.\end{aligned}$$

The last inequality uses the almost L^2 orthogonality among the $\tilde{\tilde{P}}_N$'s. $\square$

We also record here the Bernstein type inequalities that will be useful in the sequel.

Proposition 8 (Corollary 2.2 in [**BGT04**]). *Let d be the dimension of M. Then for all $1 \le p \le r \le \infty$,*

$$\|P_Nf\|_{L^r(M)} \lesssim N^{d(\frac{1}{p}-\frac{1}{r})}\|f\|_{L^p(M)}. \tag{1.4}$$

Remark 9. *Note that the above proposition in particular implies that (1.2) holds for $N \lesssim 1$ or $p = \infty$.*

2. Reduction to a Finite Cover

Proposition 10. *Let $\pi : (\tilde{M}, \tilde{g}) \to (M, g)$ be a Riemannian covering map between compact Riemannian manifolds (then automatically with finite fibers). Let $\Delta_{\tilde{g}}$, Δ_g be the Laplace-Beltrami operators on $(\tilde{M}, \tilde{g})$ and (M, g) respectively and let $\tilde{\mu}$ and μ be the normalized volume form measures respectively, which define the L^p spaces. Let π^* be the pull back map. Define $C_\pi^\infty(\tilde{M}) := \pi^*(C^\infty(M))$, and similarly define $C_\pi(\tilde{M})$, $L_\pi^p(\tilde{M})$ and $H_\pi^s(\tilde{M})$. Then the following statement hold.*
(i) $\pi^* : C(M) \to C_\pi(\tilde{M})$ and $\pi^* : C^\infty(M) \to C_\pi^\infty(\tilde{M})$ are well-defined and linear isomorphisms.
(ii) For any $f \in C(M)$, we have $\int_M f \, d\mu = \int_{\tilde{M}} \pi^* f \, d\tilde{\mu}$. This implies $\pi^* : L^p(M) \to L_\pi^p(\tilde{M})$ is well-defined and an isometry.
(iii) $\Delta_{\tilde{g}}$ maps $C_\pi^\infty(\tilde{M})$ into $C_\pi^\infty(\tilde{M})$ and the diagram

$$\begin{array}{ccc} C^\infty(M) & \xrightarrow{\pi^*} & C_\pi^\infty(\tilde{M}) \\ {\scriptstyle\Delta_g}\downarrow & & \downarrow{\scriptstyle\Delta_{\tilde{g}}} \\ C^\infty(M) & \xrightarrow{\pi^*} & C_\pi^\infty(\tilde{M}) \end{array}$$

commutes.
(iv) $e^{it\Delta_{\tilde{g}}}$ maps $L_\pi^2(\tilde{M})$ into $L_\pi^2(\tilde{M})$ and is an isometry, and the diagrams

$$\begin{array}{ccc} L^2(M) & \xrightarrow{\pi^*} & L_\pi^2(\tilde{M}) \\ {\scriptstyle e^{it\Delta_g}}\downarrow & & \downarrow{\scriptstyle e^{it\Delta_{\tilde{g}}}} \\ L^2(M) & \xrightarrow{\pi^*} & L_\pi^2(\tilde{M}) \end{array} \qquad \begin{array}{ccc} L^2(M) & \xrightarrow{\pi^*} & L_\pi^2(\tilde{M}) \\ {\scriptstyle P_N}\downarrow & & \downarrow{\scriptstyle P_N} \\ L^2(M) & \xrightarrow{\pi^*} & L_\pi^2(\tilde{M}) \end{array} \tag{2.1}$$

commutes, where P_N stands for both $\varphi(N^{-2}\Delta_g)$ and $\varphi(N^{-2}\Delta_{\tilde{g}})$.
(v) $\pi^* : H^s(M) \to H_\pi^s(\tilde{M})$ is well-defined and an isometry.

Proof. (i)(ii)(iii) are direct consequences of the definition of a Riemannian covering map. For (iv), note that (i)(ii)(iii) together imply that the triples $(L^2(M), C^\infty(M), \Delta_g)$ and $(L_\pi^2(\tilde{M}), C_\pi^\infty(\tilde{M}), \Delta_{\tilde{g}})$ are isometric as essentially self-adjoint operators on Hilbert spaces, thus have isometric functional calculus. This implies (iv). Note that the $H^s(M)$ and $H_\pi^s(\tilde{M})$ norms are also defined in terms of the isometric functional calculus of $(L^2(M), C^\infty(M), \Delta_g)$ and $(L_\pi^2(\tilde{M}), C_\pi^\infty(\tilde{M}), \Delta_{\tilde{g}})$ respectively, which implies (v). $\square$

Combining Proposition 6 and 10 and Remark 9, the Main Conjecture 4 is reduced to the following.

Conjecture 11. *Let $\tilde{M}$ be a universal covering compact symmetric space as in Definition 1, equipped with a rational metric as in Definition 2. Then*

$$\|P_N e^{it\Delta} f\|_{L^p(I\times\tilde{M})} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}} \|f\|_{L^2(\tilde{M})} \tag{2.2}$$

holds for $p > 2 + \frac{4}{r}$ and $N \gtrsim 1$.

3. Littlewood-Paley Projections of the Product Type

Let (M, g) be the Riemannian product of the compact Riemannian manifolds (M_j, g_j), $j = 0, \cdots, m$. Any eigenfunction of the Laplace-Beltrami operator Δ on M with the eigenvalue $\lambda \leq 0$ is of the form $\prod_{j=0}^m \phi_{\lambda_j}$, where each ϕ_{λ_j} is an eigenfunction of Δ_j on M_i with eigenvalue $\lambda_j \leq 0$, $j = 0, \cdots, m$, such that $\lambda = \lambda_0 + \cdots + \lambda_n$.

Given any bump function φ on $\mathbb{R}$, there always exist bump functions φ_j's, $j = 0, \cdots, m$, such that for all $(x_0, \cdots, x_m) \in \mathbb{R}^{m+1}_{\leq 0}$ with $\varphi(x_0 + \cdots + x_m) \neq 0$, $\prod_{j=0}^m \varphi_j(x_j) = 1$. In particular,

$$\varphi \cdot \prod_{j=0}^m \varphi_j(x_j) = \varphi.$$

For $N \geq 1$, define

$$P_N := \varphi(N^{-2}\Delta),$$
$$\mathbf{P}_N := \varphi_0(N^{-2}\Delta_0) \otimes \cdots \otimes \varphi_m(N^{-2}\Delta_m),$$

as bounded operators on $L^2(M)$, where $\varphi_0(N^{-2}\Delta_0) \otimes \cdots \otimes \varphi_m(N^{-2}\Delta_m)$ is defined to map $\prod_{j=0}^m \phi_{\lambda_j}$ to $\prod_{j=0}^m \varphi_j(N^{-2}\lambda_j)\phi_{\lambda_j}$. We call $\mathbf{P}_N$ a *Littlewood-Paley projection of the product type.* We have

$$\mathbf{P}_N \circ P_N = P_N.$$

This implies that we can further reduce Conjecture 11 into the following.

Conjecture 12. *Let $M = \mathbb{T}^n \times U_1/K_1 \times \cdots \times U_m/K_m$ be a universal covering compact symmetric space equipped with a rational metric. Let $\Delta_0, \cdots, \Delta_m$ be respectively the Laplace-Beltrami operators on $\mathbb{T}^n, U_1/K_1, \cdots, U_m/K_m$. Let φ_j be any bump function for each $j = 0, \cdots, m$, $N \geq 1$, and let $\mathbf{P}_N = \otimes_{j=0}^m \varphi_j(N^{-2}\Delta_j)$. Then*

$$\|\mathbf{P}_N e^{it\Delta} f\|_{L^p(I\times M)} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}} \|f\|_{L^2(M)} \tag{3.1}$$

holds for $p > 2 + \frac{4}{r}$ and $N \gtrsim 1$.

On the other hand, similarly, for each Littlewood-Paley projection $\mathbf{P}_N$ of the product type, there exists a bump function φ such that $P_N = \varphi(N^{-2}\Delta)$ satisfies $P_N \circ \mathbf{P}_N = \mathbf{P}_N$. Noting that $\|\mathbf{P}_N f\|_{L^2} \lesssim \|f\|_{L^2}$, (1.4) then implies

$$\|\mathbf{P}_N f\|_{L^r(M)} \lesssim N^{d(\frac{1}{2}-\frac{1}{r})} \|f\|_{L^2(M)}. \tag{3.2}$$

for all $2 \leq r \leq \infty$.

CHAPTER 3

Harmonic Analysis on Compact Symmetric Spaces

In this chapter, we review harmonic analysis on compact symmetric spaces. Most of the material can be found in [**Hel84**], [**Hel01**], [**Hel08**], [**HS94**], [**Kna01**], [**Tak94**], [**Var84**].

1. Spherical Fourier Series

Let U/K be a symmetric space of compact type, equipped with the push forward measure of the normalized Haar measure du of U. Let (δ, V_δ) be an irreducible unitary representation of U and let V_δ^K be the space of vectors $v \in V_\delta$ fixed under $\delta(K)$. We say δ is *spherical* if $V_\delta^K \neq 0$. Let δ be such an irreducible spherical representation of U. Then V_δ^K is spanned by a single unit vector $\mathbf{e}$, and let

$$H_\delta(U/K) = \{\langle \delta(u)\mathbf{e}, v\rangle_{V_\delta} : v \in V_\delta\}. \tag{1.1}$$

Let $\widehat{U}_K$ be the set of equivalence classes of spherical representations of U with respect to K. The theory of Peter-Weyl gives the Hilbert space decomposition

$$L^2(U/K) = \bigoplus_{\delta \in \widehat{U}_K} H_\delta(U/K).$$

Define the *spherical functions*

$$\Phi_\delta(u) := \langle \delta(u)\mathbf{e}, \mathbf{e}\rangle_{V_\delta} \in H_\delta(U/K),$$

then the L^2 projections $P_\delta : L^2(U/K) \to H_\delta(U/K)$ can be realized by convolution with $d_\delta \Phi_\delta$, so we have the L^2 *spherical Fourier* series

$$f = \sum_{\delta \in \widehat{U}_K} d_\delta f * \Phi_\delta = \sum_{\delta \in \widehat{U}_K} d_\delta \Phi_\delta * f.$$

Here the convolution on U/K is defined by pulling back the functions to U and then applying the standard convolution on U.

Example 13. *Let M be a compact simply connected simple Lie group and continue the notations in Example 3. Then the set $\widehat{M}$ of irreducible unitary representations of M correspond to the set $\widehat{U}_K$ of irreducible spherical representations of U with respect to K, by*

$$\widehat{M} \ni \delta \mapsto \delta \otimes \delta^* \in \widehat{U}_K,$$

where δ^* is the contragradient representation associated to δ. Let χ_δ be the character of δ. We have

$$\Phi_{\delta\otimes\delta^*} = \frac{1}{d_\delta}\chi_\delta,$$
$$d_{\delta\otimes\delta^*} = d_\delta^2.$$

Note that convolution operations with respect to M and U/K do not necessarily match, but we always have $f * \Phi_{\delta\otimes\delta^*} = \frac{1}{d_\delta} f * \chi_\delta$, thus the spherical Fourier series reduces to the Fourier series

$$f = \sum_{\delta\in\widehat{M}} d_\delta f * \chi_\delta = \sum_{\delta\in\widehat{M}} d_\delta \chi_\delta * f.$$

More generally, let $M = \mathbb{T}^n \times U_1/K_1 \times \cdots \times U_m/K_m$ be a universal covering compact symmetric space where $\mathbb{T}^n = \mathbb{R}^n/2\pi\Gamma$. Let Λ be the dual lattice of Γ. Define the *Fourier dual* $\widehat{M}$ of M

$$\widehat{M} = \Lambda \times \widehat{U_1}_{K_1} \times \cdots \times \widehat{U_m}_{K_m}.$$

Let $\delta = (\lambda_0, \delta_1, \cdots, \delta_m) \in \widehat{M}$, $(H_0, x_1, \cdots, x_m) \in M$, and let

$$\Phi_\delta(H_0, x_1, \cdots, x_m) = e^{i\langle\lambda_0, H_0\rangle}\Phi_{\delta_1}\cdots\Phi_{\delta_m},$$
$$d_\delta = d_{\delta_1}\cdots d_{\delta_m}.$$

Then the spherical Fourier series reads

$$f = \sum_{\delta\in\widehat{M}} d_\delta \Phi_\delta * f = \sum_{\delta\in\widehat{M}} d_\delta f * \Phi_\delta,$$

where the convolution is defined component-wise. This gives the *Plancherel identity*

$$\|f\|^2_{L^2(M)} = \sum_{\delta\in\widehat{M}} d_\delta^2 \|\Phi_\delta * f\|_{L^2(M)}.$$

The Young's convolution inequalities hold on compact symmetric spaces

$$\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q},\ \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1,\ 1 \leq r, p, q \leq \infty.$$

This implies the Hausdorff-Young type inequality

$$\|f * \Phi_\delta\|_{L^2} \leq \|f\|_{L^1}\|\Phi_\delta\|_{L^2} = d_\delta^{-\frac{1}{2}}\|f\|_{L^1},\ \forall \delta \in \widehat{M}. \tag{1.2}$$

Let $g = \sum_{\delta\in\widehat{M}} c_\delta d_\delta \Phi_\delta$, then $f * g = \sum_{\delta\in\widehat{M}} c_\delta d_\delta f * \Phi_\delta$, which implies

$$\|f * g\|^2_{L^2} = \sum_{\delta\in\widehat{M}} |c_\delta|^2 d_\delta^2 \|f * \Phi_\delta\|^2, \tag{1.3}$$

$$\|f * g\|_{L^2} \leq (\sup_{\delta\in\widehat{M}} |c_\delta|) \cdot \|f\|_{L^2}. \tag{1.4}$$

2. Restricted Root Systems

Let U/K be a simply connected Riemannian globally symmetric space of compact type. Let G/K be the dual symmetric space of noncompact type, and $G^{\mathbb{C}}$ be the complexification of U and G, and $\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}, \mathfrak{u}, \mathfrak{k}$ be the Lie algebra of $G^{\mathbb{C}}, G, U, K$ respectively. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition and $\mathfrak{a}$ be the maximal abelian subspace of $\mathfrak{p}$. Then we have the *restricted root space decomposition*

$$\mathfrak{g} = \mathfrak{a} + \mathfrak{c} + \sum_{\lambda\in\Sigma} \mathfrak{g}_\lambda$$

where $\mathfrak{c} = \{X \in \mathfrak{k} : [X, H] = 0,\ \forall H \in \mathfrak{a}\}$, and Σ consists of nonzero real-valued linear functions λ on $\mathfrak{a}$ such that $\mathfrak{g}_\lambda := \{X \in \mathfrak{g} : [H, X] = \lambda(H)X, \forall H \in \mathfrak{a}\} \neq 0$. Let $\mathfrak{b}$ be the maximal abelian subspace of $\mathfrak{c}$, then $\mathfrak{h} = i\mathfrak{a} + \mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{u}$, and then the complexification $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{h}$ becomes a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. We also have the *root space decomposition*

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + \sum_{\alpha \in \Phi} \mathfrak{g}^{\mathbb{C}}_{\alpha}$$

where Φ consists of nonzero complex-valued linear functionals α on $\mathfrak{h}^{\mathbb{C}}$ such that $\mathfrak{g}^{\mathbb{C}}_{\alpha} := \{X \in \mathfrak{g}^{\mathbb{C}} : [H, X] = \alpha(H)X, \forall H \in \mathfrak{h}^{\mathbb{C}}\} \neq 0$. For $\alpha \in \Phi$, $\alpha|_{\mathfrak{a}}$ is either 0 or belongs to Σ. For each $\lambda \in \Sigma$, define the *multiplicity function* $m_\lambda := |\{\alpha \in \Phi : \alpha|_{\mathfrak{a}} = \lambda\}|$. $\mathfrak{g}^{\mathbb{C}}_{\alpha}$ is of one complex dimension for any $\alpha \in \Phi$ and $\mathfrak{g}_\lambda = \mathfrak{g} \cap (\sum_{\alpha|_{\mathfrak{a}} = \lambda} \mathfrak{g}^{\mathbb{C}}_{\alpha})$, which implies $\mathfrak{g}_\lambda$ is of real dimension equal to m_λ.

Let $\mathfrak{h}_{\mathbb{R}} = \mathfrak{a} + i\mathfrak{b}$. The Cartan-Killing form on $\mathfrak{g}^{\mathbb{C}}$ induces an inner product on $\mathfrak{a}^*$ and ${\mathfrak{h}_{\mathbb{R}}}^*$ respectively, under which both Σ and Φ become *root systems* respectively. We state the axiomatic description of a root system which will be needed in the sequel. A root system is a finite set Δ in a finite dimensional real inner product space $(V, \langle \ , \ \rangle)$ such that

$$\text{(2.1)} \qquad \begin{cases} \text{(i)} & \Delta = -\Delta; \\ \text{(ii)} & s_\alpha \Delta = \Delta \text{ for all } \alpha \in \Delta; \\ \text{(iii)} & 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \text{ for all } \alpha, \beta \in \Delta. \end{cases}$$

Here $s_\alpha : V \to V$ is the reflection

$$s_\alpha(x) := x - 2\frac{\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha, \ \forall x \in V.$$

If in addition it holds

$$\text{(2.2)} \qquad \text{(iv)} \ \ \alpha \in \Delta, k \in \mathbb{R}, k\alpha \in \Delta \Rightarrow k = \pm 1,$$

then we call it a *reduced* root system. Φ is reduced but not necessarily for Σ.

For $\alpha \in V$, let $\alpha^\perp := \{\beta \in V : \langle \alpha, \beta \rangle = 0\}$. Then the *Weyl chambers* are defined to be the connected components of $V \setminus \cup_{\alpha \in \Phi} \alpha^\perp$, and each $\alpha^\perp$ is called a *Weyl chamber wall*. The s_α's generate the *Weyl group* W, which acts simply transitively on the set of *Weyl chambers*, the set of *positive roots*, and the set of *simple roots* respectively. Note that the identification $V \cong V^*$ by the inner product $\langle \ , \ \rangle$ induces an isomorphic root system in $(V^*, \langle \ , \ \rangle)$, for which we have the isomorphic objects of Weyl chambers, Weyl group, positive roots, and simple roots.

Let Σ^+ denote a set of positive restricted roots in Σ. Then we have the Iwasawa decomposition

$$\text{(2.3)} \qquad \mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$$

where $\mathfrak{n} = \sum_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$. Let r and d be the rank and dimension of U/K respectively. Recall that the real dimension of $\mathfrak{g}_\lambda$ is m_λ for $\lambda \in \Sigma$, then the Iwasawa decomposition implies that

$$\text{(2.4)} \qquad \sum_{\lambda \in \Sigma^+} m_\lambda = d - r.$$

Let

$$\text{(2.5)} \qquad \Sigma_* := \{\alpha \in \Sigma : 2\alpha \notin \Sigma\}.$$

Then Σ_* is a reduced root system. Define the *weight lattice* Λ by

$$\text{(2.6)} \qquad \Lambda := \{\lambda \in \mathfrak{a}^* : \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}, \ \text{for all } \alpha \in \Sigma_*\}.$$

Let Γ be the *restricted root lattice* generated by the root system 2Σ. Then $\Gamma \subset \Lambda$. Let $\Sigma_*^+ = \Sigma^+ \cap \Sigma_*$ be the set of positive roots in Σ_*. Let

$$\Lambda^+ := \{\lambda \in \mathfrak{a}^* : \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}_{\geq 0}, \text{ for all } \alpha \in \Sigma_*^+\}$$

be the set of *dominant weights*. Given any irreducible spherical representation of $\delta \in \widehat{U}_K$, the highest weight of δ vanishes on $\mathfrak{b}$ and restricts on $\mathfrak{a}$ as an element in Λ^+. This gives the isomorphism

$$\Lambda^+ \cong \widehat{U}_K. \tag{2.7}$$

We can also express Λ, Λ^+ in terms of a basis. Let $\{\alpha_1, \cdots, \alpha_r\}$ be the set of simple roots in Σ_*^+. Let $\{w_1, \cdots, w_r\}$ be the dual basis to the *coroot* basis $\{\frac{\alpha_1}{\langle \alpha_1, \alpha_1 \rangle}, \cdots, \frac{\alpha_r}{\langle \alpha_r, \alpha_r \rangle}\}$. Then

$$\Lambda = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_r,$$
$$\Lambda^+ = \mathbb{Z}_{\geq 0}w_1 + \cdots + \mathbb{Z}_{\geq 0}w_r.$$

$w_1, \cdots, w_r$ are called the *fundamental weights*. Then

$$C = \mathbb{R}_{>0}w_1 + \cdots + \mathbb{R}_{>0}w_r$$

is the *fundamental Weyl chamber*, and we have the decomposition

$$\mathfrak{a}^* = (\bigsqcup_{s \in W} sC) \bigsqcup (\bigcup_{\alpha \in \Sigma} \{\lambda \in \mathfrak{a}^* : \langle \lambda, \alpha \rangle = 0\}), \tag{2.8}$$

where $\bigsqcup$ stands for disjoint union.

Consider the map $i\mathfrak{a} \to U/K,\ iH \mapsto \exp(iH)K$. Let A denote the image of the map, then

$$A \cong i\mathfrak{a}/\Gamma^\vee$$

where $\Gamma^\vee = \{iH \in i\mathfrak{a} : \exp(iH) \in K\}$ is a lattice of $i\mathfrak{a}$. We call A a *maximal torus* of U/K. It can be shown that

$$\Gamma^\vee = 2\pi i\mathbb{Z}\frac{H_{\alpha_1}}{\langle \alpha_1, \alpha_1 \rangle} + \cdots + 2\pi i\mathbb{Z}\frac{H_{\alpha_r}}{\langle \alpha_r, \alpha_r \rangle}.$$

Here $H_{\alpha_i} \in \mathfrak{a}$ corresponds to α_i under the the identification $\mathfrak{a} \xrightarrow{\sim} \mathfrak{a}^*$ by the Cartan-Killing form. This implies the isomorphism between Λ and the character group $\widehat{A}$ of A

$$\Lambda \xrightarrow{\sim} \widehat{A},\ \lambda \mapsto e^\lambda.$$

Note that the Weyl group W on $\mathfrak{a}$ naturally falls on A also. Define the *cells* in A to be the connected components of $A \setminus \cup_{\alpha \in \Sigma}\{[iH] \in A : \langle \alpha, H \rangle \in \pi\mathbb{Z}\}$, and each $\{[iH] \in A : \langle \alpha, H \rangle \in \pi n\}$ for $n \in \mathbb{Z}$ is called a *cell wall*. Let

$$Q = \bigcap_{\alpha \in \Sigma^+} \{[iH] \in A : \langle \alpha, H \rangle \in (0, \pi)\},$$

be such a cell (often called the *fundamental cell*), the closure of which is $\bar{Q} = \bigcap_{\alpha \in \Sigma^+}\{[iH] \in A : \langle \alpha, H \rangle \in [0, \pi]\}$. It can be shown that the Weyl group W acts simply transitively on the set of cells (see Theorem 9.2 and its Corollary of Chapter II in [**Tak94**]), and $W\bar{Q}$ covers A. Moreover, it can be shown that the K-orbits of A cover the whole space U/K, combined with the fact that the K-actions on A preserving A coincide with W, we then have that the values of any K-invariant function, for example any spherical function, are determined by its restriction on $\bar{Q}$.

Example 14. *Let $M = U/K$ be a simply connected compact symmetric space of rank 1. Then the restricted root system Σ is either $\{\pm\alpha\}$ or $\{\pm\frac{\alpha}{2}, \pm\alpha\}$. In both cases, the weight lattice $\Lambda = \mathbb{Z}\alpha$. Let $A = \mathbb{R}/2\pi\mathbb{Z}$ be the maximal torus, then $e^{n\alpha} = e^{in\theta}$, $\theta \in A$. The two cells of A are $(0, \pi)$ and $(\pi, 2\pi)$. Let m_α and $m_{\frac{\alpha}{2}}$ be respectively the multiplicity of α and $\frac{\alpha}{2}$ (if the restricted root system is $\{\pm\alpha\}$, then let $m_{\frac{\alpha}{2}} = 0$). Then for $n \in \mathbb{Z}_{\geq 0} \cong \mathbb{Z}_{\geq 0}\alpha \cong \Lambda^+$, the spherical function Φ_n restricted on A is (see Theorem 4.5 of Chapter V in* [**Hel84**]*)*

$$\Phi_n = \binom{n+a}{n}^{-1} P_n^{(a,b)}(\cos\theta),$$

where $\{P_n^{(a,b)} : n \in \mathbb{Z}_{\geq 0}\}$ is the set of Jacobi polynomials (see [**Sze75**]*) with parameters*

$$a = \frac{1}{2}(m_{\frac{\alpha}{2}} + m_\alpha - 1), \; b = \frac{1}{2}(m_\alpha - 1).$$

The cases when $m_{\frac{\alpha}{2}} = 0$ correspond to spheres of dimension $d = m_\alpha + 1$, $m_\alpha \in \mathbb{N}$. If d is odd, we have explicit formulas for the Jacobi polynomials and thus for the spherical functions. Let $\{\Phi_n^{(\lambda)}, n \in \mathbb{Z}_{\geq 0}\}$ denote the spherical functions on the $(2\lambda+1)$-dimensional sphere, $\lambda \in \mathbb{N}$, then (see Equation (4.7.3) and (8.4.13) in [**Sze75**]*)*

$$\Phi_n^{(\lambda)}(\theta) = 2\binom{n+2\lambda-1}{n}^{-1} \alpha_n \sum_{\nu=0}^{\lambda-1} \alpha_\nu \frac{(1-\lambda)\cdots(\nu-\lambda)}{(n+\lambda-1)\cdots(n+\lambda-\nu)} \cdot \frac{\cos((n-\nu+\lambda)\theta - (\nu+\lambda)\pi/2)}{(2\sin\theta)^{\nu+\lambda}} \tag{2.9}$$

where $\alpha_n := \binom{n+\lambda-1}{n}$.

Example 15. *Continue Example 3 and 13. Fix a Cartan subalgebra $i\mathfrak{a}$ of $\mathfrak{m}$. The root system Δ for $\mathfrak{m}^{\mathbb{C}}$ is reduced, and can be realized as a subset of $\mathfrak{a}^*$ by restriction on $\mathfrak{a}$. We say Δ is the root system associated to the compact Lie group M. Then the root system for $\mathfrak{u}^{\mathbb{C}} = \mathfrak{m}^{\mathbb{C}} \times \mathfrak{m}^{\mathbb{C}}$ can be realized as $\Delta \times \Delta$. Let $\alpha \in \Delta$. Identifying by 1.4*

$$i\mathfrak{p} \supset \{(iH, -iH) : H \in \mathfrak{a}\} \xrightarrow{\sim} i\mathfrak{a}, \; \frac{1}{2}(iH, -iH) \mapsto iH,$$

then

$$(\alpha, 0)|_{\mathfrak{a}} = (0, \alpha)|_{\mathfrak{a}} = \frac{1}{2}(\alpha, -\alpha),$$

thus the set of restricted roots is

$$\Sigma = \left\{\lambda_\alpha := \frac{1}{2}(\alpha, -\alpha) : \alpha \in \Delta\right\},$$

with $m_\lambda = 2$ for all $\lambda \in \Sigma$. Note that $2\lambda_\alpha(\frac{1}{2}(H, -H)) = \alpha(H)$ for all $H \in \mathfrak{a}$, and in this sense we identify 2Σ and Δ as isomorphic reduced root systems, with the identical Weyl group. The restricted root lattice coincides with the root lattice Γ generated by Δ. Note that by (2.4)*,*

$$|\Delta^+| = |\Sigma^+| = \frac{d-r}{2}. \tag{2.10}$$

The maximal torus corresponding to $i\mathfrak{a}$ *is* $A = \exp(i\mathfrak{a})$. *The character* χ_λ *and dimension* d_λ *associated to the irreducible representation with highest weight* $\lambda \in \Lambda^+$ *is given by Weyl's formulas*

$$\chi_\lambda|_A = \frac{\sum_{s\in W}(\det s)e^{s(\lambda+\rho)}}{\sum_{s\in W}(\det s)e^{s\rho}}, \tag{2.11}$$

$$d_\lambda = \frac{\prod_{\alpha\in\Delta^+}\langle\lambda+\rho,\alpha\rangle}{\prod_{\alpha\in\Delta^+}\langle\rho,\alpha\rangle}, \tag{2.12}$$

where

$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha = \sum_{i=1}^{r} w_i. \tag{2.13}$$

We also record here the Weyl integral formula that will be useful in the sequel. Let $f \in L^1(M)$ *be invariant under the adjoint action of* M. *Then*

$$\int_M f\ d\mu = \frac{1}{|W|}\int_A f(a)|D_P(a)|^2\ da, \tag{2.14}$$

where the Weyl denominator $D_P = \sum_{s\in W}(\det s)\ e^{s\rho}$, *and* $d\mu, da$ *are respectively the normalized Haar measures of* M *and* A.

Continue the discussion of a general simply connected symmetric space U/K of compact type. Recall that Φ denotes the root system associated to U. Apply (2.12) to any irreducible spherical representation $\lambda \in \Lambda^+ \cong \widehat{U}_K$, we have

$$d_\lambda = \frac{\prod_{\alpha\in\Phi^+,\alpha|_{\mathfrak{a}}\neq 0}\langle\lambda+\rho',\alpha\rangle}{\prod_{\alpha\in\Phi^+,\alpha|_{\mathfrak{a}}\neq 0}\langle\rho',\alpha\rangle}, \quad \text{for } \rho' = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha. \tag{2.15}$$

Example 16. *Let* $M = SU(2)$. $SU(2)$ *is of dimension 3 and rank 1. Let* $i\mathfrak{a} = i\mathbb{R}$ *be the Cartan subalgebra and* $A = \mathbb{R}/2\pi\mathbb{Z}$ *be the maximal torus. The root system is* $\{\pm\alpha\}$, *where* α *acts on* $i\mathfrak{a}$ *by* $\alpha(i\theta) = 2i\theta$. *The fundamental weight* $w = \frac{1}{2}\alpha$. *We normalize the Cartan-Killing form so that* $|w| = 1$. *The Weyl group* W *is of order 2, and acts on* $i\mathfrak{a}$ *as well as* $\mathfrak{a}^*$ *through multiplication by* ± 1. *For* $m \in \mathbb{Z}_{\geq 0} \cong \mathbb{Z}_{\geq 0}w = \Lambda^+$, *the dimension and character corresponding to* m *are given by*

$$d_m = m + 1, \tag{2.16}$$

$$\chi_m(\theta) = \frac{e^{i(m+1)\theta} - e^{-i(m+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin(m+1)\theta}{\sin\theta}, \quad \theta \in \mathbb{R}/2\pi\mathbb{Z}. \tag{2.17}$$

3. Functional Calculus of the Laplace-Beltrami Operator

Continue the discussion of the last section. The eigenvalues of the Laplace-Beltrami operator on U/K are computed as follows.

Lemma 17. *Let* $\lambda \in \Lambda^+ \cong \widehat{U}_K$ *and* $H_\lambda(U/K)$ *be the space of matrix coefficients associated to* λ *as in* (1.1). *For any* $f \in H_\lambda(U/K)$, *we have*

$$\Delta f = (-\langle\lambda+\rho,\lambda+\rho\rangle + \langle\rho,\rho\rangle)\cdot f, \tag{3.1}$$

where

$$\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha. \tag{3.2}$$

Proof. Let λ' be the extension of λ to $\mathfrak{h}_\mathbb{R} = \mathfrak{a} + i\mathfrak{b}$ by making it 0 on $i\mathfrak{b}$. Since $H_\lambda(U/K)$ consists of matrix coefficients of the irreducible representation of U with highest weight λ', by Lemma 1 in Section 6.6

in [**Pro07**], we have for $f \in H_\lambda(U/K)$,

$$\Delta f = (-\langle \lambda' + \rho', \lambda' + \rho' \rangle + \langle \rho', \rho' \rangle) \cdot f,$$

where $\rho' = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$. Noting that $\rho'|_{\mathfrak{a}} = \rho$, $\lambda'|_{\mathfrak{a}} = \lambda$, $\lambda'|_{i\mathfrak{b}} = 0$, and that $\mathfrak{a}$ and $i\mathfrak{b}$ are orthogonal with respect to $\langle\ ,\ \rangle$, we get (3.1). $\square$

Using the spherical Fourier series, we now have the functional calculus of Δ as follows. Let $f \in L^2(U/K)$ and consider the spherical Fourier series $f = \sum_{\lambda \in \Lambda^+} d_\lambda f * \Phi_\lambda$. Then for any bounded Borel function $F : \mathbb{R} \to \mathbb{C}$, we have

$$F(\Delta) f = \sum_{\lambda \in \Lambda^+} F(-|\lambda + \rho|^2 + |\rho|^2) d_\lambda f * \Phi_\lambda.$$

In particular, we have

$$e^{it\Delta} f = \sum_{\lambda \in \Lambda^+} e^{it(-|\lambda+\rho|^2 + |\rho|^2)} d_\lambda f * \Phi_\lambda, \tag{3.3}$$

$$P_N e^{it\Delta} f = \sum_{\lambda \in \Lambda^+} \varphi(\frac{-|\lambda + \rho|^2 + |\rho|^2}{N^2}) e^{it(-|\lambda+\rho|^2 + |\rho|^2)} d_\lambda f * \Phi_\lambda. \tag{3.4}$$

In particular, let

$$K_N(t, x) = \sum_{\lambda \in \Lambda^+} \varphi(\frac{-|\lambda + \rho|^2 + |\rho|^2}{N^2}) e^{it(-|\lambda+\rho|^2 + |\rho|^2)} d_\lambda \Phi_\lambda, \tag{3.5}$$

then we have

$$P_N e^{it\Delta} f = f * K_N(t, \cdot) = K_N(t, \cdot) * f. \tag{3.6}$$

We call $K_N(t, x)$ as the *Schrödinger kernel* on U/K. If the canonical Riemannian metric g is scaled to βg for some $\beta > 0$, then the eigenvalues of Δ are scaled by the factor of β^{-1}, and the *Schrödinger kernel* becomes

$$K_N = \sum_{\lambda \in \Lambda^+} \varphi(\frac{-|\lambda + \rho|^2 + |\rho|^2}{\beta N^2}) e^{it\beta^{-1}(-|\lambda+\rho|^2 + |\rho|^2)} d_\lambda \Phi_\lambda.$$

More generally, let $M = \mathbb{T}^n \times U_1/K_1 \times \cdots \times U_m/K_m$ be a universal covering compact symmetric space equipped with a rational metric g as in Definition 2. Let Λ_j be the weight lattice for U_j/K_j and identify $\widehat{U_j}_{K_j} \cong \Lambda_j^+$, $1 \leq j \leq m$. Let $\mathbf{P}_N = \otimes_{j=0}^m \varphi_j(N^{-2}\Delta_j)$ be a Littlewood-Paley projection of the product type as described in Section 3. Define the *Schrödinger kernel* $\mathbf{K}_N$ on M by

$$\mathbf{P}_N e^{it\Delta} f = f * \mathbf{K}_N(t, \cdot) = \mathbf{K}_N(t, \cdot) * f. \tag{3.7}$$

Then

$$\mathbf{K}_N = \prod_{j=0}^m K_{N,j}, \tag{3.8}$$

where the $K_{N,j}$'s are respectively the Schrödinger kernel on each component

$$K_{N,0} = \sum_{\lambda_0 \in \Lambda} \varphi_0(\frac{-|\lambda_0|^2}{\beta_0 N^2}) e^{-it\beta_0^{-1}|\lambda_0|^2} e^{i\langle \lambda_0, H_0 \rangle},$$

$$K_{N,j} = \sum_{\lambda_j \in \Lambda_j^+} \varphi_j(\frac{-|\lambda_j + \rho_j|^2 + |\rho_j|^2}{\beta_j N^2}) e^{it\beta_j^{-1}(-|\lambda_j+\rho_j|^2 + |\rho_j|^2)} d_{\lambda_j} \Phi_{\lambda_j},$$

$j = 1, \cdots, m$. Here the ρ_j's are defined in terms of (3.2). We also write

$$\mathbf{K}_N = \sum_{\lambda \in \widehat{M}} \varphi(\lambda, N) e^{-it\|\lambda\|^2} d_\lambda \Phi_\lambda,$$

where

$$\lambda = (\lambda_0, \ldots, \lambda_m) \in \widehat{M} = \Lambda \times \Lambda_1^+ \times \cdots \times \Lambda_m^+,$$

$$-\|\lambda\|^2 = -\beta_0^{-1}|\lambda_0|^2 + \sum_{j=1}^{m} \beta_j^{-1}(-|\lambda_j + \rho_j|^2 + |\rho_j|^2), \tag{3.9}$$

$$\varphi(\lambda, N) = \varphi_0(\frac{-|\lambda_0|^2}{\beta_0 N^2}) \cdot \prod_{j=1}^{n} \varphi_j(\frac{-|\lambda_j + \rho_j|^2 + |\rho_j|^2}{\beta_j N^2}), \tag{3.10}$$

$$d_\lambda = \prod_{j=1}^{m} d_{\lambda_j}, \ \Phi_\lambda = e^{i\langle \lambda_0, H_0 \rangle} \prod_{j=1}^{m} \Phi_{\lambda_j}.$$

Lemma 18. *Let d, r be respectively the dimension and rank of M.*
(i) $|\{\lambda \in \widehat{M} : \|\lambda\|^2 \lesssim N^2\}| \lesssim N^r$.
(ii) $d_\lambda \lesssim N^{d-r}$, uniformly for all $\|\lambda\|^2 \lesssim N^2$.

Proof. Note that $\lambda \in \widehat{M}$ lies in a lattice of dimension r, then (i) is a direct consequence of the definition of $\|\lambda\|^2$. For (ii), let d_j, r_j, Σ_j be respectively the dimension, rank, and the set of restricted roots of U_j/K_j, $j = 1, \cdots, m$. For $\lambda_j \in \Lambda_j^+$, (2.15) implies that d_{λ_j} is a polynomial in λ_j of degree equal to the number of positive restricted roots counting multiplicities, which is equal to $d_j - r_j$ by (2.4). Thus $d_\lambda = d_{\lambda_1} \cdots d_{\lambda_m}$ is a polynomial in λ of degree $\sum_{j=1}^{m}(d_j - r_j) = d - r$. In view of the definition of $\|\lambda\|^2$ again, we get (ii). □

Example 19. *Continue Example 3, 13 and 15. Let M be a compact simply connected simple Lie group equipped with a rational metric. Then the Schrödinger kernel reads*

$$K_N = \sum_{\lambda \in \widehat{M}} \varphi(\frac{-|\lambda + \rho|^2 + |\rho|^2}{N^2}) e^{it(-|\lambda+\rho|^2 + |\rho|^2)} d_\lambda \chi_\lambda. \tag{3.11}$$

Example 20. *Continue Example 14. Let M be the sphere of dimension $2\lambda + 1$, $\lambda \in \mathbb{N}$. Then $\rho = \frac{1}{2} m_\alpha \alpha = \lambda \alpha$. Normalize $|\alpha| = 1$. Then the Schrödinger kernel reads*

$$K_N(t, \theta) = \sum_{n \in \mathbb{Z}_{\geq 0}} \varphi(\frac{(n + \lambda)^2 - \lambda^2}{N^2}) e^{-it[(n+\lambda)^2 - \lambda^2]} d_n \Phi_n^{(\lambda)}(\theta). \tag{3.12}$$

For the three sphere $M = SU(2)$, the Schrödinger kernel reads

$$K_N(t, \theta) = \sum_{m \in \mathbb{Z}_{\geq 0}} \varphi(\frac{(m + 1)^2 - 1}{N^2}) e^{-it[(m+1)^2 - 1]} (m + 1) \frac{\sin(m + 1)\theta}{\sin \theta}. \tag{3.13}$$

CHAPTER 4

Conditional Strichartz Estimates

1. Strichartz Estimates as Fourier Restriction Phenomena

LEMMA 21. *Let Σ be a restricted root system equipped with the Cartan-Killing form $\langle \ , \ \rangle$. Let Σ_*, Λ be the associated reduced root system and weight lattice as defined in* (2.5) *and* (2.6) *respectively. Then there exists some $D \in \mathbb{N}$, such that $\langle \alpha, \beta \rangle \in D^{-1}\mathbb{Z}$ for all $\alpha, \beta \in \Lambda$.*

PROOF. Let $\{\alpha_1, \cdots, \alpha_r\}$ be a set of simple roots for Σ_*. Let $\{w_1, \cdots, w_r\}$ be the dual basis of the coroot basis $\{\frac{\alpha_1}{\langle \alpha_1, \alpha_1 \rangle}, \cdots, \frac{\alpha_r}{\langle \alpha_r, \alpha_r \rangle}\}$ so that $\Lambda = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_r$. Then it suffices to prove that $\langle w_i, w_j \rangle \in D^{-1}\mathbb{Z}$ for all $1 \leq i, j \leq r$, for some $D \in \mathbb{N}$, which then reduces to proving the rationality of $\langle w_i, w_j \rangle$, which further reduces to proving the rationality of $\langle \alpha, \beta \rangle$ for all $\alpha, \beta \in \Sigma$. Since Σ is a root system, $2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$ for all $\alpha, \beta \in \Sigma$, thus it suffices to prove the rationality of $\langle \alpha, \alpha \rangle$ for all $\alpha \in \Sigma$. Let α be a restricted root in Σ, and let $\alpha' \in \Delta$ be a root such that $\alpha'|_{\mathfrak{a}} = \alpha$. By Lemma 4.3.5 in [**Var84**], $\langle \alpha', \alpha' \rangle$ is rational. Then by Lemma 8.4 of Ch. VII in [**Hel01**], $\langle \alpha, \alpha \rangle$ is also rational. This finishes the proof. $\square$

Let $M = \mathbb{R}^n/2\pi\Gamma \times U_1/K_1 \times \cdots \times U_m/K_m$ be a universal covering compact symmetric space equipped with a rational metric g. By the previous lemma, there exists for each $j = 1, \cdots, m$ some $D_j \in \mathbb{N}$ such that $\langle \lambda, \mu \rangle \in 2D_j^{-1}\mathbb{Z}$ for all $\lambda, \mu \in \Lambda_j^+ \cong \widehat{U_j}_{K_j}$, which implies by (3.2) that $-|\lambda_j + \rho_j|^2 + |\rho_j|^2 = -|\lambda_j|^2 - \langle \lambda_j, 2\rho_j \rangle \in D_j^{-1}\mathbb{Z}$ for all $\lambda_j \in \Lambda_j$. Also recall that we require that there exists some $D \in \mathbb{N}$ such that $\langle u, v \rangle \in D^{-1}\mathbb{Z}$ for all $u, v \in \Gamma$. This implies that there also exists some $D_0 \in \mathbb{N}$ such that $\langle \lambda, \mu \rangle \in D_0^{-1}\mathbb{Z}$ for all λ, μ in the dual lattice Λ of Γ. By Definition 2 of a rational metric, there exists some $D_* > 0$ such that

$$\beta_0^{-1}, \cdots, \beta_m^{-1} \in D_*^{-1}\mathbb{N}.$$

Define

$$T = 2\pi D_* \cdot \prod_{j=0}^{m} D_j. \tag{1.1}$$

Then (3.9) implies that $T\|\lambda\|^2 \in 2\pi\mathbb{Z}$, which then implies that the Schrödinger kernel as in (3.8) is periodic in t with a period of T. Thus we may view the time variable t as living on the circle $\mathbb{T} = [0, T)$. Now the formal dual to the operator

$$\mathbf{T} : L^2(M) \to L^p(\mathbb{T} \times M), \quad f \mapsto \mathbf{P}_N e^{it\Delta} \tag{1.2}$$

is computed to be

$$\mathbf{T}^* : L^{p'}(\mathbb{T} \times M) \to L^2(M), \quad F \mapsto \int_{\mathbb{T}} \mathbf{P}_N e^{-is\Delta} F(s, \cdot) \ \frac{ds}{T}, \tag{1.3}$$

and thus

$$\mathbf{T}\mathbf{T}^* : L^{p'}(\mathbb{T} \times M) \to L^p(\mathbb{T} \times M), \quad F \mapsto \int_{\mathbb{T}} \mathbf{P}_N^2 e^{i(t-s)\Delta} F(s, \cdot) \ \frac{ds}{T} = \tilde{\mathbf{K}}_N \times F, \tag{1.4}$$

where

$$\tilde{\mathbf{K}}_N = \sum_{\lambda\in\widehat{M}} \varphi^2(\lambda, N)e^{-it\|\lambda\|^2} d_\lambda\Phi_\lambda = \mathbf{K}_N \times \mathbf{K}_N, \tag{1.5}$$

and the symbol $\times$ is understood as convolution on the space-time $\mathbb{T}\times M$.

The cutoff function $\varphi^2(\lambda, N)$ (see (3.10)) still defines a Littlewood-Paley projection $\mathbf{P}_N$ of the product type, and $\tilde{\mathbf{K}}_N$ is the Schrödinger kernel associated to $\mathbf{P}_N$. Now the argument of $\mathbf{TT}^*$ says that the boundedness of the operators (1.2), (1.3) and (1.4) are all equivalent, thus the Strichartz estimate in (2.2) is equivalent to the following *space-time Strichartz estimate*

$$\|\mathbf{K}_N \times F\|_{L^p(\mathbb{T}\times M)} \lesssim N^{d-\frac{2(d+2)}{p}} \|F\|_{L^{p'}(\mathbb{T}\times M)}, \tag{1.6}$$

which can be interpreted as Fourier restriction estimates on the product $\mathbb{T}\times M$.

We have the *space-time spherical Fourier series* as follows. For $F \in L^2(\mathbb{T}\times M)$, we have

$$F = \sum_{\substack{n\in\frac{2\pi}{T}\mathbb{Z},\\ \lambda\in\widehat{M}}} d_\lambda F \times [e^{itn}\Phi_\lambda].$$

Let $m = \sum_{n\in\frac{2\pi}{T}\mathbb{Z}} \widehat{m}(n)e^{itn}$, then

$$m\cdot\mathbf{K}_N = \sum_{\substack{n\in\frac{2\pi}{T}\mathbb{Z},\\ \lambda\in\widehat{M}}} \varphi(\lambda, N)\widehat{m}(n+\|\lambda\|^2)d_\lambda e^{itn}\Phi_\lambda. \tag{1.7}$$

2. Conjectured Dispersive Estimates and Their Consequences

One strategy to prove (1.6) is to first explore L^∞ estimate of K_N. Throughout this section, let $\mathbb{S}^1$ stand for the standard circle of unit length, and $\|\cdot\|$ stands for the distance from 0 on $\mathbb{S}^1$. Define

$$\mathcal{M}_{a,q} := \{t\in\mathbb{S}^1 : \|t-\frac{a}{q}\| < \frac{1}{qN}\}$$

where

$$a\in\mathbb{Z}_{\geq 0},\quad q\in\mathbb{N},\quad a<q,\quad (a,q)=1,\quad q<N.$$

We call such $\mathcal{M}_{a,q}$'s as *major arcs*, which are reminiscent of the Hardy-Littlewood circle method. In [**Bou93**], the author shows that for the Schrödinger kernel on the standard $\mathbb{T}^n$

$$\mathbf{K}_N(t,\mathbf{t}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} \varphi(\mathbf{k}, N)e^{-it|\mathbf{k}|^2+i\mathbf{k}\cdot\mathbf{t}},$$

it holds that for any $D\in\mathbb{N}$,

$$|\mathbf{K}_N(t,\mathbf{t})| \lesssim \frac{N^r}{(\sqrt{q}(1+N\|\frac{t}{2\pi D}-\frac{a}{q}\|^{1/2}))^r} \tag{2.1}$$

for $\frac{t}{2\pi D}\in\mathcal{M}_{a,q}$, uniformly in $\mathbf{t}\in\mathbb{T}^n$. Inspired by this, we conjecture a general dispersive estimate as follows.

Conjecture 22. *Let M be a universal covering compact symmetric space of rank r and dimension d, equipped with a rational metric. Let $\mathbf{K}_N$ be the Schrödinger kernel* (3.8) *and T be the period* (1.1). *Then*

$$|\mathbf{K}_N(t,x)| \lesssim \frac{N^d}{(\sqrt{q}(1+N\|\frac{t}{T}-\frac{a}{q}\|^{1/2}))^r} \tag{2.2}$$

for $\frac{t}{T}\in\mathcal{M}_{a,q}$, uniformly in $x\in M$.

Noting the product structure (3.8) of $\mathbf{K}_N$, the definition of the rank of the product space M, the definition (1.1) of T, the above conjecture reduces to the irreducible components of M.

CONJECTURE 23. *Let M be an irreducible simply connected symmetric space of compact type of rank r and dimension d, equipped with a rational metric. Let Λ be the weight lattice and Λ^+ the set of positive weights. Let D be a positive number such that $\langle\lambda,\mu\rangle \in D^{-1}\mathbb{Z}$ for all $\lambda,\mu\in\Lambda$. Let K_N be the Schrödinger kernel* (3.5)*. Then*

$$|K_N(t,x)| \lesssim \frac{N^d}{(\sqrt{q}(1+N\|\frac{t}{2\pi D}-\frac{a}{q}\|^{1/2}))^r} \tag{2.3}$$

for $\frac{t}{2\pi D}\in\mathcal{M}_{a,q}$, uniformly in $x\in M$.

We will prove the following special cases of this conjecture in the next chapter.

THEOREM 24. *(1) Conjecture 23 holds when M is a simply connected compact simple Lie group.*
(2) Conjecture 23 holds with an ε-loss when M is a sphere of odd dimension $d\geq 5$. That is, we need to add an N^ε multiplicative factor to the right side of (2.3)*.*
(3) Consider $\mathbb{T}^d=\mathbb{R}^d/2\pi\Gamma$ and let Λ be the dual lattice of Γ. Let $D>0$ such that $\langle\lambda,\mu\rangle\in D^{-1}\mathbb{Z}$ for all $\lambda,\mu\in\Lambda$. Then the Schrödinger kernel $K_N=\sum_{\lambda\in\Lambda}\varphi(\frac{-|\lambda|^2}{N^2})e^{-it|\lambda|^2}e^{i\langle\lambda,H\rangle}$ satisfies (2.3) *for $\frac{t}{2\pi D}\in\mathcal{M}_{a,q}$ uniformly in $H\in\mathbb{R}^d$.*

Now we show how Conjecture 22 implies Strichartz estimates (2.2) for $p\geq 2+\frac{8}{r}$. We prove the following theorem.

THEOREM 25. *Let M be a universal covering compact symmetric space of rank r and dimension d, equipped with a rational metric. Let T be the period of Schrödinger flow as defined in* (1.1)*. Let $f\in L^2(M)$, $\lambda>0$, and define*

$$m_\lambda=\mu\{(t,x)\in\mathbb{T}\times M:|\mathbf{P}_Ne^{it\Delta}f(x)|>\lambda|\}$$

where $\mu=dt\cdot d\mu_M$, $dt,d\mu_M$ being the canonical normalized measures on $\mathbb{T}=\mathbb{R}/T\mathbb{Z}$ and M respectively. Let

$$p_0=\frac{2(r+2)}{r}.$$

Assuming the truthfulness of Conjecture 22 *, the following statements hold true .*
Part I.

$$m_\lambda\lesssim_\varepsilon N^{\frac{dp_0}{2}-(d+2)+\varepsilon}\lambda^{-p_0}\|f\|_{L^2(M)}^{p_0},\quad \textit{for all } \lambda\gtrsim N^{\frac{d}{2}-\frac{r}{4}},\quad \varepsilon>0.$$

Part II.

$$m_\lambda\lesssim N^{\frac{dp}{2}-(d+2)}\lambda^{-p}\|f\|_{L^2(M)}^{p},\quad \textit{for all } \lambda\gtrsim N^{\frac{d}{2}-\frac{r}{4}},\quad p>p_0.$$

Part III.

$$\|\mathbf{P}_Ne^{it\Delta}f\|_{L^p(\mathbb{T}\times M)}\lesssim N^{\frac{d}{2}-\frac{d+2}{p}}\|f\|_{L^2(M)} \tag{2.4}$$

holds for all $p\geq 2+\frac{8}{r}$.
Part IV. *Assume it holds that*

$$\|\mathbf{P}_Ne^{it\Delta}f\|_{L^p(\mathbb{T}\times M)}\lesssim_\varepsilon N^{\frac{d}{2}-\frac{d+2}{p}+\varepsilon}\|f\|_{L^2(M)} \tag{2.5}$$

for some $p>p_0$, then (2.4) *holds for all $q>p$.*
Assuming the only truthfulness of Conjecture (22) **with ε-loss**, *then Part I holds, Part II and Part*

III hold with an ε-loss (i.e. adding an N^ε multiplicative factor to the right side of the inequalities), while Part IV fails.

Note that Theorem 24 implies Conjecture 22 (or its ε-loss version, respectively) for those spaces M described in Theorem 5. Then Case 1 and Case 2 of Theorem 5 follows by Part III (or its ε-loss version, respectively) of Theorem 25. For Case 3 of Theorem 5, note that we already have its ε-loss version from [**BD15**]. Then Part IV of Theorem 25 says that this ε-loss can in fact be removed.

We now follow closely the Stein-Tomas type argument in [**Bou93**] to prove Theorem 25. We generalize its argument for tori to the general setting of compact symmetric spaces. We will only write out the details of the proof for the case assuming the truthfulness of Conjecture 22, while the proof for the ε-loss version is entirely similar.

Let $\omega \in C_c^\infty(\mathbb{R})$ such that $\omega \geq 0$, $\omega(x) = 1$ for all $|x| \leq 1$ and $\omega(x) = 0$ for all $|x| \geq 2$. Let $N \in 2^{\mathbb{N}}$. Define

$$\begin{aligned}\omega_{\frac{1}{N^2}} &:= \omega(N^2\cdot),\\ \omega_{\frac{1}{NM}} &:= \omega(NM\cdot) - \omega(2NM\cdot),\end{aligned}$$

where

$$M < N, \quad M \in 2^{\mathbb{N}}.$$

Let

$$N_1 = \frac{N}{2^{10}}, \quad Q < N_1, \quad Q \in 2^{\mathbb{N}}.$$

Then

$$\sum_{Q \leq M \leq N} \omega_{\frac{1}{NM}} = 1, \quad \text{on} \quad \left[-\frac{1}{NQ}, \frac{1}{NQ}\right], \tag{2.6}$$

$$\sum_{Q \leq M \leq N} \omega_{\frac{1}{NM}} = 0, \quad \text{outside} \quad \left[-\frac{2}{NQ}, \frac{2}{NQ}\right]. \tag{2.7}$$

Write

$$1 = \sum_{1 \leq Q \leq N_1} \sum_{Q \leq M \leq N} \left[\left(\sum_{\substack{(a,q)=1,\\ Q \leq q < 2Q}} \delta_{a/q}\right) * \omega_{\frac{1}{NM}}\right](\frac{t}{T}) + \rho(t). \tag{2.8}$$

Note the major arc disjointness property

$$\left(\frac{a_1}{q_1} + \left[-\frac{2}{NQ_1}, \frac{2}{NQ_1}\right]\right) \bigcap \left(\frac{a_2}{q_2} + \left[-\frac{2}{NQ_2}, \frac{2}{NQ_2}\right]\right) = \varnothing$$

for $(a_i, q_i) = 1$, $Q_i \leq q_i < 2Q_i$, $i = 1, 2$, $Q_1 \leq Q_2 \leq N_1$. This in particular implies that

$$0 \leq \rho(t) \leq 1, \quad \text{for all } t \in [0, T), \tag{2.9}$$

$$\left[\left(\sum_{\substack{(a,q)=1,\\ Q \leq q < 2Q}} \delta_{a/q}\right) * \omega_{\frac{1}{NM}}(\frac{\cdot}{T})\right]^\wedge(0) = \frac{1}{T}\int_0^T \left(\sum_{\substack{(a,q)=1,\\ Q \leq q < 2Q}} \delta_{a/q}\right) * \omega_{\frac{1}{NM}}(\frac{t}{T})\, dt \leq \frac{2Q^2}{NM}, \tag{2.10}$$

which implies

$$1 \geq |\widehat{\rho}(0)| \geq 1 - \sum_{1\leq Q\leq N_1}\sum_{Q\leq M\leq N}\left|\left[\left(\sum_{\substack{(a,q)=1,\\ Q\leq q<2Q}}\delta_{a/q}\right)*\omega_{\frac{1}{NM}}(\frac{\cdot}{T})\right]^{\wedge}(0)\right| \geq 1-\frac{8N_1}{N}\geq\frac{1}{2}. \tag{2.11}$$

By Dirichlet's lemma on rational approximations, for any $\frac{t}{T}\in\mathbb{S}^1$, there exists a,q with $a\in\mathbb{Z}_{\geq 0}$, $q\in\mathbb{N}$, $(a,q)=1$, $q\leq N$, such that $|\frac{t}{T}-\frac{a}{q}|<\frac{1}{qN}$. If $\rho(\frac{t}{T})\neq 0$, then (2.6) implies that $q>N_1=\frac{N}{2^{10}}$. This implies by (2.3) and (2.9) that

$$\|\rho\mathbf{K}_N\|_{L^\infty(\mathbb{T}\times M)}\lesssim N^{d-\frac{r}{2}}. \tag{2.12}$$

Now define coefficients $\alpha_{Q,M}$ such that

$$\left[\left(\sum_{\substack{(a,q)=1,\\ Q\leq q<2Q}}\delta_{a/q}\right)*\omega_{\frac{1}{NM}}(\frac{\cdot}{T})\right]^{\wedge}(0)=\alpha_{Q,M}\widehat{\rho}(0), \tag{2.13}$$

then (2.10) and (2.11) imply that

$$\alpha_{Q,M}\lesssim\frac{Q^2}{NM}. \tag{2.14}$$

Write

$$\begin{aligned}\mathbf{K}_N(t,x)=&\sum_{Q\leq N_1}\sum_{Q\leq M\leq N}\mathbf{K}_N(t,x)\left[\left(\left(\sum_{(a,q)=1,Q\leq q<2Q}\delta_{a/q}\right)*\omega_{\frac{1}{NM}}(\frac{\cdot}{T})\right)-\alpha_{Q,M}\rho\right](t)\\&+\left(1+\sum_{Q,M}\alpha_{Q,M}\right)\mathbf{K}_N(t,x)\rho(t),\end{aligned}$$

and define

$$\Lambda_{Q,M}(t,x):=\mathbf{K}_N(t,x)\left[\left(\left(\sum_{(a,q)=1,Q\leq q<2Q}\delta_{a/q}\right)*\omega_{\frac{1}{NM}}(\frac{\cdot}{T})\right)-\alpha_{Q,M}\rho\right](t). \tag{2.15}$$

Then from (2.3), (2.12), (2.14), we have

$$\|\Lambda_{Q,M}\|_{L^\infty(\mathbb{T}\times M)}\lesssim N^{d-\frac{r}{2}}(\frac{M}{Q})^{r/2}. \tag{2.16}$$

Next, we estimate $\widehat{\Lambda}_{Q,M}$. From (1.7), we have

$$\Lambda_{Q,M}=\sum_{\substack{n\in\frac{2\pi}{T}\mathbb{Z},\\ \lambda\in\widehat{M}}}\lambda_{Q,M}(n,\lambda)d_\lambda e^{itn}\Phi_\lambda. \tag{2.17}$$

where

$$\lambda_{Q,M}(n,\lambda)=\varphi(\lambda,N)\left[\left(\sum_{(a,q)=1,Q\leq q<2Q}\delta_{a/q}\right)^{\wedge}\cdot\widehat{\omega}_{\frac{1}{NM}}(T\cdot)-\alpha_{Q,M}\widehat{\rho}\right](n+\|\lambda\|^2). \tag{2.18}$$

Note that (2.13) immediately implies

$$\lambda_{Q,M}(n,\lambda)=0,\quad\text{for } n+\|\lambda\|^2=0. \tag{2.19}$$

Let $d(m,Q)$ denote the number of divisors of m less than Q, using Lemma 3.33 in [**Bou93**],

$$|(\sum_{(a,q)=1,Q\le q<2Q}\delta_{a/q})^\wedge(Tn)|\lesssim_\varepsilon d(\frac{Tn}{2\pi},Q)Q^{1+\varepsilon},\quad n\ne 0,\ \varepsilon>0, \tag{2.20}$$

we get

$$|\lambda_{Q,M}(n,\lambda)|\lesssim_\varepsilon \varphi(\lambda,N)\frac{Q^{1+\varepsilon}}{NM}d(\frac{T(n+k_\lambda)}{2\pi},Q)+\frac{Q^2}{NM}|\widehat{\rho}(n+\|\lambda\|^2)|. \tag{2.21}$$

Using the divisor bound

$$d(m,Q)\lesssim_\varepsilon m^\varepsilon,$$

and (2.20), (2.8), we have

$$|\widehat{\rho}(n)|\le\sum_{1\le Q\le N_1}\sum_{Q\le M\le N}\frac{d(\frac{Tn}{2\pi},Q)Q^{1+\varepsilon}}{NM}\lesssim\frac{N^\varepsilon}{N},\quad\text{for } n\ne 0, |n|\lesssim N^2, \tag{2.22}$$

thus

$$\begin{aligned}|\lambda_{Q,M}(n,\lambda)|&\lesssim_\varepsilon\varphi(\lambda,N)\frac{Q}{NM}\left[Q^\varepsilon d(\frac{T(n+\|\lambda\|^2)}{2\pi},Q)+\frac{Q}{N^{1-\varepsilon}}\right]\\&\lesssim_\varepsilon\varphi(\lambda,N)\frac{QN^\varepsilon}{NM},\quad\text{for } |n|\lesssim N^2.\end{aligned} \tag{2.23}$$

Proposition 26. *(i) Assume that $f\in L^1(\mathbb{T}\times M)$. Then*

$$\|f\times\Lambda_{Q,M}\|_{L^\infty(\mathbb{T}\times M)}\lesssim N^{d-\frac{r}{2}}(\frac{M}{Q})^{r/2}\|f\|_{L^1(\mathbb{T}\times M)}. \tag{2.24}$$

(ii) Assume that $f\in L^2(\mathbb{T}\times M)$. Assume also

$$f\times[e^{itn}\Phi_\lambda]=0,\quad \textit{for all } n\in\frac{2\pi}{T}\mathbb{Z}\ \textit{such that } |n|\gtrsim N^2. \tag{2.25}$$

Then

$$\|f\times\Lambda_{Q,M}\|_{L^2(\mathbb{T}\times M)}\lesssim_\varepsilon\frac{QN^\varepsilon}{NM}\|f\|_{L^2(\mathbb{T}\times M)}, \tag{2.26}$$

and

$$\|f\times\Lambda_{Q,M}\|_{L^2(\mathbb{T}\times M)}\lesssim_{\tau,B}\frac{Q^{1+2\tau}L}{NM}\|f\|_{L^2(\mathbb{T}\times M)}+M^{-1}L^{-B/2}N^{d/2}\|f\|_{L^1(\mathbb{T}\times M)}. \tag{2.27}$$

for all

$$L>1,\quad 0<\tau<1,\quad B>\frac{6}{\tau},\quad N>(LQ)^B. \tag{2.28}$$

Proof. Using (2.16), we have

$$\|f\times\Lambda_{Q,M}\|_{L^\infty(\mathbb{T}\times M)}\le\|f\|_{L^1(\mathbb{T}\times M)}\|\Lambda_{Q,M}\|_{L^\infty(\mathbb{T}\times M)}\lesssim N^{d-\frac{r}{2}}(\frac{M}{Q})^{r/2}\|f\|_{L^1(\mathbb{T}\times M)}.$$

This proves (i). (2.26) is a consequence of (1.4), (2.17), and (2.23). To prove (2.27), we use (1.3) and (2.17) to get

$$\|f\times\Lambda_{Q,M}\|_{L^2(\mathbb{T}\times M)}=\left(\sum_{n,\lambda}d_\lambda^2\|f\times[e^{itn}\Phi_\lambda]\|^2_{L^2(\mathbb{T}\times M)}\cdot|\lambda_{Q,M}(n,\lambda)|^2\right)^{1/2},$$

which combined with (2.19), (2.21), and (2.22) yields

$$\|f\times\Lambda_{Q,M}\|_{L^2(\mathbb{T}\times M)}\lesssim_\varepsilon\frac{Q^{1+\varepsilon}}{NM}\left(\sum_{n,\lambda}\varphi(\lambda,N)^2d_\lambda^2\|f\times[e^{itn}\Phi_\lambda]\|^2_{L^2(\mathbb{T}\times M)}d(\frac{T(n+\|\lambda\|^2)}{2\pi},Q)^2\right)^{1/2} + \frac{Q^2}{MN^{2-\varepsilon}}\|f\|_{L^2(\mathbb{T}\times M)}.$$

Using Lemma 3.47 in [**Bou93**], we have

$$\begin{aligned}&\left|\{(n,\lambda):|n|,\|\lambda\|^2\lesssim N^2,d(\frac{T(n+\|\lambda\|^2)}{2\pi},Q)>D\}\right|\\ &\lesssim_{\tau,B}(D^{-B}Q^\tau N^2+Q^B)\cdot\max_{|m|\lesssim N^2}|\{(n,\lambda):n+\|\lambda\|^2=m\}|\\ &\lesssim_{\tau,B}(D^{-B}Q^\tau N^2+Q^B)\cdot|\{\lambda\in\widehat{M}:\|\lambda\|^2\lesssim N^2\}|\\ &\lesssim_{\tau,B}(D^{-B}Q^\tau N^2+Q^B)\cdot N^r.\end{aligned}\tag{2.29}$$

Here we used (i) of Lemma 18.

Now (1.2) gives

$$\|f\times[e^{itn}\Phi_\lambda]\|_{L^2(\mathbb{T}\times M)}\le d_\lambda^{-\frac12}\|f\|_{L^1(\mathbb{T}\times M)},$$

which together with (2.29), $d(\cdot,Q)\le Q$, and (ii) of Lemma 18 implies

$$\|f\times\Lambda_{Q,M}\|_{L^2(\mathbb{T}\times M)}\lesssim_{\tau,B}(\frac{Q^{1+\varepsilon}D}{NM}+\frac{Q^2}{MN^{2-\varepsilon}})\|f\|_{L^2(\mathbb{T}\times M)} + \frac{Q^{1+\varepsilon}}{NM}\cdot Q\cdot(D^{-B/2}Q^\tau N+Q^{B/2})N^{d/2}\|f\|_{L^1(\mathbb{T}\times M)}.$$

This implies (2.27) assuming the conditions in (2.28). □

Now interpolating (2.24) and (2.26), we get

$$\|f\times\Lambda_{Q,M}\|_{L^p(\mathbb{T}\times M)}\lesssim_\varepsilon N^{d-\frac r2-\frac{2d-r+2}{p}+\varepsilon}M^{\frac r2-\frac{r+2}{p}}Q^{-\frac r2+\frac{r+2}{p}}\|f\|_{L^{p'}(\mathbb{T}\times M)}.\tag{2.30}$$

Interpolating (2.24) and (2.27) (see Lemma A.1 in the appendix) for

$$p>\frac{2(r+2)}{r}+10\tau,\quad\text{(which implies }\sigma:=\frac r2-\frac{r+2+4\tau}{p}>0)\tag{2.31}$$

we get

$$\|f\times\Lambda_{Q,M}\|_{L^p(\mathbb{T}\times M)}\lesssim_{\tau,B}N^{d-\frac r2-\frac{2d-r+2}{p}}M^{\frac r2-\frac{r+2}{p}}Q^{-\sigma}L^{\frac2p}\|f\|_{L^{p'}(\mathbb{T}\times M)} + Q^{-\frac2r(1-\frac2p)}M^{\frac r2-\frac{r+2}{p}}L^{-\frac Bp}N^{d-\frac r2-\frac{d-r}{p}}\|f\|_{L^1(\mathbb{T}\times M)}.\tag{2.32}$$

Now we are ready to prove Theorem 25.

Proof of Theorem 25. Without loss of generality, we assume that $\|f\|_{L^2(M)}=1$. Then for $F=\mathbf{P}_Ne^{it\Delta}f$, (3.2) implies that

$$\|F\|_{L^2_x}\lesssim 1,\tag{2.33}$$

$$\|F\|_{L^\infty_x}\lesssim N^{\frac d2}.\tag{2.34}$$

For $\lambda > 0$, let

$$H = \chi_{|F|>\lambda} \cdot \frac{F}{|F|}. \tag{2.35}$$

Let $\tilde{\tilde{\mathbf{P}}}_N$ be a Littlewood-Paley projection of the product type such that $\tilde{\tilde{\mathbf{P}}}_N \circ \mathbf{P}_N = \mathbf{P}_N$. Let $\tilde{\tilde{\mathbf{K}}}_N$ be the Schrödinger kernel associated to $\tilde{\tilde{\mathbf{P}}}_N e^{it\Delta}$. Then

$$F \times \tilde{\tilde{\mathbf{K}}}_N = F.$$

Let P_{N^2} be the self-adjoint Littlewood-Paley projection operator on $L^2(\mathbb{T} \times M)$ defined by

$$P_{N^2}H := \sum_{n,\lambda} \varphi(\frac{-\|\lambda\|^2 - n^2}{N^4}) d_\lambda H \times [e^{itn}\Phi_\lambda]$$

for some bump function φ, such that $P_{N^2} \circ \mathbf{P}_N = \mathbf{P}_N$. Then $F = P_{N^2}F$ so that

$$\langle F, H\rangle_{L^2_{t,x}} = \langle P_{N^2}F, H\rangle_{L^2_{t,x}} = \langle F, P_{N^2}H\rangle_{L^2_{t,x}}.$$

Then we can write

$$\lambda m_\lambda \leq \langle F, H\rangle_{L^2_{t,x}} = \langle F \times \tilde{\tilde{\mathbf{K}}}_N, P_{N^2}H\rangle_{L^2_{t,x}}.$$

Noting that convolution with $\tilde{\tilde{\mathbf{K}}}_N$ is also a self-adjoint operator on $L^2(\mathbb{T} \times M)$, then we have

$$\begin{aligned}\lambda m_\lambda &\leq \langle F, P_{N^2}H \times \tilde{\tilde{\mathbf{K}}}_N\rangle_{L^2_{t,x}} \leq \|F\|_{L^2_{t,x}} \|P_{N^2}H \times \tilde{\tilde{\mathbf{K}}}_N\|_{L^2_{t,x}} \\ &\lesssim \|P_{N^2}H \times \tilde{\tilde{\mathbf{K}}}_N\|_{L^2_{t,x}} = \langle P_{N^2}H \times \tilde{\tilde{\mathbf{K}}}_N, P_{N^2}H \times \tilde{\tilde{\mathbf{K}}}_N\rangle_{L^2_{t,x}} = \langle P_{N^2}H, P_{N^2}H \times (\tilde{\tilde{\mathbf{K}}}_N \times \tilde{\tilde{\mathbf{K}}}_N)\rangle_{L^2_{t,x}}.\end{aligned} \tag{2.36}$$

Let

$$H' = P_{N^2}H, \quad \tilde{\mathbf{K}}_N = \tilde{\tilde{\mathbf{K}}}_N \times \tilde{\tilde{\mathbf{K}}}_N.$$

Note that H' by definition satisfies the assumption in (2.25) and we can apply Proposition 26. Also note that $\tilde{\mathbf{K}}_N$ is still a Schrödinger kernel associated to a Littlewood-Paley projection operator of the product type (see (1.5)). Finally note that the Bernstein type inequalities (1.4) and the definition (2.35) of H give

$$\|H'\|_{L^p_{t,x}} \lesssim \|H\|_{L^p_{t,x}} \lesssim m_\lambda^{\frac{1}{p}}. \tag{2.37}$$

Write

$$\Lambda = \sum_{1\leq Q\leq N_1} \sum_{Q\leq M\leq N} \Lambda_{Q,M}, \quad \tilde{\mathbf{K}}_N = \Lambda + (\tilde{\mathbf{K}}_N - \Lambda),$$

where $\Lambda_{Q,M}$ is defined as in (2.15) except that $\mathbf{K}_N$ is replaced by $\tilde{\mathbf{K}}_N$. We have by (2.36)

$$\begin{aligned}\lambda^2 m_\lambda^2 &\lesssim \langle H', H' \times \Lambda\rangle_{L^2_{t,x}} + \langle H', H' \times (\tilde{\mathbf{K}}_N - \Lambda)\rangle_{L^2_{t,x}} \\ &\lesssim \|H'\|_{L^{p'}_{t,x}} \|H' \times \Lambda\|_{L^p_{t,x}} + \|H'\|^2_{L^1_{t,x}} \|\tilde{\mathbf{K}}_N - \Lambda\|_{L^\infty_{t,x}}.\end{aligned} \tag{2.38}$$

Using (2.30) for $p = p_0 := \frac{2(r+2)}{r}$, then summing over Q, M, and noting (2.37), we have

$$\|H'\|_{L^{p'}_{t,x}} \|H' \times \Lambda\|_{L^p_{t,x}} \lesssim N^{d-\frac{2d+4}{p_0}+\varepsilon} \|H'\|^2_{L^{p_0'}_{t,x}} \lesssim N^{d-\frac{2d+4}{p_0}+\varepsilon} m_\lambda^{\frac{2}{p_0'}}.$$

From (2.12) and (2.14) we get

$$\|\tilde{\mathbf{K}}_N - \Lambda\|_{L^\infty_{t,x}} \lesssim N^{d-\frac{r}{2}}, \tag{2.39}$$

which implies

$$\|H'\|^2_{L^1_{t,x}} \|\tilde{\mathbf{K}}_N - \Lambda\|_{L^\infty_{t,x}} \lesssim N^{d-\frac{r}{2}} \|H'\|^2_{L^1_{t,x}} \lesssim N^{d-\frac{r}{2}} m_\lambda^2. \tag{2.40}$$

Then we have

$$\lambda^2 m_\lambda^2 \lesssim N^{d-\frac{2d+4}{p_0}+\varepsilon} m_\lambda^{\frac{2}{p_0'}} + N^{d-\frac{r}{2}} m_\lambda^2,$$

which implies for $\lambda \gtrsim N^{\frac{d}{2}-\frac{r}{4}}$

$$m_\lambda \lesssim_\varepsilon N^{p_0(\frac{d}{2}-\frac{d+2}{p_0})+\varepsilon} \lambda^{-p_0}.$$

Thus **Part I** is proved. To prove **Part II** for some fixed p, using **Part I** and (2.34), it suffices to prove it for $\lambda \gtrsim N^{\frac{d}{2}-\varepsilon}$. Summing (2.32) over Q, M in the range indicated by (2.28), we get

$$\|H' \times \Lambda_1\|_{L^p_{t,x}} \lesssim L N^{d-\frac{2d+4}{p}} \|H'\|_{L^{p'}_{t,x}} + L^{-B/p} N^{d-\frac{d+2}{p}} \|H'\|_{L^1_{t,x}}, \tag{2.41}$$

where

$$\Lambda_1 := \sum_{Q<Q_1, Q \le M \le N} \Lambda_{Q,M}$$

and Q_1 is the largest Q-value satisfying (2.28). For values $Q \ge Q_1$, use (2.30) to get

$$\|H' \times (\Lambda - \Lambda_1)\|_{L^p_{t,x}} \lesssim_\varepsilon N^{d-\frac{2d+4}{p}+\varepsilon} Q_1^{-(\frac{r}{2}-\frac{r+2}{p})} \|H'\|_{L^{p'}_{t,x}}. \tag{2.42}$$

Using (2.38), (2.40), (2.41) and (2.42), we get

$$\lambda^2 m_\lambda^2 \lesssim N^{d-\frac{2(d+2)}{p}} (L + \frac{N^\varepsilon}{Q_1^{\frac{r}{2}-\frac{r+2}{p}}}) m_\lambda^{2/p'} + L^{-B/p} N^{d-\frac{d+2}{p}} m_\lambda^{1+\frac{1}{p'}} + N^{d-\frac{r}{2}} m_\lambda^2.$$

For $\lambda \gtrsim N^{\frac{d}{2}-\frac{r}{4}}$, the last term of the above inequality can be dropped. Let $Q_1 = N^\delta$ such that $\delta > 0$ and

$$(LN^\delta)^B < N \tag{2.43}$$

such that (2.28) holds. Note that

$$L > 1 > \frac{N^\varepsilon}{Q_1^{\frac{r}{2}-\frac{r+2}{p}}}$$

for $p > p_0 + 10\tau$ and ε sufficiently small, thus

$$\lambda^2 m_\lambda^2 \lesssim N^{d-\frac{2(d+2)}{p}} L m_\lambda^{2/p'} + L^{-B/p} N^{d-\frac{d+2}{p}} m_\lambda^{1+\frac{1}{p'}}.$$

This implies

$$\begin{aligned} m_\lambda &\lesssim N^{p(\frac{d}{2}-\frac{d+2}{p})} L^{\frac{p}{2}} \lambda^{-p} + N^{p(d-\frac{d+2}{p})} L^{-B} \lambda^{-2p} \\ &\lesssim N^{-d-2} (\frac{N^{d/2}}{\lambda})^p L^{\frac{p}{2}} + N^{-d-2} (\frac{N^{d/2}}{\lambda})^{2p} L^{-B}. \end{aligned}$$

Let

$$L = (\frac{N^{d/2}}{\lambda})^\tau, \quad B > \frac{p}{\tau}$$

and δ sufficiently small so that (2.43) holds, then

$$m_\lambda \lesssim N^{-d-2} (\frac{N^{d/2}}{\lambda})^{p+\frac{p\tau}{2}}.$$

Note that conditions for p, τ indicated in (2.31) implies that $p + \frac{p\tau}{2}$ can take any exponent $> p_0 = \frac{2(r+2)}{r}$. This completes the proof of **Part II**.

The proof of **Part III** and **Part IV** is almost identical to the proof of Proposition 3.110 and 3.113 respectively in [**Bou93**]. The proof of **Part III** is an interpolation between the result of **Part II** with the trivial subcritical Strichartz estimates $\|\mathbf{P}_N e^{it\Delta} f\|_{L^2_{t,x}} \lesssim \|f\|_{L^2_x}$. The proof of **Part IV** is similarly an interpolation between the result of **Part II** with the assumption (2.5). We omit the details. $\square$

CHAPTER 5

Dispersive Estimates – General Theory

In this chapter, we start to prove Theorem 24. First note that the Schrödinger kernel $K_N(t,\cdot)$ in (3.5) as a function on $M = U/K$ is a linear combination of spherical functions which are K-invariant, whence $K_N(t,\cdot)$ is also K-invariant, thus the values of $K_N(t,\cdot)$ are determined by its restriction on any maximal torus (more precisely, on the closure of any cell in a maximal torus, see Section 2). Thus it suffices to prove (2.3) uniformly on a fixed maximal torus. By Proposition 9.4 of Ch. III in [**Hel08**], the spherical function Φ_λ for $\lambda \in \Lambda^+$ on a maximal torus equals

$$\Phi_\lambda = \sum_{i=1}^{q} c_i e^{\lambda_i},\ \ \lambda_i \in \Lambda, c_i \geq 0.$$

This puts the Schrödinger kernel (3.5) in the perfect form of an exponential sum. To be able to estimate the size of such an exponential sum, we need to decompose and assemble the terms rightly in order to exploit the oscillation in them.

1. Weyl Type Sums on Rational Lattices

Definition 27. *Let $L = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_r$ be a lattice on an inner product space $(V, \langle\ ,\ \rangle)$. We say L is a rational lattice provided that there exists some $D > 0$ such that $\langle w_i, w_j\rangle \in D^{-1}\mathbb{Z}$. We call the number D a period of L.*

By Lemma 21, the weight lattice Λ of U/K is a rational lattice with respect to the Cartan-Killing form. As a sublattice of Λ, the restricted root lattice Γ is also rational.

Let f be a function on $\mathbb{Z}^r$ and define the *difference operator* D_i's by

$$D_i f(n_1,\cdots,n_r) := f(n_1,\cdots,n_{i-1},n_i+1,n_{i+1},\cdots,n_r) - f(n_1,\cdots,n_r) \tag{1.1}$$

for $i = 1,\cdots,r$. The Leibniz rule for D_i reads

$$D_i(\prod_{j=1}^{n} f_j) = \sum_{l=1}^{n} \sum_{1\leq k_1<\cdots<k_l\leq n} D_i f_{k_1}\cdots D_i f_{k_l} \cdot \prod_{\substack{j\neq k_1,\cdots,k_l\\ 1\leq j\leq n}} f_j. \tag{1.2}$$

Note that there are $2^n - 1$ terms in the right side of the above formula.

Definition 28. *Let $L \cong \mathbb{Z}^r$ be a lattice of rank r. Given $A \in \mathbb{R}$, we say a function f on L is a pseudo-polynomial of degree A provided for each $n \in \mathbb{Z}_{\geq 0}$,*

$$|D_{i_1}\cdots D_{i_n} f(n_1,\cdots,n_r)| \lesssim N^{A-n} \tag{1.3}$$

holds uniformly in $|n_i| \lesssim N$, $i = 1,\cdots,r$, for all $i_j = 1,\cdots,r$, $j = 1,\cdots,n$, and $N \geq 1$.

A direct application of the Leibniz rule (1.2) gives the following lemma.

Lemma 29. *Let L be a lattice and f, g two functions on L. Assume f, g are pseudo-polynomials of degree A, B respectively. Then $f \cdot g$ is a pseudo-polynomial of degree $A + B$.*

Now we have the following estimate on Weyl type sums, which generalizes the classical Weyl type inequality in one dimension (as in Lemma 3.18 of [**Bou93**]).

LEMMA 30. *Let $L = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_r$ be a rational lattice in the inner product space $(V, \langle\ ,\ \rangle)$ with a period D. Let φ be a bump function on $\mathbb{R}$ and $N \geq 1$, $A \in \mathbb{R}$. Suppose f an N-pseudo-polynomial of degree A. Let*

$$F(t,H) = \sum_{\lambda \in L} e^{-it|\lambda|^2 + i\langle \lambda, H\rangle} \varphi(\frac{|\lambda|^2}{N^2}) \cdot f \tag{1.4}$$

for $t \in \mathbb{R}$ and $H \in V$. Then for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$, we have

$$|F(t,H)| \lesssim \frac{N^{A+r}}{(\sqrt{q}(1 + N\|\frac{t}{2\pi D} - \frac{a}{q}\|^{1/2}))^r} \tag{1.5}$$

uniformly in $H \in V$.

Note that part (3) of Theorem 24 is a direct consequence of this lemma.

PROOF. By the Weyl differencing trick, write

$$\begin{aligned}
|F|^2 &= \sum_{\lambda_1, \lambda_2 \in L} e^{-it(|\lambda_1|^2 - |\lambda_2|^2) + i\langle \lambda_1 - \lambda_2, H\rangle} \varphi(\frac{|\lambda_1|^2}{N^2}) \varphi(\frac{|\lambda_2|^2}{N^2}) f(\lambda_1)\overline{f(\lambda_2)} \\
&= \sum_{\mu = \lambda_1 - \lambda_2} e^{-it|\mu|^2 + i\langle \mu, H\rangle} \sum_{\lambda = \lambda_2} e^{-i2t\langle \mu, \lambda\rangle} \varphi(\frac{|\mu+\lambda|^2}{N^2}) \varphi(\frac{|\lambda|^2}{N^2}) f(\mu+\lambda)\overline{f(\lambda)} \\
&\leq \sum_{|\mu| \lesssim N} |\sum_{\lambda} e^{-i2t\langle \mu, \lambda\rangle} \varphi(\frac{|\mu+\lambda|^2}{N^2}) \varphi(\frac{|\lambda|^2}{N^2}) f(\mu+\lambda)\overline{f(\lambda)})|.
\end{aligned}$$

Now write

$$\lambda = \sum_{i=1}^{r} n_i w_i,$$

and

$$g = \varphi(\frac{|\mu+\lambda|^2}{N^2}) \varphi(\frac{|\lambda|^2}{N^2}) f(\mu+\lambda)\overline{f(\lambda)}.$$

Note that

$$|D_{i_1} \cdots D_{i_n} \varphi(\frac{|\mu+\lambda|^2}{N^2})| \lesssim N^{-n}, \quad |D_{i_1} \cdots D_{i_n} \varphi(\frac{|\lambda|^2}{N^2})| \lesssim N^{-n}$$

for all $n \in \mathbb{Z}_{\geq 0}$ uniformly in $|n_i| \lesssim N$, $i = 1, \cdots, r$, which combined with (1.3) and the Leibniz rule (1.2) for the D_i's implies

$$|D_{i_1} \cdots D_{i_n} g| \lesssim N^{2A-n}. \tag{1.6}$$

Write

$$\sum_{\lambda \in L} e^{-i2t\langle \mu, \lambda\rangle} g = \sum_{n_1, \cdots, n_r \in \mathbb{Z}} (\prod_{i=1}^{r} e^{-itn_i\langle \mu, 2w_i\rangle}) g. \tag{1.7}$$

By summation by parts twice, we have

$$\sum_{n_1 \in \mathbb{Z}} e^{-itn_1\langle \mu, 2w_1\rangle} g = (\frac{e^{-it\langle \mu, 2w_1\rangle}}{1 - e^{-it\langle \mu, 2w_1\rangle}})^2 \sum_{n_1 \in \mathbb{Z}} e^{-itn_1\langle \mu, 2w_1\rangle} D_1^2 g(n_1, \cdots, n_r), \tag{1.8}$$

then (1.7) becomes

$$\sum_{\lambda\in L} e^{-i2t\langle\mu,\lambda\rangle} g = (\frac{e^{-it\langle\mu,2w_1\rangle}}{1-e^{-it\langle\mu,2w_1\rangle}})^2 \sum_{n_1,\cdots,n_r\in\mathbb{Z}} (\prod_{i=1}^r e^{-itn_i\langle\mu,2w_i\rangle}) D_1^2 g(n_1,\cdots,n_r).$$

Then we can carry out the procedure of summation by parts twice with respect to other variables $n_2,\cdots,n_r$. But we require that only when $|1-e^{-it\langle\mu,2w_i\rangle}|\geq\frac{1}{N}$ do we carry out the procedure to the variable n_i. Using (1.6), then we obtain

$$\begin{aligned}
&|\sum_{\lambda} e^{-i2t\langle\mu,\lambda\rangle}\varphi(\frac{|\mu+\lambda|^2}{N^2})\varphi(\frac{|\lambda|^2}{N^2})f(\mu+\lambda)\overline{f(\lambda)}| \\
&\lesssim N^{2A-r}\prod_{i=1}^r \frac{1}{(\max\{1-e^{-it\langle\mu,2w_i\rangle},\frac{1}{N}\})^2} \\
&\lesssim N^{2A-r}\prod_{i=1}^r \frac{1}{(\max\{\|\frac{1}{2\pi}t\langle\mu,2w_i\rangle\|,\frac{1}{N}\})^2}.
\end{aligned}$$

Write $\mu=\sum_{j=1}^r m_j w_j$, $m_j\in\mathbb{Z}$, then we have

$$|F|^2\lesssim N^{2A-r}\sum_{\substack{|m_j|\lesssim N,\\ j=1,\cdots,r}}\prod_{i=1}^r \frac{1}{(\max\{\|\frac{1}{2\pi}t\sum_{j=1}^r m_j\langle w_j,2w_i\rangle\|,\frac{1}{N}\})^2}.$$

Let

$$n_i=\sum_{j=1}^r m_j\langle w_j,2w_i\rangle\cdot D,\ i=1,\cdots,r, \tag{1.9}$$

where $D>0$ is the period of L so that $\langle w_j,w_i\rangle\in D^{-1}\mathbb{Z}$. Then $n_i\in\mathbb{Z}$. Noting that the matrix $(\langle w_j,2w_i\rangle D)_{i,j}$ is non-degenerate, which implies that for each vector $(n_1,\cdots,n_r)\in\mathbb{Z}^r$, there exists at most one vector $(m_1,\cdots,m_r)\in\mathbb{Z}^r$ so that (1.9) holds, thus

$$\begin{aligned}
|F|^2 &\lesssim N^{2A-r}\sum_{\substack{|n_i|\lesssim N,\\ i=1,\cdots,r}}\prod_{i=1}^r \frac{1}{(\max\{\|\frac{t}{2\pi D}n_i\|,\frac{1}{N}\})^2} \\
&\lesssim N^{2A-r}\prod_{i=1}^r\left(\sum_{|n_i|\lesssim N}\frac{1}{(\max\{\|\frac{t}{2\pi D}n_i\|,\frac{1}{N}\})^2}\right).
\end{aligned}$$

Then by Lemma B.1 in the appendix, we have

$$\sum_{|n_i|\lesssim N}\frac{1}{(\max\{\|\frac{t}{2\pi D}n_i\|,\frac{1}{N}\})^2}\lesssim\frac{N^3}{(\sqrt{q}(1+N\|\frac{t}{2\pi D}-\frac{a}{q}\|^{1/2}))^2},$$

which implies the desired result

$$|F|^2\lesssim\frac{N^{2A+2r}}{(\sqrt{q}(1+N\|\frac{t}{2\pi D}-\frac{a}{q}\|^{1/2}))^{2r}}.$$

□

We also have a variant of Lemma 30.

LEMMA 31. *Let $L=\mathbb{Z}w_1+\cdots+\mathbb{Z}w_r$ be a rational lattice in the inner product space $(V,\langle\ ,\ \rangle)$ with a period D. Let $L^+=\mathbb{Z}_{\geq 0}w_1+\cdots+\mathbb{Z}_{\geq 0}w_r$. Let φ be a bump function on $\mathbb{R}$ and $N\geq 1$. Let f be a function* *on $L^+ \cong (\mathbb{Z}_{\geq 0})^r$, with the requirement that*

$$|D_1^{\varepsilon_1} \cdots D_r^{\varepsilon_r} f(n_1, \cdots, n_r)| \lesssim N^{A-\varepsilon_1-\cdots-\varepsilon_r} \tag{1.10}$$

for all $(\varepsilon_1, \ldots, \varepsilon_r) \in \{0,1\}^r$, uniformly for $0 \leq n_i \lesssim N$, $i = 1, \ldots, r$, where A is a universal constant. Let

$$F(t, H) = \sum_{\lambda \in L^+} e^{-it|\lambda|^2 + i\langle \lambda, H\rangle} \varphi(\frac{|\lambda|^2}{N^2}) \cdot f \tag{1.11}$$

for $t \in \mathbb{R}$ and $H \in V$. Then for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$, we have

$$|F(t,H)| \lesssim_{\varepsilon > 0} \frac{N^{A+r+\varepsilon}}{[\sqrt{q}(1 + N\|\frac{t}{2\pi D} - \frac{a}{q}\|^{1/2})]^r} \tag{1.12}$$

uniformly in $H \in V$.

Proof. This proof is similar to that of Lemma 30. By Weyl's differencing technique,

$$\begin{aligned}
|F(t,H)|^2 &= \sum_{\lambda_1 \in L^+, \lambda_2 \in L^+} e^{-it(|\lambda_1|^2 - |\lambda_2|^2) + i\langle \lambda_1 - \lambda_2, H\rangle} \varphi(\frac{|\lambda_1|^2}{N^2})\varphi(\frac{|\lambda_2|^2}{N^2}) f(\lambda_1)\overline{f(\lambda_2)} \\
&= \sum_{\substack{\mu \in L^+ \\ (\mu = \lambda_1 + \lambda_2)}} e^{it|\mu|^2 - i\langle \mu, H\rangle} \sum_{\substack{\lambda \in L^+ \cap (\mu - L^+) \\ (\lambda = \lambda_1)}} e^{2i[\langle \lambda, H\rangle - t\langle \lambda, \mu\rangle]} \varphi(\frac{|\mu|^2}{N^2})\varphi(\frac{|\mu - \lambda|^2}{N^2}) f(\mu)\overline{f(\mu - \lambda)} \\
&\lesssim \sum_{\substack{\mu \in L^+ \\ (\mu = \lambda_1 + \lambda_2)}} \left| \sum_{\substack{\lambda \in L^+ \cap (\mu - L^+) \\ (\lambda = \lambda_1)}} e^{2i[\langle \lambda, H\rangle - t\langle \lambda, \mu\rangle]} \varphi(\frac{|\mu|^2}{N^2})\varphi(\frac{|\mu - \lambda|^2}{N^2}) f(\mu)\overline{f(\mu - \lambda)} \right|. \qquad (1.13)
\end{aligned}$$

For $\mu \in L^+$, write

$$\mu = n_1^\mu w_1 + \cdots + n_r^\mu w_r.$$

Then

$$\lambda \in \mu^+ \cap (\mu - L^+) \text{ if and only if } \lambda = n_1 w_1 + \cdots + n_r w_r,\ 0 \leq n_j \leq n_j^\mu,\ j = 1, \ldots, r.$$

For $\lambda = n_1 w_1 + \cdots + n_r w_r$, let

$$g(n_1, \ldots, n_r) = g(\lambda) = g(\lambda, N, \mu) := \phi(\frac{|\lambda|^2}{N^2})\phi(\frac{|\mu - \lambda|^2}{N^2}) f_\lambda f_{\mu - \lambda}.$$

By the assumption on f and the Leibiniz rule for difference operators,

$$|D_1^{\varepsilon_1} \cdots D_r^{\varepsilon_r} g(n_1, \cdots, n_r)| \lesssim N^{2A - \varepsilon_1 - \cdots - \varepsilon_r} \tag{1.14}$$

for all $(\varepsilon_1, \ldots, \varepsilon_r) \in \{0,1\}^r$, uniformly for $0 \leq n_i \lesssim N$, $i = 1, \ldots, r$. Now let $F^\mu = F^\mu(t,H)$ be the sum in (1.13) inside of the absolute value. Then

$$F^\mu = \sum_{0 \leq n_r \leq n_r^\mu} e^{in_r\theta_r} \cdots \sum_{0 \leq n_1 \leq n_1^\mu} e^{in_1\theta_1} g(\lambda)$$

where

$$\theta_j = \theta_j(t, H, \mu) := 2[\langle w_j, H\rangle - t\langle w_j, \mu\rangle],\ j = 1, \ldots, r.$$

We can perform summation by parts on F^μ with respect to the variable n_1

$$\sum_{0\le n_1\le n_1^\mu} e^{in_1\theta_1} g(\lambda) = \frac{1}{1-e^{i\theta_1}} \sum_{0\le n_1\le n_1^\mu} e^{i(n_1+1)\theta_1} D_1 g(\lambda)$$
$$+ \frac{1}{1-e^{i\theta_1}} g(0,n_2,\dots,n_r) - \frac{e^{i(n_1^\mu+1)\theta_1}}{1-e^{i\theta_1}} g(n_1^\mu+1,n_2,\dots,n_r).$$

Then we can perform summation by parts with respect to other variables $n_2,\dots,n_r$. But we require that only when

$$|1-e^{i\theta_j}| \ge N^{-1},$$

do we carry out the procedure to the variable n_j. Using (1.14), what we we end up with is an estimate

$$\begin{aligned} |F^\mu|^2 &\lesssim N^{2A} \prod_{j=1}^r \frac{1}{\max\{\frac{1}{N}, |1-e^{i\theta_j}|\}} \\ &\lesssim N^{2A} \prod_{j=1}^r \frac{1}{\max\{\frac{1}{N}, \|\frac{\theta_j}{2\pi}\|\}} \\ &\lesssim N^{2A} \prod_{j=1}^r \frac{1}{\max\{\frac{1}{N}, \|\frac{w_j(H)}{\pi} - \frac{t\langle w_j,\mu\rangle}{\pi}\|\}}. \end{aligned}$$

Since D is a period of the lattice L, $-2\langle w_j,\mu\rangle \in D^{-1}\mathbb{Z},\ \forall \mu\in L,\ j=1,\dots,r$. Let

$$m_j = -2\langle w_j,\mu\rangle \cdot D,\ j=1,\dots,r.$$

Since the map $\Lambda \ni \mu \mapsto (m_1,\dots,m_r)\in\mathbb{Z}^r$ is one-one, we can write (1.13) into

$$\begin{aligned} |F(t,H)|^2 &\lesssim N^{2A} \sum_{|m_1|\lesssim N,\dots,|m_j|\lesssim N} \prod_{j=1}^r \frac{1}{\max\{\frac{1}{N}, \|m_j\frac{t}{2\pi D} + \frac{w_j(H)}{\pi}\|\}} \\ &\lesssim N^{2A} \prod_{j=1}^r \left(\sum_{|m_j|\lesssim N} \frac{1}{\max\{\frac{1}{N}, \|m_j\frac{t}{2\pi D} + \frac{w_j(H)}{\pi}\|\}} \right). \end{aligned}$$

By Remark B.2 in the appendix, we get

$$\sum_{|m_j|\lesssim N} \frac{1}{\max\{\frac{1}{N}, \|m_j\frac{t}{2\pi D} + \frac{w_j(H)}{\pi}\|\}} \lesssim \frac{N^2\log N}{[\sqrt{q}(1+N\|\frac{t}{2\pi D}-\frac{a}{q}\|^{1/2})]^2}$$

for $\frac{t}{2\pi D}$ lying on the major arc $\mathcal{M}_{a,q}$. Hence

$$|F(t,H)|^2 \lesssim \frac{N^{2A+2r}\log^r N}{[\sqrt{q}(1+N\|\frac{t}{2\pi D}-\frac{a}{q}\|^{1/2})]^{2r}}.$$

□

REMARK 32. *Let λ_0 be any constant vector in V and C any constant real number. Then we can slightly generalize the form of the function $F(t,H)$ in Lemma 30 and 31 into*

$$F(t,H) = \sum_{\lambda\in L(or\ L^+)} e^{-it|\lambda+\lambda_0|^2 + i\langle\lambda,H\rangle} \varphi(\frac{|\lambda+\lambda_0|^2+C}{N^2}) \cdot f$$

such that the proofs still work and the results still hold.

We have our first application of Lemma 31. Let U/K be a simply connected irreducible symmetric space of compact type. Specializing the Schrödinger kernel (3.5) to $x = K$, noting that $\Phi_\lambda(K) = 1$, we have

$$K_N(t, K) = \sum_{\lambda\in\Lambda^+} \varphi(\frac{-|\lambda+\rho|^2 + |\rho|^2}{N^2}) e^{it(-|\lambda+\rho|^2+|\rho|^2)} d_\lambda. \tag{1.15}$$

Proposition 33. *Let d, r be respectively the dimension and rank of U/K. Let D be a period of the weight lattice. Then we have*

$$|K_N(t, K)| \lesssim_{\varepsilon>0} \frac{N^{d+\varepsilon}}{(\sqrt{q}(1 + N\|\frac{t}{2\pi D} - \frac{a}{q}\|^{1/2}))^r}$$

for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$.

Proof. Recall that d_λ is a polynomial in $\lambda \in \Lambda$ of degree $d - r$. Thus d_λ as a function on $\Lambda^+ \cong \mathbb{Z}^r_{\geq 0}$ satisfies (1.10) with $A = d - r$. The result is now a consequence of Lemma 31. □

2. On a N^{-1}-Neighborhood of K

We strengthen Proposition 33.

Theorem 34. *Let d, r be respectively the dimension and rank of U/K. Let D be a period of the weight lattice. Let $d(\ ,\ )$ be the distance function on U/K. Then we have*

$$|K_N(t, x)| \lesssim_{\varepsilon>0} \frac{N^{d+\varepsilon}}{(\sqrt{q}(1 + N\|\frac{t}{2\pi D} - \frac{a}{q}\|^{1/2}))^r} \tag{2.1}$$

for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$, uniformly for $d(x, K) \lesssim N^{-1}$.

The proof hinges on an integral representation of spherical functions in a neighborhood of K. Continue the notations in Section 2. Let $\mathfrak{n}^{\mathbb{C}}, \mathfrak{a}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}$ be respectively the complexification of $\mathfrak{n}, \mathfrak{a}, \mathfrak{k}$. By Section 9.2 Ch. III in [**Hel08**], the mapping

$$(X, H, T) \mapsto \exp X \exp H \exp T,\ \ X \in \mathfrak{n}^{\mathbb{C}}, H \in \mathfrak{a}^{\mathbb{C}}, T \in \mathfrak{k}^{\mathbb{C}}$$

is a holomorphic diffeomorphism of a neighborhood $\mathcal{U}^{\mathbb{C}}$ of $G^{\mathbb{C}}$ such that $\mathcal{U} = \mathcal{U}^{\mathbb{C}} \cap U$ is invariant under the maps $u \mapsto kuk^{-1}$, $k \in K$. This induces the map

$$A : \exp X \exp H \exp T \to H$$

that sends $\mathcal{U}^{\mathbb{C}}$ into $\mathfrak{a}^{\mathbb{C}}$. Let Φ_λ be the spherical function associated to $\lambda \in \Lambda^+$. By Lemma 9.2 of Ch. III in [**Hel08**],

$$\Phi_\lambda(u) = \int_K e^{-\lambda(A(kuk^{-1}))}\, dk,\ \ u \in \mathcal{U}. \tag{2.2}$$

Note that the map $u \mapsto kuk^{-1}$ preserves the distance $d_U(\cdot, e)$ to the identity e of U. Let $N \geq 1$ be large enough so that $\{u \in U : d_U(u, e) \lesssim N^{-1}\} \subset \mathcal{U}$. Then

$$|A(kuk^{-1})| \lesssim N^{-1} \tag{2.3}$$

uniformly for $d_U(u, e) \lesssim N^{-1}$ and $k \in K$. Here the norm on $\mathfrak{a}^{\mathbb{C}}$ of course comes from the Cartan-Killing form. Write $\lambda = n_1w_1 + \cdots + n_rw_r$, $n_i \in \mathbb{Z}_{\geq 0}$, viewing $\Phi_\lambda(u) = \Phi(\lambda, u)$ as a function of $\lambda \in \mathbb{Z}^r_{\geq 0}$, (2.2) and (2.3) imply that $\Phi(\lambda, u)$ satisfies an equality of the type (1.10) as follows.

LEMMA 35.

$$|D_{i_1}\cdots D_{i_n}\Phi(n_1,\cdots,n_r,u)|\lesssim N^{-n}$$

holds uniformly in $0\le n_i\lesssim N$ *and* $d_U(u,e)\lesssim N^{-1}$, *for all* $i_j=1,\cdots,r$ *and* $n\in\mathbb{Z}_{\geq 0}$.

PROOF OF THEOREM 34. The proof is very similar to that of Proposition 33. Using Lemma 35, the dimension formula (2.15), and (1.2), we have that $f=d_\lambda\Phi_\lambda$ satisfies (1.10) with $A=d-r$. Then we can apply Lemma 31 to get the result. □

3. On a N^{-1}-Neighborhood of any Corner

Continue the notations in Section 2.

DEFINITION 36. *Recall that* $A=i\mathfrak{a}/\Gamma^\vee$ *is the maximal torus of* $M=U/K$ *where* $\Gamma^\vee=2\pi i\mathbb{Z}\frac{H_{\alpha_1}}{\langle\alpha_1,\alpha_1\rangle}+\cdots+2\pi i\mathbb{Z}\frac{H_{\alpha_r}}{\langle\alpha_r,\alpha_r\rangle}$. *For* $H\in\mathfrak{a}$, *we say* $[iH]\in A$ *is a corner if* $\alpha(H)\in\pi\mathbb{Z}$ *for all* $\alpha\in\Sigma$.

Note that this definition is well defined and there are finitely many corners in A. In fact, let $w_1,\cdots,w_r$ be the fundamental weights associated to the set of positive roots $\{\alpha_1,\cdots,\alpha_r\}$, and let $\Lambda^\vee=\pi i\mathbb{Z}\frac{H_{w_1}}{\langle\alpha_1,\alpha_1\rangle}+\cdots+\pi i\mathbb{Z}\frac{H_{w_r}}{\langle\alpha_r,\alpha_r\rangle}$. Then $\Gamma^\vee\subset\Lambda^\vee$ and the set of corners is isomorphic to the finite set $\Lambda^\vee/\Gamma^\vee$.

EXAMPLE 37. *Continue Example 14. Then the only corners are* $\theta=0,\pi$.
Continue Example 15. Since $\Delta=2\Sigma$, $[iH]\in A$ *is a corner if and only if* $\alpha(H)\in 2\pi\mathbb{Z}$ *for all* $\alpha\in\Delta$.

THEOREM 38. *Let* $[iH_0]\in A$ *be any corner. Then* (2.1) *holds for* $\frac{t}{2\pi D}\in\mathcal{M}_{a,q}$, *uniformly for* $d(x,[iH_0])\lesssim N^{-1}$, $x\in A$.

REMARK 39. *It can be shown that any corner is fixed by the left actions by* K. *By the invariance of the Schrödinger kernel under* K, *the above theorem can be slightly generalized as such that* (2.1) *holds uniformly for* $d(x,[iH_0])\lesssim N^{-1}$, $x\in U/K$.

To prove this theorem, we describe an important characterization of spherical functions. For $\mu,\lambda\in\Lambda$, let $\mu\le\lambda$ denote the statement that $\lambda-\mu\in 2\mathbb{Z}_{\geq 0}\alpha_1+\cdots+2\mathbb{Z}_{\geq 0}\alpha_r$. For $\mu\in\Lambda^+$, define

$$M(\mu)=\sum_{s\in W}e^{s\mu}.$$

Then define the *Heckman-Opdam polynomials* $P(\lambda)$, $\lambda\in\Lambda^+$, by

$$P(\lambda)=\sum_{\mu\in\Lambda^+,\mu\le\lambda}c_{\lambda,\mu}M(\mu),\ c_{\lambda,\lambda}=1$$

such that

$$\int_A P(\lambda)\cdot\overline{M(\mu)}\cdot\left|\prod_{\alpha\in\Sigma^+}(e^\alpha-e^{-\alpha})^{m_\alpha}\right|da=0,\ \forall\mu\in\Lambda^+,\ \mu<\lambda.$$

Here da is the normalized Haar measure on the A. Let e denote the identity of the maximal torus A. Normalize $P(\lambda)$ by

$$\tilde{P}(\lambda)=\frac{P(\lambda)}{P(\lambda)(e)}.$$

THEOREM 40 (Corollary 5.2.3 in Part I of [**HS94**]). *The spherical functions on* U/K *restricted on* A *are given by the normalized Heckman-Opdam polynomials:*

$$\Phi_\lambda=\tilde{P}(\lambda),\ \forall\lambda\in\Lambda^+.$$

Corollary 41. *Let $[iH_0]\in A$ be a corner. Then*

$$\Phi_\lambda(iH+iH_0)=e^{i\lambda(H_0)}\Phi_\lambda(iH),\ \forall H\in\mathfrak{a},\ \forall\lambda\in\Lambda^+.$$

Proof. By the above theorem and the definition of Heckman-Opdam polynomials, it suffices to show that for any $\lambda\in\Lambda^+$,

$$e^{i(s\mu)(H_0)}=e^{i\lambda(H_0)},\ \forall\mu\leq\lambda,\ \forall s\in W.$$

This is reduced to showing $(s\mu-\lambda)(H_0)\in 2\pi\mathbb{Z}$, and by the definition of $[iH_0]$ as a corner, it is further reduced to $s\mu-\lambda\in 2\mathbb{Z}\alpha_1+\cdots+2\mathbb{Z}\alpha_r$. By the fact $\mu\leq\lambda$, it then suffices to show $s\mu-\mu\in 2\mathbb{Z}\alpha_1+\cdots+2\mathbb{Z}\alpha_r$ for any $\mu\in\Lambda$ and $s\in W$. But this is a fact by Corollary 4.13.3 in [**Var84**]. □

Let $\Gamma=2\mathbb{Z}\alpha_1+\cdots+2\mathbb{Z}\alpha_r$. The above corollary implies that for $\lambda\in\Gamma$ and $\mu\in\Lambda^+$ such that $\lambda+\mu\in\Lambda^+$,

$$\Phi_{\lambda+\mu}(iH+iH_0)=e^{i\mu(H_0)}\Phi_{\lambda+\mu}(iH). \tag{3.1}$$

This inspires a decomposition of Λ^+ and thus of the Schrödinger kernel (3.5), which makes applicable the techniques in proving Theorem 34 for the proof of Theorem 38.

Proof of Theorem 38. The definition of the weight lattice and Axiom (iii) of the root system in (2.1) imply that any of the fundamental weights $w_1,\cdots,w_r$ is a rational linear combination of roots. Thus there exists some $B\in\mathbb{N}$ such that $Bw_i\in\Gamma$ for all i. Define

$$\Gamma_1^+=\mathbb{Z}_{\geq0}Bw_1+\cdots+\mathbb{Z}_{\geq0}Bw_r.$$

Let $\Lambda^+/\Gamma_1^+=\{n_1w_1+\cdots+n_rw_r:n_i=0,\cdots,B-1,i=1,\cdots,r\}$ and decompose

$$\Lambda^+=\bigsqcup_{\mu\in\Lambda^+/\Gamma_1^+}(\Gamma_1^++\mu).$$

This yields decomposition of the Schrödinger kernel

$$K_N=\sum_{\mu\in\Lambda^+/\Gamma_1^+}K_N^\mu, \tag{3.2}$$
$$K_N^\mu=\sum_{\lambda\in\Gamma_1^+}\varphi(\frac{-|\lambda+\mu+\rho|^2+|\rho|^2}{N^2})e^{it(-|\lambda+\mu+\rho|^2+|\rho|^2)}d_{\lambda+\mu}\Phi_{\lambda+\mu}.$$

By the finiteness of Λ^+/Γ_1^+, it suffices to prove (2.1) replacing K_N by K_N^μ. By (3.1),

$$K_N^\mu(t,iH+iH_0)=e^{i\mu(H_0)}\sum_{\lambda\in\Gamma_1^+}\varphi(\frac{-|\lambda+\mu+\rho|^2+|\rho|^2}{N^2})e^{it(-|\lambda+\mu+\rho|^2+|\rho|^2)}d_{\lambda+\mu}\Phi_{\lambda+\mu}(iH).$$

Now for $x=[iH+iH_0]$, the assumption $d(x,[iH_0])\lesssim N^{-1}$ gives $|H|\lesssim N^{-1}$. We apply Lemma 31 to $f(\lambda)=d_{\lambda+\mu}\Phi_{\lambda+\mu}(iH)$, and then the rest of the proof follows exactly as the proof of Theorem 34. □

4. Away From the Corners

We do not have a general theory yet to prove (2.3) uniformly for all $x\in A$ that stays away from the corners by a distance $\gtrsim N^{-1}$ for a general symmetric space of compact type. The main obstacle is the lack of explicit formulas or even approximate formulas for the general spherical functions or say the Heckman-Opdam polynomials (for research in this direction, see for example [**EFK95**], [**Obl04**], [**vD03**]). In this section, we deal with the special case of odd dimensional spheres required in Theorem 24, for which explicit formulas of spherical functions exist and are useful. The other case of compact Lie groups required in Theorem 24 of which the spherical functions are given explicitly by Weyl's character and dimension formulas, is to be dealt with next chapter.

Let U/K be the sphere of dimension $d = 2\lambda + 1$, $\lambda \in \mathbb{N}$. Continue the notations in Example 14. To prove (2.3) with ε-loss for the Schrödinger kernel (3.12), first realize that $K_N(t,\theta)$ is invariant under the Weyl group action $\theta \mapsto 2\pi - \theta$, thus it suffices to prove (2.3) uniformly for θ in the closed cell $[0,\pi]$. Then Theorem 38 implies (2.3) with ε-loss uniformly for $|\theta| \lesssim N^{-1}$ or $|\theta - \pi| \lesssim N^{-1}$, thus it suffices to prove (2.3) with ε-loss uniformly for θ away from $0, \pi$ by a distance $\gtrsim N^{-1}$. By (2.9), it then suffices to prove for any $\nu = 0, 1, \ldots, \lambda - 1$, we have

$$|K_N^{(\nu)}(t,\theta)| \lesssim_\varepsilon \frac{N^{2\lambda+1+\varepsilon}}{\sqrt{q}(1 + N\|\frac{t}{2\pi D} - \frac{a}{q}\|^{1/2})}$$

for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$, uniformly in $CN^{-1} \le \theta \le \pi - CN^{-1}$, $C > 0$, where

$$K_N^{(\nu)}(t,\theta) = \frac{2}{(2\sin\theta)^{\nu+\lambda}} \sum_{n \in \mathbb{Z}_{\ge 0}} \varphi(\frac{(n+\lambda)^2 - \lambda^2}{N^2}) e^{-it[(n+\lambda)^2 - \lambda^2]} d_n C_{n,\nu} \cos((n - \nu + \lambda)\theta - (\nu + \lambda)\pi/2),$$

with

$$C_{n,\nu} = \binom{n + 2\lambda - 1}{n}^{-1} \binom{n + \lambda - 1}{n} \binom{\nu + \lambda - 1}{\nu} \frac{(1 - \lambda)\cdots(\nu - \lambda)}{(n + \lambda - 1)\cdots(n + \lambda - \nu)}.$$

As $CN^{-1} \le \theta \le \pi - CN^{-1}$, we have $|\frac{2}{(2\sin\theta)^{\nu+\lambda}}| \lesssim N^{\nu+\lambda}$, $\nu = 0, \cdots, \lambda - 1$. Rewriting $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$, it then suffices to prove

$$|\sum_{n \in \mathbb{Z}_{\ge 0}} \varphi(\frac{(n+\lambda)^2 - \lambda^2}{N^2}) e^{-it[(n+\lambda)^2 - \lambda^2] \pm i(n - \nu + \lambda)\theta \mp i(\nu + \lambda)\pi/2} d_n C_{n,\nu}| \lesssim_\varepsilon \frac{N^{\lambda - \mu + 1 + \varepsilon}}{\sqrt{q}(1 + N\|\frac{t}{2\pi D} - \frac{a}{q}\|^{1/2})} \tag{4.1}$$

uniformly in $\theta \in [CN^{-1}, \pi - CN^{-1}]$. Note that d_n is polynomial in n of degree $d - 1 = 2\lambda$, then we can write $d_n C_{n,v} = \frac{f(n)}{g(n)}$ such that $f(n)$ and $g(n)$ are polynomials of degree $3\lambda - 1$ and $2\lambda - 1 + \nu$ respectively. This implies that $d_n C_{n,v}$ satisfies estimate of the form (1.10)

$$|D^\mu(d_n C_{n,\nu})| \lesssim N^{\lambda - \nu - \mu}$$

for $\mu = 0, 1$, uniformly for $0 \le n \lesssim N$. We can now apply Lemma 31 to (4.1) and finish the proof.

Remark 42. *We have the following partial result on* (2.3) *for general symmetric spaces of compact type of rank 1. Continue Example 14. Let M be a simply connected symmetric space of compact type of dimension d and rank 1. The Schrödinger kernel reads*

$$K_N(t,\theta) = \sum_{n \in \mathbb{Z}_{\ge 0}} \varphi(\frac{-(n+\rho)^2 + \rho^2}{N^2}) e^{it(-(n+\rho)^2 + \rho^2)} d_n \binom{n + a}{n}^{-1} P_n^{(a,b)}(\cos\theta)$$

where d_n is polynomial in n of degree $d - 1$, $\rho = \frac{1}{2}m_\alpha + \frac{1}{4}m_{\frac{\alpha}{2}}$. We have the asymptotics for the Jacobi polynomials (Theorem 8.21.8 in [**Sze75**]*)*

$$P_n^{(a,b)}(\cos\theta) = (n\pi)^{-\frac{1}{2}} (\sin\frac{\theta}{2})^{-a-\frac{1}{2}} (\cos\frac{\theta}{2})^{-b-\frac{1}{2}} \cos([n + (a + b + 1)/2]\theta - (a + \frac{1}{2})\pi/2) + O(n^{-\frac{3}{2}}),$$

where the bound for the error term holds uniformly in the interval $[c, \pi - c]$, $c > 0$. Fix such a constant $c > 0$. Note that

$$n^{a-\varepsilon} \lesssim_{\varepsilon>0} \left|\binom{n+a}{n}\right| \lesssim_{\varepsilon>0} n^{a+\varepsilon}, \text{ uniformly in } n \in \mathbb{N}.$$

This implies

$$\left|\varphi(\frac{-(n+\rho)^2+\rho^2}{N^2})e^{it(-(n+\rho)^2+\rho^2)}d_n\binom{n+a}{n}^{-1}P_n^{(a,b)}(\cos\theta)\right|\lesssim_{c,\varepsilon} n^{d-1-a-\frac{1}{2}+\varepsilon}.$$

Now either $a=0$ *or* $a\geq\frac{1}{2}$. *For* $a\geq\frac{1}{2}$, *the above estimate directly implies*

$$|K_N(t,\theta)|\lesssim_{c,\varepsilon} N^{d-1+\varepsilon}$$

which satisfies (2.3) *uniformly for* $\theta\in[c,\pi-c]$ *(noting that* $\sqrt{q}(1+N\|\frac{t}{2\pi D}-\frac{a}{q}\|^{1/2})\lesssim N^{\frac{1}{2}}$*). For* $a=0$, *if* $d=2$, *which is the case of the two sphere, then the above estimate gives*

$$|K_N(t,\theta)|\lesssim_{c,\varepsilon} N^{\frac{1}{2}+\varepsilon}$$

which satisfies (2.3) *with an* ε*-loss for* $\theta\in[c,\pi-c]$. *If* $d\geq 3$ *for* $a=0$, *then*

$$\left|D_{i_1}\cdots D_{i_k}\left(\binom{n+a}{n}^{-1}d_n n^{-\frac{1}{2}}\right)\right|\lesssim_{\varepsilon} n^{d-\frac{3}{2}-k+\varepsilon}\leq n^{d-1-k}.$$

Since $d-1\geq 2$, *an application of part (i) of Lemma 31 implies* (2.3) *for* $\theta\in[c,\pi-c]$. *In conclusion, for all symmetric space of compact type of rank 1,* (2.3) *holds (with an* ε*-loss for the special case of the two sphere) uniformly for* $\theta\in[c,\pi-c]$. *Recall from Theorem 38 we also have that* (2.3) *holds (with an* ε*-loss for the two sphere) uniformly for* θ *close to the corners 0 and* π *by a distance of* $\lesssim N^{-1}$. *But the estimate is still missing for other values of* θ.

CHAPTER 6

Dispersive Estimates on Compact Lie Groups

In this chapter, we finish proof of part (1) of Theorem 24 . Let M be a simply connected compact simple Lie group and continue the notations in Example 15.

Let $Q = \bigcap_{\alpha\in\Delta^+}\{[iH]\in A : \langle\alpha, H\rangle \in [0, 2\pi]\}$ be the fundamental cell in the maximal torus A. In Section 3 of last chapter, we proved that (2.3) holds uniformly for $x = [iH] \in Q$ that stays within a distance of $\lesssim N^{-1}$ from some corner, that is, if we use $\|\cdot\|$ to denote the distance from 0 in the unit circle $[0, 1)$, when $\|\frac{1}{2\pi}\langle\alpha, H\rangle\| \lesssim N^{-1}$ for all $\alpha \in \Delta$. The key ingredient in proving this is the *pseudo-polynomial* behavior of spherical functions as well as characters. Then it suffices to prove it for the cases when x stays away from all the corners by a distance of $\gtrsim N^{-1}$. We will first prove it for the special case when $x = [iH]$ stays away from all the cell walls, that is, when $\|\frac{1}{2\pi}\langle\alpha, H\rangle\| \gtrsim N^{-1}$ for all $\alpha\in\Delta$, by exploiting the *oscillatory* behavior of characters for such x's. The general case when x is close to some cell walls within a distance of $\lesssim N^{-1}$ but away from other cell walls by a distance of $\gtrsim N^{-1}$ will be dealt with combining both the polynomial-like and the oscillatory behavior of characters.

1. From a Chamber to the Whole Weight Lattice

First, we rewrite the Schrödinger kernel as a sum over the whole weight lattice instead of just a chamber of the lattice, so to make Lemma 30 applicable. Let M be a compact simply connected simple Lie group of dimension d and r. Apply the notations as in Example 3, 13, 15 and 19. We make the identification $\mathfrak{a} \cong \mathfrak{a}^*$, so that for $\lambda\in\mathfrak{a}^*$ and $H \in\mathfrak{a}$, $\lambda(H) = \langle\lambda, H\rangle$. Then the Weyl's character formula (2.11) becomes

$$\chi_\lambda(iH) = \frac{\sum_{s\in W}\det(s)e^{i\langle s(\lambda+\rho),H\rangle}}{\sum_{s\in W}\det(s)e^{i\langle s(\rho),H\rangle}}$$

for $\lambda\in\Lambda^+$, $H\in\mathfrak{a}$. By Lemma 4.13.4 of Chapter 4 in [**Var84**], the Weyl denominator $D_P = \sum_{s\in W}(\det s)e^{i\langle s(\rho),H\rangle}$ can be rewritten as

$$D_P = e^{-i\langle\rho,H\rangle}\prod_{\alpha\in\Delta^+}(e^{i\langle\alpha,H\rangle} - 1). \tag{1.1}$$

LEMMA 43. *Let $D_P = e^{-i\langle\rho,H\rangle}\prod_{\alpha\in P}(e^{i\langle\alpha,H\rangle} - 1)$ be the Weyl denominator. We have*

$$K_N(t, [iH]) = \frac{e^{it|\rho|^2}}{(\prod_{\alpha\in P}\langle\alpha,\rho\rangle)D_P}\sum_{\lambda\in\Lambda} e^{-it|\lambda|^2+i\langle\lambda,H\rangle}\varphi(\frac{-|\lambda|^2+|\rho|^2}{N^2})\prod_{\alpha\in P}\langle\alpha,\lambda\rangle \tag{1.2}$$

$$= \frac{e^{it|\rho|^2}}{(\prod_{\alpha\in P}\langle\alpha,\rho\rangle)|W|}\sum_{\lambda\in\Lambda} e^{-it|\lambda|^2}\varphi(\frac{-|\lambda|^2+|\rho|^2}{N^2})\prod_{\alpha\in P}\langle\alpha,\lambda\rangle\frac{\sum_{s\in W}\det(s)e^{i\langle s(\lambda),H\rangle}}{D_P} \tag{1.3}$$

PROOF. We first prove (1.2). Recall that $\rho = w_1 + \cdots + w_r$ where $\{w_1, \cdots, w_r\}$ is a set of fundamental weights such that $\Lambda^+ = \mathbb{Z}_{\geq 0}w_1 + \cdots + \mathbb{Z}_{\geq 0}w_r$ and $\Lambda = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_r$. Recall that the fundamental chamber is $C = \mathbb{R}_{>0}w_1 + \cdots + \mathbb{R}_{>0}w_r$. Thus we have

$$\Lambda^+ + \rho = \Lambda\cap C.$$

Then we can rewrite the Schrödinger kernel as

$$K_N = \sum_{\lambda\in\Lambda\cap C} \varphi(\frac{-|\lambda|^2+|\rho|^2}{N^2}) e^{it(-|\lambda|^2+|\rho|^2)} \frac{\prod_{\alpha\in P}\langle\alpha,\lambda\rangle}{\prod_{\alpha\in P}\langle\alpha,\rho\rangle} \frac{\sum_{s\in W}\det(s) e^{i\langle s\lambda,H\rangle}}{D_P}.$$

Recall that from Proposition 46, $\prod_{\alpha\in P}\langle\alpha,\cdot\rangle$ is an anti-invariant polynomial so that for $s\in W$

$$\prod_{\alpha\in P}\langle\alpha, s(\lambda)\rangle = \det(s)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle. \tag{1.4}$$

Then recall that the Weyl group W acts on $\mathfrak{a}^*$ as a group of isometries so that

$$|s(\lambda)| = |\lambda|, \quad \text{for all } s\in W, \quad \lambda\in\mathfrak{a}^*. \tag{1.5}$$

Using the above two formulas, we rewrite the Schrödinger kernel

$$\begin{aligned}
K_N(t,[iH]) &= \frac{e^{it|\rho|^2}}{(\prod_{\alpha\in P}\langle\alpha,\rho\rangle)D_P} \sum_{\lambda\in\Lambda\cap C} e^{-it|\lambda|^2}\varphi(\frac{-|\lambda|^2+|\rho|^2}{N^2}) \prod_{\alpha\in P}\langle\alpha,\lambda\rangle \sum_{s\in W}\det(s)e^{i\langle s(\lambda),H\rangle}\\
&= \frac{e^{it|\rho|^2}}{(\prod_{\alpha\in P}\langle\alpha,\rho\rangle)D_P} \sum_{s\in W}\sum_{\lambda\in\Lambda\cap C} e^{-it|\lambda|^2}\varphi(\frac{-|\lambda|^2+|\rho|^2}{N^2}) \prod_{\alpha\in P}\langle\alpha,s(\lambda)\rangle e^{i\langle s(\lambda),H\rangle}\\
&= \frac{e^{it|\rho|^2}}{(\prod_{\alpha\in P}\langle\alpha,\rho\rangle)D_P} \sum_{s\in W}\sum_{\lambda\in\Lambda\cap C} e^{-it|s(\lambda)|^2}\varphi(\frac{-|s(\lambda)|^2+|\rho|^2}{N^2}) \prod_{\alpha\in P}\langle\alpha,s(\lambda)\rangle e^{i\langle s(\lambda),H\rangle}\\
&= \frac{e^{it|\rho|^2}}{(\prod_{\alpha\in P}\langle\alpha,\rho\rangle)D_P} \sum_{\lambda\in\bigsqcup_{s\in W}s(\Lambda\cap C)} e^{-it|\lambda|^2}\varphi(\frac{-|\lambda|^2+|\rho|^2}{N^2}) \prod_{\alpha\in P}\langle\alpha,\lambda\rangle e^{i\langle\lambda,H\rangle}.
\end{aligned}$$

Now (2.8) implies

$$\Lambda = (\bigsqcup_{s\in W} s(\Lambda\cap C)) \bigsqcup (\bigcup_{\alpha\in\Sigma}\{\lambda\in\Lambda : \langle\lambda,\alpha\rangle = 0\}),$$

using which we rewrite

$$K_N(t,x) = \frac{e^{it|\rho|^2}}{(\prod_{\alpha\in P}\langle\alpha,\rho\rangle)D_P} \sum_{\lambda\in\Lambda} e^{-it|\lambda|^2}\varphi(\frac{-|\lambda|^2+|\rho|^2}{N^2}) \prod_{\alpha\in P}\langle\alpha,\lambda\rangle e^{i\langle\lambda,H\rangle}.$$

This proves (1.2). To prove (1.3), using $s\Lambda=\Lambda$ for all $s\in W$, write

$$\sum_{\lambda\in\Lambda} e^{-it|\lambda|^2+i\langle\lambda,H\rangle}\varphi(\frac{-|\lambda|^2+|\rho|^2}{N^2}) \prod_{\alpha\in P}\langle\alpha,\lambda\rangle = \sum_{\lambda\in\Lambda} e^{-it|s(\lambda)|^2+i\langle s(\lambda),H\rangle}\varphi(\frac{-|s(\lambda)|^2+|\rho|^2}{N^2}) \prod_{\alpha\in P}\langle\alpha,s(\lambda)\rangle, \tag{1.6}$$

which implies using (1.4) and (1.5) that

$$\sum_{\lambda\in\Lambda} e^{-it|\lambda|^2+i\langle\lambda,H\rangle}\varphi(\frac{-|\lambda|^2+|\rho|^2}{N^2}) \prod_{\alpha\in P}\langle\alpha,\lambda\rangle = \det(s)\sum_{\lambda\in\Lambda} e^{-it|\lambda|^2+i\langle s(\lambda),H\rangle}\varphi(\frac{-|\lambda|^2+|\rho|^2}{N^2}) \prod_{\alpha\in P}\langle\alpha,\lambda\rangle,$$

which further implies

$$\begin{aligned}
&\sum_{\lambda\in\Lambda} e^{-it|\lambda|^2+i\langle\lambda,H\rangle}\varphi(\frac{-|\lambda|^2+|\rho|^2}{N^2}) \prod_{\alpha\in P}\langle\alpha,\lambda\rangle\\
&= \frac{1}{|W|}\sum_{\lambda\in\Lambda} e^{-it|\lambda|^2}\varphi(\frac{-|\lambda|^2+|\rho|^2}{N^2}) \prod_{\alpha\in P}\langle\alpha,\lambda\rangle \sum_{s\in W}\det(s)e^{i\langle s(\lambda),H\rangle}.
\end{aligned}$$

This combined with (1.2) yields (1.3). □

2. Approaches to Theorem 34 for Compact Lie Groups

Before we go into the case when x stays away from some corner, we present here two more approaches to Theorem 34 for compact Lie groups. These new approaches will be useful later on. Instead of using the integral formula (2.2) to establish Lemma 35, these two approaches are based on the Weyl's formula (2.11) to establish a simlar result for the characters.

LEMMA 44. *Let $\mu \in \mathfrak{a}^*$. For $\lambda \in \mathfrak{a}^*$, define*

$$\chi^\mu(\lambda, H) = \frac{\sum_{s\in W} \det(s) e^{i\langle s(\lambda+\mu), H\rangle}}{\sum_{s\in W} \det(s) e^{i\langle s(\rho), H\rangle}}.$$

Let $L \cong \mathbb{Z}^r$ be the weight lattice or the root lattice (or any sublattice of full rank of the weight lattice), and viewing $\chi^\mu(\lambda, H)$ as a function on $\lambda \in L$, we have

$$|D_{i_1}\cdots D_{i_k}\chi^\mu(n_1,\cdots,n_r,H)| \lesssim N^{\frac{d-r}{2}-k} \tag{2.1}$$

holds uniformly in $|n_i| \lesssim N$ and $|H| \lesssim N^{-1}$, for all $i_j = 1,\cdots,r$ and $k \in \mathbb{Z}_{\geq 0}$. In other words, $\chi^\mu(\lambda, H)$ is a pseudo-polynomial of degree $\frac{d-r}{2}$ in λ uniformly in $|H| \lesssim N^{-1}$.

This lemma implies Theorem 34 for the case of compact Lie groups, by a direct application of Lemma 30 to the Schrödinger kernel in the form of (1.3).

We now prove Lemma 44 for $L \cong \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_r$ being the weight lattice (the case for the root lattice or any other sublattice can be proved similarly). First note that as $|H| \lesssim N^{-1}$, for N large enough, we have

$$\left|\frac{\prod_{\alpha\in\Delta^+}\langle\alpha, H\rangle}{D_P}\right| \approx 1.$$

Thus it suffices to show (2.1) replacing $\chi^\mu(\lambda, H)$ by

$$\chi_1^\mu(\lambda, H) = \frac{\sum_{s\in W} \det(s) e^{i\langle s(\lambda+\mu), H\rangle}}{\prod_{\alpha\in\Delta^+}\langle\alpha, H\rangle}. \tag{2.2}$$

2.1. Approach 1: via BGG-Demazure Operators.

The idea is to expand the numerator of $\chi_1^\mu(\lambda, H)$ into a power series of polynomials in $H \in \mathfrak{a}$, which are *anti-invariant* with respect to the Weyl group W, and then to estimate the quotients of these polynomial over the denominator $\prod_{\alpha\in\Delta^+}\langle\alpha, H\rangle$. We will see that these quotients are in fact polynomials in $H \in \mathfrak{a}$, and can be more or less explicitly computed by the *BGG-Demazure operators*. We now review the basic definitions and facts of the BGG-Demazure operators and the related invariant theory. A good reference is Chapter IV in [**Hil82**]. The following theory works for any reduced root system $\Delta \subset \mathfrak{a}^*$.

Let $P(\mathfrak{a})$ be the space of polynomial functions on $\mathfrak{a}$. The orthogonal group $O(\mathfrak{a})$ with respect to the inner product on $\mathfrak{a}$, in particular the Weyl group, acts on $P(\mathfrak{a})$ by

$$(sf)(H) := f(s^{-1}H), \quad s \in O(\mathfrak{a}), \quad f \in P(\mathfrak{a}), \quad H \in \mathfrak{a}.$$

DEFINITION 45. *For $\alpha \in \mathfrak{a}^*$, let $s_\alpha : \mathfrak{a} \to \mathfrak{a}$ denote the reflection about the hyperplane $\{H \in \mathfrak{a} : \alpha(H) = 0\}$, that is,*

$$s_\alpha(H) := H - 2\frac{\alpha(H)}{\langle\alpha,\alpha\rangle}H_\alpha$$

where $H \in \mathfrak{a}$. Here H_α corresponds to α through the identification $\mathfrak{a} \xrightarrow{\sim} \mathfrak{a}^*$. Define the BGG-Demazure operator $\Delta_\alpha : P(\mathfrak{a}) \to P(\mathfrak{a})$ associated to $\alpha \in \mathfrak{a}^*$ by

$$\Delta_\alpha(f) = \frac{f - s_\alpha(f)}{\alpha}.$$

As an example, we compute $\Delta_\alpha(\lambda^m)$ for $\lambda \in \mathfrak{a}^*$.

$$\begin{aligned}\Delta_\alpha(\lambda^m) &= \frac{\lambda^m - \lambda(\cdot - 2\frac{\alpha}{\langle\alpha,\alpha\rangle}H_\alpha)^m}{\alpha}\\ &= \frac{\lambda^m - (\lambda - 2\frac{\langle\lambda,\alpha,\rangle}{\langle\alpha,\alpha\rangle}\alpha)^m}{\alpha}\\ &= \sum_{i=1}^m (-1)^{i-1}\binom{m}{i}\frac{2^i}{\langle\alpha,\alpha\rangle^i}\langle\lambda,\alpha\rangle^i\alpha^{i-1}\lambda^{m-i}. \end{aligned} \tag{2.3}$$

This computation in particular implies that for any $f \in P(\mathfrak{a})$, $\Delta_\alpha(f)$ lowers the degree of f by at least 1.

Let $P(\mathfrak{a})^W$ denote the subspace of $P(\mathfrak{a})$ that are invariant under the action of the Weyl group W, that is,

$$P(\mathfrak{a})^W := \{f \in P(\mathfrak{a}) \mid sf = f \text{ for all } s \in W\}.$$

We call $P(\mathfrak{a})^W$ the space of *invariant polynomials*. We also define

$$P(\mathfrak{a})^W_{\det} := \{f \in P(\mathfrak{a}) \mid sf = (\det s)f \text{ for all } s \in W\}.$$

We call $P(\mathfrak{a})^W_{\det}$ the space of *anti-invariant polynomials*. We have the following proposition which tells that $P(\mathfrak{a})^W_{\det}$ is a free $P(\mathfrak{a})^W$-module of rank 1.

Proposition 46 (Chapter II, Proposition 4.4 in [**Hil82**]). *Define $d_{\det} \in P(\mathfrak{a})$ by*

$$d_{\det} = \prod_{\alpha\in\Delta^+}\alpha.$$

Then $d_{\det} \in P(\mathfrak{a})^W_{\det}$ and

$$P(\mathfrak{a})^W_{\det} = d_{\det}\cdot P(\mathfrak{a})^W.$$

By the above proposition, given any anti-invariant polynomial f, we have $f = d\cdot g$ where g is invariant. We call g the *invariant part* of f. The BGG-Demazure operators provide a procedure that computes the invariant part of any anti-invariant polynomial. We describe this procedure as follows. The Weyl group W is generated by the reflections $s_{\alpha_1},\cdots,s_{\alpha_r}$ where $S = \{\alpha_1,\cdots,\alpha_r\}$ is the set of simple roots. Define the *length* of $s \in W$ to be the smallest number k such that s can be written as $s = s_{\alpha_{i_1}}\cdots s_{\alpha_{i_k}}$. The longest element s in W is of length $L = |\Delta^+| = \frac{d-r}{2}$, and such s is unique (see Section 1.8 in [**Hum90**]). Write $s = s_{\alpha_{i_1}}\cdots s_{\alpha_{i_L}}$. Define

$$\delta = \Delta_{\alpha_{i_1}}\cdots\Delta_{\alpha_{i_L}}$$

and it is well defined in the sense it does not depend on the particular choice of the decomposition $s = s_{\alpha_{i_1}}\cdots s_{\alpha_{i_L}}$ (see Chapter IV, Proposition 1.7 in [**Hil82**]).

Proposition 47 (Chapter IV, Proposition 1.6 in [**Hil82**]). *We have*

$$\delta f = \frac{|W|}{d_{\det}}\cdot f$$

for all $f \in P(\mathfrak{a})^W_{\det}$.

That is, the operator δ produces the invariant part of any anti-invariant polynomial (modulo a multiplicative constant). Now we compute $\delta = \Delta_{\alpha_{i_1}}\cdots\Delta_{\alpha_{i_L}}$ on λ^m. Proceed inductively using (2.3), we arrive at the following proposition.

PROPOSITION 48. *Let $m \geq L$. Then*

$$\delta(\lambda^m) = \sum_{\theta, a(\alpha,\beta), b(\gamma), c(\zeta), \eta \in \mathbb{Z}} (-1)^\theta \prod_{\alpha \leq \beta} \langle \alpha_{i_\alpha}, \alpha_{i_\beta} \rangle^{a(\alpha,\beta)} \prod_\gamma \langle \lambda, \alpha_{i_\gamma} \rangle^{b(\gamma)} \prod_\zeta \alpha_{i_\zeta}^{c(\zeta)} \lambda^\eta$$

such that the following statements are true.
(1) In each term of the sum, $\sum_\gamma b(\gamma) + \eta = m$.
(2) In each term of the sum, $\sum_\zeta c(\zeta) + \eta = m - L$.
(3) In each term of the sum, $\sum_\gamma b(\gamma) - \sum_\zeta c(\zeta) = L$.
(4) In each term of the sum, $|a(\alpha,\beta)| \leq mL$, $0 \leq b(\gamma), c(\zeta), \eta \leq m$.
(5) There are in total less than 3^{mL} terms in the sum.

Note that since each BGG-Demazure operator $\Delta_{\alpha_{i_j}}$ in $\delta = \Delta_{\alpha_{i_1}} \cdots \Delta_{\alpha_{i_L}}$ lowers the degree of polynomials by at least 1, δ lowers the degree by at least L. Thus

$$\delta(\lambda^m) = 0, \quad \text{for } m < L. \tag{2.4}$$

EXAMPLE 49. *We specialize the discussion to the case $M = SU(2)$. Recall that $\mathfrak{a}^* = \mathbb{R}w$ where w is the fundamental weight, and $\Delta = \{\pm\alpha\}$ with $\alpha = 2w$. $P(\mathfrak{a})$ consists of polynomials in the variable $\lambda \in \underset{1 \mapsto w}{\mathbb{R} \cong \mathbb{R}w}$. For $\lambda \in \underset{1 \mapsto w}{\mathbb{R} \cong \mathbb{R}w}$, and $f \in P(\mathfrak{a})$, we have*

$$\begin{aligned} (\delta f)(\lambda) &= \frac{f(\lambda) - f(-\lambda)}{2\lambda}, \\ \delta(\lambda^m) &= \begin{cases} \lambda^{m-1}, & m \text{ odd}, \\ 0, & m \text{ even}, \end{cases} \\ d_{\det}(\lambda) &= 2\lambda. \end{aligned} \tag{2.5}$$

We can now finish the proof of (2.1).

PROOF OF LEMMA 44. Recall that it suffices to prove (2.1) replacing $\chi^\mu(\lambda, H)$ by $\chi_1^\mu(\lambda, H)$ in (2.2). Using power series expansions, write

$$\begin{aligned} \sum_{s \in W} (\det s) e^{i\langle \lambda + \mu, H \rangle} &= \sum_{s \in W} \det s \sum_{m=0}^\infty \frac{1}{m!} (i \langle s(\lambda + \mu), H \rangle)^m \\ &= \sum_{m=0}^\infty \frac{i^m}{m!} \sum_{s \in W} \det s \ \langle s(\lambda + \mu), H \rangle^m. \end{aligned} \tag{2.6}$$

Note that

$$f_m(H) = f_m(\lambda) = f_m(\lambda, H) := \sum_{s \in W} \det s \ \langle s(\lambda + \mu), H \rangle^m \tag{2.7}$$

is an anti-invariant polynomial in H with respect to the Weyl group W, thus by Proposition 47,

$$f_m(H) = \frac{d_{\det}(H)}{|W|} \cdot \delta f_m(H) = \frac{\prod_{\alpha \in \Delta^+} \langle \alpha, H \rangle}{|W|} \cdot \delta f_m(H).$$

This implies that we can rewrite (2.2) into

$$\chi\mu_1(\lambda, H) = \frac{1}{|W|} \sum_{m=0}^\infty \frac{i^m}{m!} \delta f_m(H).$$

Thus to prove (2.1), it suffices to prove that

$$\sum_{m=0}^{\infty} \frac{1}{m!} \left| D_{i_1} \cdots D_{i_k} \left(\delta f_m(\lambda) \right) \right| \lesssim N^{L-k},$$

for all $k \in \mathbb{Z}_{\geq 0}$, uniformly in $|n_i| \lesssim N$, where $\lambda = n_1 w_1 + \cdots + n_r w_r$. Then by (2.7), it suffices to prove that

$$\sum_{m=0}^{\infty} \frac{1}{m!} \left| D_{i_1} \cdots D_{i_k} \left(\delta \left[(s(\lambda+\mu))^m \right] \right) \right| \lesssim N^{L-k}, \quad \forall s \in W.$$

Without loss of generality, it suffices to show

$$\sum_{m=0}^{\infty} \frac{1}{m!} \left| D_{i_1} \cdots D_{i_k} \left(\delta \left[(\lambda+\mu)^m \right] \right) \right| \lesssim N^{L-k}. \tag{2.8}$$

Noting (2.4), it suffices to consider cases when $m \geq L$. We apply Proposition 48 to write

$$\begin{aligned} &\delta((\lambda+\mu)^m)(H) \\ &= \sum_{\theta, a(\alpha,\beta), b(\gamma), c(\zeta), \eta} (-1)^\theta \prod_{\alpha \leq \beta} \langle \alpha_{i_\alpha}, \alpha_{i_\beta} \rangle^{a(\alpha,\beta)} \prod_\gamma \langle \lambda+\mu, \alpha_{i_\gamma} \rangle^{b(\gamma)} \prod_\zeta \langle \alpha_{i_\zeta}, H \rangle^{c(\zeta)} \langle \lambda+\mu, H \rangle^\eta. \end{aligned} \tag{2.9}$$

First note that for $\lambda = n_1 w_1 + \cdots + n_r w_r$, $|n_i| \lesssim N$, $i = 1, \cdots, r$, we have

$$1 \lesssim |\langle \alpha_i, \alpha_j \rangle| \lesssim 1, \quad |\langle \lambda+\mu, \alpha_i \rangle| \lesssim N, \tag{2.10}$$

and by the assumption $|H| \lesssim N^{-1}$,

$$|\langle \alpha_i, H \rangle| \lesssim N^{-1}, \quad |\langle \lambda+\mu, H \rangle| = \left| \left(\sum_{i=1}^r n_i \langle w_i, H \rangle \right) + \langle \mu, H \rangle \right| \lesssim 1. \tag{2.11}$$

These imply that

$$|\delta((\lambda+\mu)^m)(H)| \leq \sum_{\theta, a(\alpha,\beta), b(\gamma), c(\zeta), \eta} C^{\sum_{\alpha,\beta} |a(\alpha,\beta)| + \sum_\gamma b(\gamma) + \sum_\zeta c(\zeta) + \eta} N^{\sum_\gamma c(\gamma) - \sum_\zeta c(\zeta)} \tag{2.12}$$

for some constant C independent of m. Now we derive a similar estimate for $D_i \left(\delta \left[(\lambda+\mu)^m \right] \right)(H)$. By (2.9),

$$\begin{aligned} D_i \left(\delta \left[(\lambda+\mu)^m \right] \right)(H) = &\sum_{\theta, a(\alpha,\beta), b(\gamma), c(\zeta), \eta} (-1)^\theta \prod_{\alpha \leq \beta} \langle \alpha_{i_\alpha}, \alpha_{i_\beta} \rangle^{a(\alpha,\beta)} \prod_\zeta \langle \alpha_{i_\zeta}, H \rangle^{c(\zeta)} \\ &\cdot D_i \left(\prod_\gamma \langle \lambda+\mu, \alpha_{i_\gamma} \rangle^{b(\gamma)} \langle \lambda+\mu, H \rangle^\eta \right). \end{aligned} \tag{2.13}$$

For $\lambda = n_1 w_1 + \cdots + n_r w_r$, we compute that

$$D_i \left(\langle \lambda+\mu, \alpha_{i_\gamma} \rangle \right) = \langle \alpha_i, \alpha_{i_\gamma} \rangle,$$
$$D_i \left(\langle \lambda+\mu, H \rangle \right) = \langle \alpha_i, H \rangle.$$

The above two formulas combined with (2.10), (2.11), and the Leibniz rule (1.2) for D_i imply that

$$\left| D_i \left(\prod_\gamma \langle \lambda+\mu, \alpha_{i_\gamma} \rangle^{b(\gamma)} \langle \lambda+\mu, H \rangle^\eta \right) \right| \leq C^{\sum_\gamma b(\gamma) + \eta} N^{\sum_\gamma b(\gamma) - 1}.$$

This combined with (2.10), (2.11) and (2.13) implies that

$$\left| D_i \left(\delta \left[(\lambda+\mu)^m \right] \right)(H) \right| \lesssim \sum_{\theta, a(\alpha,\beta), b(\gamma), c(\zeta), \eta} C^{\sum_{\alpha,\beta} |a(\alpha,\beta)| + \sum_\gamma b(\gamma) + \sum_\zeta c(\zeta) + \eta} N^{\sum_\gamma b(\gamma) - \sum_\zeta c(\zeta) - 1}.$$

Inductively, we have

$$|D_{i_1}\cdots D_{i_k}\left(\delta\left[(\lambda+\mu)^m\right]\right)(H)|\lesssim\sum_{\theta,a(\alpha,\beta),b(\gamma),c(\zeta),\eta}C^{\sum_{\alpha,\beta}|a(\alpha,\beta)|+\sum_\gamma b(\gamma)+\sum_\zeta c(\zeta)+\eta}N^{\sum_\gamma b(\gamma)-\sum_\zeta c(\zeta)-k},$$

for some constant C independent of m. This by Proposition 48 then implies

$$|D_{i_1}\cdots D_{i_k}\left(\delta\left[(\lambda+\mu)^m\right]\right)(H)|\leq 3^{mL}C^{CmL}N^{L-k}\leq C^mN^{L-k}$$

for some positive constant C independent of m. This estimate implies (2.8), noting that

$$\sum_{m=0}^\infty\frac{C^m}{m!}\lesssim 1. \tag{2.14}$$

This finishes the proof. ☐

2.2. Approach 2: via Harish-Chandra's Integral Formula. This very short approach expresses $\chi_1^\mu(\lambda,H)$ as an integral over the group M, similar to the approach in Section 2 for general symmetric spaces U/K of compact type where the spherical functions are expressed as an integral over K (see (2.2)). We apply the Harish-Chandra's integral formula (see [**HC57**]), which reads

$$\sum_{s\in W}\det(s)e^{\langle s\lambda,\mu\rangle}=\frac{\prod_{\alpha\in\Delta^+}\langle\alpha,\lambda\rangle\cdot\prod_{\alpha\in\Delta^+}\langle\alpha,\mu\rangle}{\prod_{\alpha\in\Delta^+}\langle\alpha,\rho\rangle}\int_M e^{\langle\mathrm{Ad}_m(\lambda),\mu\rangle}\,dm.$$

for $\lambda,\mu\in\mathfrak{m}^{\mathbb{C}}$, and dm is the normalized Haar measure on M. Then we can rewrite $\chi_1^\mu(\lambda,H)$ as

$$\chi_1^\mu(\lambda,H)=\frac{i^{|\Delta^+|}\prod_{\alpha\in\Delta^+}\langle\alpha,\lambda+\rho\rangle}{\prod_{\alpha\in\Delta^+}\langle\alpha,\rho\rangle}\int_M e^{i\langle\lambda+\rho,\mathrm{Ad}_m(H)\rangle}\,dm.$$

Note that $\frac{i^{|\Delta^+|}\prod_{\alpha\in\Delta^+}\langle\alpha,\lambda+\rho\rangle}{\prod_{\alpha\in\Delta^+}\langle\alpha,\rho\rangle}$ is a polynomial in $\lambda\in\Lambda$ of degree $|\Delta^+|=\frac{d-r}{2}$. Also, as $|H|\lesssim N^{-1}$, $|Ad_m(H)|\lesssim N^{-1}$ uniformly in $m\in M$, which implies that the integral $f(\lambda)=\int_M e^{i\langle\lambda+\rho,\mathrm{Ad}_m(H)\rangle}\,dm$ as a function of $\lambda=n_1w_1+\cdots+n_rw_r$ is a pseudo-polynomial of degree 0, uniformly in $|H|\lesssim N^{-1}$. Then by the Leibniz rule, $\chi'(\lambda,H)$ as a function of λ is a pseudo-polynomial of degree $\frac{d-r}{2}$, uniformly in $|H|\lesssim N^{-1}$. This finishes the proof of Lemma 44.

Remark 50. *Note that Lemma 44 can be stated purely in terms of a reduced root system without mentioning the ambient compact Lie group. And it is still true this way. It can be seen either by the approach via BGG-Demazure operators which is purely a root system theoretic argument, or by the fact that, for any reduced root system Δ, there associates to it a unique compact simply connected semisimple Lie group equipped with this root system, thus the approach via Harish-Chandra's integral formula still works, even though the argument explicitly involves the group.*

3. Away From All the Cell Walls

From now on, let P denote the set Δ^+ of positive roots. Using (2.11), (2.12) and (1.1), the Schrödinger kernel (3.11) reads

$$K_N=\sum_{\lambda\in\Lambda^+}\varphi(\frac{-|\lambda+\rho|^2+|\rho|^2}{N^2})e^{it(-|\lambda+\rho|^2+|\rho|^2)}\frac{\prod_{\alpha\in P}\langle\alpha,\lambda+\rho\rangle}{\prod_{\alpha\in P}\langle\alpha,\rho\rangle}\frac{\sum_{s\in W}\det(s)e^{i\langle s(\lambda+\rho),H\rangle}}{e^{-i\langle\rho,H\rangle}\prod_{\alpha\in P}(e^{i\langle\alpha,H\rangle}-1)}.$$

Proposition 51. *We have*

$$|K_N(t,[iH])|\lesssim\frac{N^d}{(\sqrt{q}(1+N\|\frac{t}{2\pi D}-\frac{a}{q}\|^{1/2}))^r} \tag{3.1}$$

for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$*, uniformly for* $\|\frac{1}{2\pi}\langle\alpha, H\rangle\| \gtrsim N^{-1}$ *for all* $\alpha \in \Delta$*,* $H \in \mathfrak{a}$*.*

Proof of Proposition 51. Under the condition that $\|\frac{1}{2\pi}\langle\alpha, H\rangle\| \gtrsim N^{-1}$ for all $\alpha \in \Delta = P \cup (-P)$, we have

$$\left| e^{-i\langle\rho,H\rangle} \prod_{\alpha\in P} (e^{i\langle\alpha,H\rangle} - 1) \right| \gtrsim N^{-L}$$

where $L = |P| = \frac{d-r}{2}$. Using (1.2) and the above inequality, it suffices to prove

$$\left| \sum_{\lambda\in\Lambda} e^{-it|\lambda|^2 + i\lambda(H)} \varphi(\frac{-|\lambda|^2 + |\rho|^2}{N^2}) \prod_{\alpha\in P} \langle\alpha, \lambda\rangle \right| \lesssim \frac{N^{\frac{d+r}{2}}}{(\sqrt{q}(1 + N\|\frac{t}{2\pi D} - \frac{a}{q}\|^{1/2}))^r}.$$

for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$. This is then a direct consequence of Lemma 30. □

Example 52. *We summarize the techniques in Chapter 5 and Section 3 to prove for the special case* $M = SU(2)$ *that*

$$|K_N(t, \theta)| \lesssim \frac{N^3}{\sqrt{q}(1 + N\|\frac{t}{2\pi} - \frac{a}{q}\|^{1/2})} \tag{3.2}$$

for $\frac{t}{2\pi} \in \mathcal{M}_{a,q}$*, uniformly for* θ *lying in the cell* $[0, \pi]$ *(then automatically in the whole maximal torus* $[0, 2\pi)$*). Specialize* (1.2) *and* (1.3) *to the Schrödinger kernel* (3.13)*, we get*

$$K_N(t, \theta) = \frac{e^{it}}{e^{i\theta} - e^{-i\theta}} \sum_{m\in\mathbb{Z}} e^{-itm^2 + im\theta} \varphi(\frac{m^2 - 1}{N^2}) m \tag{3.3}$$

$$= \frac{e^{it}}{2} \sum_{m\in\mathbb{Z}} e^{-itm^2} \varphi(\frac{m^2 - 1}{N^2}) m \cdot \frac{e^{im\theta} - e^{-im\theta}}{e^{i\theta} - e^{-i\theta}}, \quad \theta \in \mathbb{R}/2\pi\mathbb{Z}. \tag{3.4}$$

Scenario 1*:* θ *is away from the two corners* $0, \pi$ *by a distance of* $\gtrsim N^{-1}$*. Then* (3.2) *follows directly from Lemma 30, noting that* $|e^{i\theta} - e^{-i\theta}| \gtrsim N^{-1}$*.*
Scenario 2*:* θ *is close to 0 or* π *by a distance of* $\lesssim N^{-1}$*. Recall that* $\Lambda = \mathbb{Z}w$*,* $\Gamma = \mathbb{Z}\alpha$ *with* $\alpha = 2w$*, thus* $\Lambda/\Gamma \cong \{0, 1\} \cdot w$*. Similar to* (3.2)*, we decompose*

$$K_N(t, \theta) = \frac{e^{it}}{2} \left(K_N^0(t, \theta) + K_N^1(t, \theta) \right),$$

where

$$K_N^0 = \sum_{\substack{m=2k,\\ k\in\mathbb{Z}}} e^{-itm^2} \varphi(\frac{m^2 - 1}{N^2}) m \cdot \frac{e^{im\theta} - e^{-im\theta}}{e^{i\theta} - e^{-i\theta}},$$

$$K_N^1 = \sum_{\substack{m=2k+1,\\ k\in\mathbb{Z}}} e^{-itm^2} \varphi(\frac{m^2 - 1}{N^2}) m \cdot \frac{e^{im\theta} - e^{-im\theta}}{e^{i\theta} - e^{-i\theta}}.$$

Write $\theta = \theta_1 + \theta_2$*, where* $|\theta_1| \lesssim N^{-1}$*, and* $\theta_2 = 0, \pi$*. Then for* $m = 2k$*,* $k \in \mathbb{Z}$*,*

$$\begin{aligned}
\chi_m(\theta) &= \frac{1}{e^{-i\theta}(e^{i2\theta_1} - 1)} \cdot (e^{im\theta_1} - e^{-im\theta_1}) \\
&= \frac{1}{e^{-i\theta}(e^{i2\theta_1} - 1)} \cdot \sum_{n=0}^{\infty} \frac{i^n}{n!} ((m\theta_1)^n - (-m\theta_1)^n) \\
&= \frac{\theta_1}{e^{-i\theta}(e^{i2\theta_1} - 1)} \cdot \sum_{n\ odd} \frac{i^n}{n!} (2\theta_1^{n-1} m^n),
\end{aligned}$$

and similarly for $m = 2k+1$, $k \in \mathbb{Z}$,

$$\chi_m(\theta) = \frac{e^{i\theta_2}\theta_1}{e^{-i\theta}(e^{i2\theta_1}-1)} \cdot \sum_{n\ odd} \frac{i^n}{n!}(2\theta_1^{n-1}m^n).$$

Note that we have been implicitly applying the special case of Proposition 47 that

$$f_n(\theta_1) := (m\theta_1)^n - (-m\theta_1)^n = \theta_1 \cdot \delta f_n = \begin{cases} \theta_1 \cdot 2\theta_1^{n-1}m^n, & n\ odd, \\ 0, & n\ even. \end{cases}$$

If $|k| \lesssim N$, noting that $\left|\frac{\theta_1}{e^{i2\theta_1}-1}\right| \lesssim 1$, then

$$|D^L\chi_{2k}| \lesssim N^{1-L}, \quad |D^L\chi_{2k+1}| \lesssim N^{1-L}, \quad L \in \mathbb{Z}_{\geq 0},$$

where D is the difference operator with respect to the variable k. These two inequalities will give the desired estimates for K_N^0 and K_N^1 respectively and thus for K_N, using Lemma 30.

4. Root Subsystems

To finish proof of part (1) of Theorem 24, considering Theorem 38 and Proposition 51, it suffices to prove 2.3 in the scenarios when $[iH] \in Q$ is away from some of the cell walls by a distance of $\gtrsim N^{-1}$ but stays close to the other cell walls within a distance of $\lesssim N^{-1}$. We will identify these other walls as belonging to a *root subsystem* of the original root system Δ, and then we will decompose the character, the weight lattice as well as the Schrödinger kernel according to this root subsystem, so to make Lemma 30 applicable.

4.1. Identifying Root Subsystems and Rewriting the Character. Fix any $H \in \mathfrak{a}$, let R_H be the subset of the set Δ of roots defined by

$$R_H := \{\alpha \in \Delta : \|\frac{1}{2\pi}\langle\alpha, H\rangle\| \leq N^{-1}\}.$$

Thus

$$\Delta \setminus R_H = \{\alpha \in \Delta : \|\frac{1}{2\pi}\langle\alpha, H\rangle\| > N^{-1}\}.$$

Define

$$\Delta_H := \{\alpha \in \Delta : \alpha \text{ lies in the } \mathbb{Z}\text{-linear span of } R_H\}, \tag{4.1}$$

then $\Delta_H \supset R_H$, and

$$\|\frac{1}{2\pi}\langle\alpha, H\rangle\| \lesssim N^{-1}, \quad \forall \alpha \in \Delta_H, \tag{4.2}$$

with the implicit constant independent of H, and

$$\|\frac{1}{2\pi}\langle\alpha, H\rangle\| > N^{-1}, \quad \forall \alpha \in \Delta \setminus \Delta_H. \tag{4.3}$$

Note that Δ_H is $\mathbb{Z}$-*closed* in Δ, that is, no element in $\Delta \setminus \Delta_H$ lies in the $\mathbb{Z}$-linear span of Δ_H.

PROPOSITION 53. *Δ_H is a reduced root system.*

PROOF. We check the requirements for a reduced root system listed in (2.1) and (2.2). (iii) and (iv) are automatic from the fact that Δ_H is a subset of Δ. (i) comes from the fact that Δ_H is a $\mathbb{Z}$-linear space. (ii) follows from the fact that $s_\alpha\beta$ is a $\mathbb{Z}$-linear combination of α and β, for all $\alpha, \beta \in \Delta_H$, and the fact that Δ_H is a $\mathbb{Z}$-linear space. □

Then we say that Δ_H is a reduced root subsystem of Δ.

Let W_H be the Weyl group associated to Δ_H. W_H is generated by reflections s_α for $\alpha \in \Delta_H$ and thus W_H is considered a subgroup of the Weyl group W of Δ. Let P be a positive system of roots of Δ and define $P_H = P \cap \Delta_H$. Then P_H is a positive system of roots of Δ_H. We rewrite the Weyl character

$$\begin{aligned}\chi_\lambda([iH]) &= \frac{\sum_{s\in W} \det s\ e^{i\langle s(\lambda),H\rangle}}{e^{-i\langle\rho,H\rangle}\prod_{\alpha\in P}(e^{i\langle\alpha,H\rangle}-1)}\\ &= \frac{\frac{1}{|W_H|}\sum_{s_H\in W_H}\sum_{s\in W}\det(s_Hs)\ e^{i\langle (s_Hs)(\lambda),H\rangle}}{e^{-i\langle\rho,H\rangle}\Big(\prod_{\alpha\in P\setminus P_H}(e^{i\langle\alpha,H\rangle}-1)\Big)\big(\prod_{\alpha\in P_H}(e^{i\langle\alpha,H\rangle}-1)\big)}\\ &= \frac{1}{|W_H|e^{-i\langle\rho,H\rangle}\prod_{\alpha\in P\setminus P_H}(e^{i\langle\alpha,H\rangle}-1)}\sum_{s\in W}\det s\cdot\frac{\sum_{s_H\in W_H}\det s_H\ e^{i\langle s_H(s(\lambda)),H\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H\rangle}-1)}\\ &= C(H)\sum_{s\in W}\det s\cdot\frac{\sum_{s_H\in W_H}\det s_H\ e^{i\langle s_H(s(\lambda)),H\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H\rangle}-1)},\end{aligned}$$

where

$$C(H) := \frac{1}{|W_H|e^{-i\langle\rho,H\rangle}\prod_{\alpha\in P\setminus P_H}(e^{i\langle\alpha,H\rangle}-1)}. \tag{4.4}$$

Then by (4.3),

$$|C(H)| \lesssim N^{|P\setminus P_H|}. \tag{4.5}$$

Let V_H be the $\mathbb{R}$-linear span of Δ_H in $\mathfrak{a}^*$ and let $H^\parallel$ be the orthogonal projection of $H\in\mathfrak{a}$ on V_H. Let $H^\perp = H - H^\parallel$. Then $H^\perp$ is orthogonal to V_H and we have

$$\begin{aligned}\chi_\lambda &= C(H)\sum_{s\in W}\det s\cdot\frac{\sum_{s_H\in W_H}\det s_H\ e^{i\langle s_H(s(\lambda)),H^\perp+H^\parallel\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^\perp+H^\parallel\rangle}-1)}\\ &= C(H)\sum_{s\in W}\det s\cdot\frac{\sum_{s_H\in W_H}\det s_H\ e^{i\langle s(\lambda),s_H^{-1}(H^\perp)\rangle}e^{i\langle s(\lambda),s_H^{-1}(H^\parallel)\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^\parallel\rangle}-1)}.\end{aligned}$$

Note that since $H^\perp$ is orthogonal to every root in Δ_H, $H^\perp$ is fixed by the reflection s_α for any $\alpha\in\Delta_H$, which in turn implies that $H^\perp$ is fixed by any $s_H\in W_H$, that is, $s_H(H^\perp)=H^\perp$. Then

$$\chi_\lambda = C(H)\sum_{s\in W}\det s\cdot e^{i\langle s(\lambda),H^\perp\rangle}\cdot\frac{\sum_{s_H\in W_H}\det s_H\ e^{i\langle s(\lambda),s_H^{-1}(H^\parallel)\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^\parallel\rangle}-1)}.$$

Note that by the definition of $H^\parallel$, we have

$$\|\frac{1}{2\pi}\langle\alpha,H^\parallel\rangle\| \lesssim N^{-1},\quad \forall\alpha\in\Delta_H. \tag{4.6}$$

This means that $[iH^\parallel]$ is a corner of the maximal torus associated to Δ_H. We will exploit the oscillatory behavior of χ_λ embodied in the term $e^{i\langle s(\lambda),H^\perp\rangle}$ as well as the polynomial-like behavior embodied in the term $\frac{\sum_{s_H\in W_H}\det s_H\ e^{i\langle s(\lambda),s_H^{-1}(H^\parallel)\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^\parallel\rangle}-1)}$ (similar to the treatment in Section 3, see Lemma **??** below) so to make Lemma 30 applicable.

Using the above formula, we rewrite the Schrödinger kernel (1.3)

$$K_N = \frac{C(H)e^{it|\rho|^2}}{(\prod_{\alpha\in P}\langle\alpha,\rho\rangle)|W|}\sum_{s\in W}\det s\ \cdot K_{N,s} \tag{4.7}$$

where

$$K_{N,s} = \sum_{\lambda\in\Lambda} e^{i\langle s(\lambda),H^\perp\rangle - it|\lambda|^2}\varphi(\frac{-|\lambda|^2+|\rho|^2}{N^2})\left(\prod_{\alpha\in P}\langle\alpha,\lambda\rangle\right)\frac{\sum_{s_H\in W_H}\det s_H\ e^{i\langle s(\lambda),s_H^{-1}(H^\parallel)\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^\parallel\rangle}-1)}.$$

Using (1.4), (1.5) and $s(\Lambda)=\Lambda$ for all $s\in W$, we have

$$K_{N,s} = \det s\ K_{N,\mathbb{1}}$$

where $\mathbb{1}$ is the identity element in W. Then (4.7) becomes

$$K_N = \frac{C(H)e^{it|\rho|^2}}{(\prod_{\alpha\in P}\langle\alpha,\rho\rangle)}K_{N,\mathbb{1}}. \tag{4.8}$$

Proposition 54. *Recall that*

$$K_{N,\mathbb{1}}(t,[iH]) = \sum_{\lambda\in\Lambda} e^{i\langle\lambda,H^\perp\rangle - it|\lambda|^2}\varphi(\frac{-|\lambda|^2+|\rho|^2}{N^2})\left(\prod_{\alpha\in P}\langle\alpha,\lambda\rangle\right)\frac{\sum_{s_H\in W_H}\det s_H\ e^{i\langle\lambda,s_H^{-1}(H^\parallel)\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^\parallel\rangle}-1)}. \tag{4.9}$$

Then

$$|K_{N,\mathbb{1}}(t,[iH])| \lesssim \frac{N^{d-|P\setminus P_H|}}{\left(\sqrt{q}(1+N|\frac{t}{2\pi D}-\frac{a}{q}|^{1/2})\right)^r} \tag{4.10}$$

for $\frac{t}{2\pi D}\in\mathcal{M}_{a,q}$, *uniformly in* $H\in\mathfrak{a}$.

Noting (4.5) and (4.8), the above proposition directly implies part (1) of Theorem 24.

Example 55. *The following Figure 1 is an illustration of the decomposition of the maximal torus of* $SU(3)$ *according to the values of* $\|\frac{1}{2\pi}\langle\alpha,H\rangle\|$, $\alpha\in\Delta$. *Here* $\Delta^+=\{\alpha_1,\alpha_2,\alpha_3=\alpha_1+\alpha_2\}$. *The three proper subsystems of* Δ *are* $\{\pm\alpha_i\}$, $i=1,2,3$. *The association of* Δ_H *to* H *is as follows.*

$[iH]\in$ *regions of color* (red) $\Leftrightarrow\ \Delta_H=\Delta$,

$[iH]\in$ *regions of color* (yellow) $\Leftrightarrow\ \Delta_H=\{\pm\alpha_1\}$,

$[iH]\in$ *regions of color* (pink) $\Leftrightarrow\ \Delta_H=\{\pm\alpha_2\}$,

$[iH]\in$ *regions of color* (blue) $\Leftrightarrow\ \Delta_H=\{\pm\alpha_3\}$,

$[iH]\in$ *regions of color* (green) $\Leftrightarrow\ \Delta_H=\varnothing$.

4.2. Decomposition of the Weight Lattice. To prove Proposition 54, we now make a decomposition of the weight lattice Λ according to the reduced root subsystem Δ_H. Let Proj_U denote the orthogonal projection map from the ambient inner product space onto any subspace U.

Lemma 56. *Let* Φ *be a reduced root system in the space* V *with the associated weight lattice* Λ_Φ. *Let* Ψ *be a reduced root subsystem of* Φ. *Then let* Γ_Ψ *and* Λ_Φ *be the root lattice and weight lattice associated to* Ψ *respectively. Let* V_Ψ *be the* $\mathbb{R}$*-linear span of* Ψ *in* V. *Let* Υ_Ψ *be the image of the orthogonal projection of* Λ_Φ *onto* V_Ψ. *Then the following statements hold true.*
(1) Υ_Ψ *is a lattice and* $\Gamma_\Psi\subset\Upsilon_\Psi\subset\Lambda_\Psi$. *In particular, the rank of* Υ_Ψ *equals the rank of* Γ_Ψ *as well as* Λ_Ψ.
(2) Let the rank of Υ_Ψ *and* Λ_Φ *be* r *and* R *respectively. Let* $\{w_1,\cdots,w_r\}$ *be a* $\mathbb{Z}$*-basis of* Υ_Ψ. *Pick any*

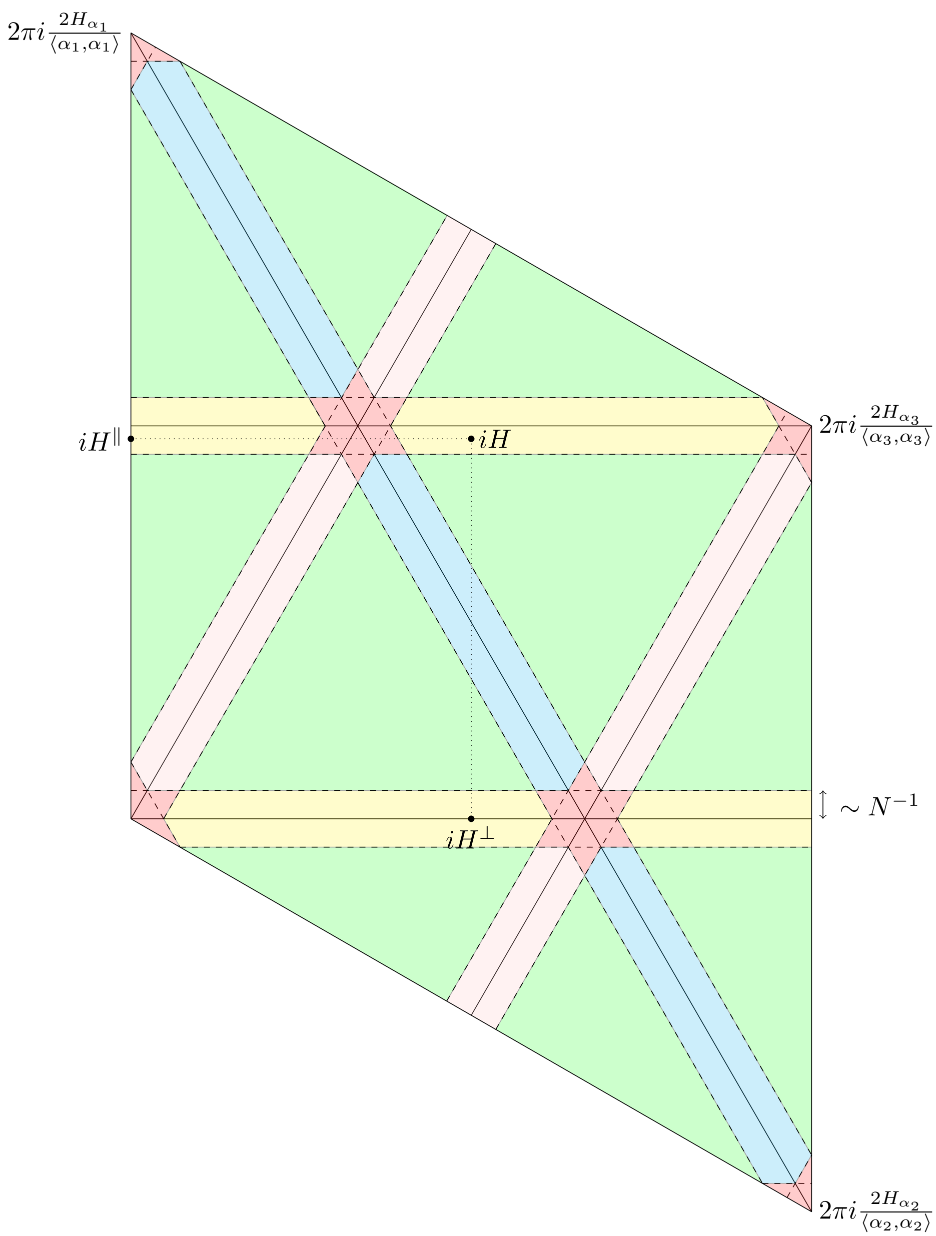

FIGURE 1. Decomposition of the maximal torus of SU(3) according to the values of $\|\frac{1}{2\pi}\langle\alpha, H\rangle\|$, $\alpha \in \Delta$

$\{u_1, \cdots, u_r\} \subset \Lambda_\Phi$ *such that* $Proj_{V_\Psi}(u_i) = w_i$, $i = 1, \cdots, r$. *Then we can extend* $\{u_1, \cdots, u_r\}$ *into a basis* $\{u_1, \cdots, u_r, u_{r+1}, \cdots, u_R\}$ *of* Λ_Φ. *Furthermore, we can pick* $\{u_{r+1}, \cdots, u_R\}$ *such that* $Proj_{V_\Psi}(u_i) = 0$ *for* $i = r+1, \cdots, R$.

PROOF. Part (1). It's clear that Υ_Ψ is a lattice. Let Γ_Φ be the root lattice associated to Φ. Then $\Gamma_\Psi \subset \Gamma_\Phi$. Then

$$\Gamma_\Psi = \mathrm{Proj}_{V_\Psi}(\Gamma_\Psi) \subset \mathrm{Proj}_{V_\Psi}(\Gamma_\Phi) \subset \mathrm{Proj}_{V_\Psi}(\Lambda_\Phi) = \Upsilon_\Psi.$$

On the other hand, for any $\mu \in \Lambda_\Phi$, $\alpha \in \Gamma_\Psi$, $\langle \mathrm{Proj}_{V_\Psi}(\mu), \alpha\rangle = \langle \mu, \alpha\rangle$. This in particular implies that

$$2\frac{\langle \mathrm{Proj}_{V_\Psi}(\mu), \alpha\rangle}{\langle \alpha, \alpha\rangle} = 2\frac{\langle \mu, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z}, \quad \text{for all } \mu \in \Lambda_\Phi,\ \alpha \in \Gamma_\Psi.$$

This implies that $\mathrm{Proj}_{V_\Psi}(\mu) \in \Lambda_\Psi$ for all $\mu \in \Lambda_\Phi$, that is, $\Upsilon_\Psi = \mathrm{Prof}_{V_\Psi}(\Lambda_\Phi) \subset \Lambda_\Psi$.

Part (2). Let $S_\Phi := \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_r$, then S_Φ is a sublattice of Λ_Φ of rank r. By the theory of sublattices (see Chapter II, Theorem 1.6 in [**Hun80**]), there exists a basis $\{u_1', \cdots, u_R'\}$ of Λ_Φ and positive integers $d_1|d_2|\cdots|d_r$ such that $\{d_1u_1', \cdots, d_ru_r'\}$ is a basis of S_Φ. Then we must have $d_1 = d_2 = \cdots = d_r = 1$, since

$$\begin{aligned}\mathbb{Z}d_1\mathrm{Proj}_{V_\Psi}(u_1') + \cdots + \mathbb{Z}d_r\mathrm{Proj}_{V_\Psi}(u_r') &= \mathrm{Proj}_{V_\Psi}(S_\Phi)\\ &= \mathrm{Proj}_{V_\Psi}(\Lambda_\Phi) \supset \mathbb{Z}\mathrm{Proj}_{V_\Psi}(u_1') + \cdots + \mathbb{Z}\mathrm{Proj}_{V_\Psi}(u_r')\end{aligned}$$

and that $u_1', \cdots, u_r'$ are $\mathbb{R}$-linear independent. Thus we have

$$S_\Phi = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_r = \mathbb{Z}u_1' + \cdots + \mathbb{Z}u_r'$$

and then $\{u_1, \cdots, u_r, u_{r+1}', \cdots, u_R'\}$ is also a basis of Λ_Φ. Furthermore, by adding a $\mathbb{Z}$-linear combination of $u_1, \cdots, u_r$ to each of $u_{r+1}', \cdots, u_R'$, we can assume that $\mathrm{Proj}_{V_\Psi}(u_i') = 0$, for $i = r+1, \cdots, R$. □

EXAMPLE 57. *Continue the example of $SU(3)$. Recall that the three proper subsystems of the root system $\Delta = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3\}$ are $\{\pm\alpha_i\}$, $i = 1, 2, 3$. Then the weight lattice of Δ projects on $\mathbb{R}\alpha_i$ to be the weight lattice $\mathbb{Z}\frac{\alpha_i}{2}$ associated to the root system $\{\pm\alpha_i\}$, $i = 1, 2, 3$.*

We apply the above lemma to the reduced root subsystem Δ_H of Δ. Recall that V_H denotes the $\mathbb{R}$-linear span of Δ_H in $\mathfrak{a}^*$. Let Γ_H be the root lattice for Φ_H, and let

$$\Upsilon_H := \mathrm{Proj}_{V_H}(\Lambda). \tag{4.11}$$

Then by the above lemma, we have

$$\Upsilon_H \supset \Gamma_H. \tag{4.12}$$

Let r_H be the rank of Δ_H as well as of Γ_H and Υ_H, and let $\{w_1, \cdots, w_{r_H}\} \subset \Upsilon_H$ such that

$$\Upsilon_H = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_{r_H}.$$

Pick $\{u_1, \cdots, u_{r_H}\} \subset \Lambda$ such that

$$\mathrm{Proj}_{V_H}(u_i) = w_i, \quad i = 1, \cdots, r_H.$$

Then by the above lemma, we can extend $\{u_1, \cdots, u_{r_H}\}$ into a basis $\{u_1, \cdots, u_r\}$ of Λ, such that

$$\mathrm{Proj}_{V_H}(u_i) = 0, \quad i = r_H + 1, \cdots, r, \tag{4.13}$$

with

$$\Lambda = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_r.$$

Denote

$$\Upsilon_H' = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_{r_H} \subset \Lambda,$$

then

$$\mathrm{Proj}_{V_H} : \Upsilon_H' \xrightarrow{\sim} \Upsilon_H.$$

Recalling (4.12), let Γ'_H be the sublattice of Υ'_H corresponding to $\Gamma_H \subset \Upsilon_H$ under this isomorphism. More precisely, let $\{\alpha_1, \cdots, \alpha_{r_H}\}$ be a simple system of roots for Γ_H, then

$$\mathrm{Proj}_{V_H} : \Gamma'_H = \mathbb{Z}\alpha'_1 + \cdots + \mathbb{Z}\alpha'_{r_H} \xrightarrow{\sim} \Gamma_H = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_{r_H}, \quad \alpha'_i \mapsto \alpha_i, \quad i = 1, \cdots, r_H, \tag{4.14}$$

and we have

$$\Upsilon'_H/\Gamma'_H \cong \Upsilon_H/\Gamma_H, \quad |\Upsilon'_H/\Gamma'_H| = |\Upsilon_H/\Gamma_H| < \infty. \tag{4.15}$$

We decompose the weight lattice

$$\Lambda = \bigsqcup_{\mu \in \Upsilon'_H/\Gamma'_H} \left(\mu + \Gamma'_H + \mathbb{Z}u_{r_H+1} + \cdots + \mathbb{Z}u_r\right),$$

then

$$K_{N,\mathbb{1}} = \sum_{\substack{\mu \in \Upsilon'_H/\Gamma'_H, \\ \lambda'_1 = n_1\alpha'_1 + \cdots + n_{r_H}\alpha'_{r_H}, \\ \lambda_2 = n_{r_H+1}u_{r_H+1} + \cdots + n_r u_r}} e^{i\langle \mu+\lambda'_1+\lambda_2, H^\perp\rangle - it|\mu+\lambda'_1+\lambda_2|^2} \varphi\left(\frac{-|\mu+\lambda'_1+\lambda_2|^2 + |\rho|^2}{N^2}\right)$$
$$\cdot \left(\prod_{\alpha \in P} \langle \alpha, \mu + \lambda'_1 + \lambda_2\rangle\right) \frac{\sum_{s_H \in W_H} \det s_H \; e^{i\langle \mu+\lambda'_1+\lambda_2, s_H^{-1}(H^\parallel)\rangle}}{\prod_{\alpha \in P_H} (e^{i\langle \alpha, H^\parallel\rangle} - 1)}.$$

Note that (4.13) implies for $\lambda_2 = n_{r_H+1}u_{r_H+1} + \cdots + n_r u_r$ that

$$\langle \lambda_2, s_H^{-1}(H^\parallel)\rangle = 0,$$

and (4.14) implies for $\lambda'_1 = n_1\alpha'_1 + \cdots + n_{r_H}\alpha'_{r_H}$ that

$$\langle \lambda'_1, s_H^{-1}(H^\parallel)\rangle = \langle \lambda_1, s_H^{-1}(H^\parallel)\rangle = \langle s_H(\lambda_1), H^\parallel\rangle$$

where $\lambda_1 = n_1\alpha_1 + \cdots + n_{r_H}\alpha_{r_H} \in V_H$. Similarly, also note that

$$\langle \mu, s_H^{-1}(H^\parallel)\rangle = \langle \mu^\parallel, s_H^{-1}(H^\parallel)\rangle = \langle s_H(\mu^\parallel), H^\parallel\rangle, \quad \text{where } \mu^\parallel := \mathrm{Proj}_{V_H}(\mu).$$

Thus we rewrite

$$K_{N,\mathbb{1}} = \sum_{\mu \in \Upsilon'_H/\Gamma'_H} \sum_{\substack{(n_1,\cdots,n_r) \in \mathbb{Z}^r, \\ \lambda'_1 = n_1\alpha'_1 + \cdots + n_{r_H}\alpha'_{r_H}, \\ \lambda_1 = n_1\alpha_1 + \cdots + n_{r_H}\alpha_{r_H}, \\ \lambda_2 = n_{r_H+1}u_{r_H+1} + \cdots + n_r u_r}} e^{i\langle \mu+\lambda'_1+\lambda_2, H^\perp\rangle - it|\mu+\lambda'_1+\lambda_2|^2} \varphi\left(\frac{-|\mu+\lambda'_1+\lambda_2|^2 + |\rho|^2}{N^2}\right)$$
$$\cdot \left(\prod_{\alpha \in P} \langle \alpha, \mu + \lambda'_1 + \lambda_2\rangle\right) \frac{\sum_{s_H \in W_H} \det s_H \; e^{i\langle s_H(\mu^\parallel+\lambda_1), H^\parallel\rangle}}{\prod_{\alpha \in P_H} (e^{i\langle \alpha, H^\parallel\rangle} - 1)}.$$

Remark 58. *We have that in the above formula*

$$\chi_{\mu^\parallel+\lambda_1}(H^\parallel) := \frac{\sum_{s_H \in W_H} \det s_H \; e^{i\langle s_H(\mu^\parallel+\lambda_1), H^\parallel\rangle}}{\prod_{\alpha \in P_H} (e^{i\langle \alpha, H^\parallel\rangle} - 1)} \tag{4.16}$$

is a character associated to the weight $\mu^\parallel + \lambda_1$ of the reduced root subsystem Δ_H, noting that $\mu^\parallel \in \mathrm{Proj}_{V_H}(\Lambda)$ lies in the weight lattice of Δ_H by Lemma 56.

Noting (4.15), Proposition 54 reduces to the following.

PROPOSITION 59. *For $\mu \in \Upsilon'_H/\Gamma'_H$, let*

$$K^{\mu}_{N,\mathbb{1}}(t,[iH]) := \sum_{\substack{(n_1,\cdots,n_r)\in\mathbb{Z}^r,\\ \lambda'_1=n_1\alpha'_1+\cdots+n_{r_H}\alpha'_{r_H},\\ \lambda_1=n_1\alpha_1+\cdots+n_{r_H}\alpha_{r_H},\\ \lambda_2=n_{r_H+1}u_{r_H+1}+\cdots+n_r u_r,\\ n_1,\cdots,n_r\in\mathbb{Z}}} e^{i\langle\mu+\lambda'_1+\lambda_2,H^{\perp}\rangle-it|\mu+\lambda'_1+\lambda_2|^2}\varphi\Big(\frac{-|\mu+\lambda'_1+\lambda_2|^2+|\rho|^2}{N^2}\Big)$$

$$\cdot\left(\prod_{\alpha\in P}\langle\alpha,\mu+\lambda'_1+\lambda_2\rangle\right)\frac{\sum_{s_H\in W_H}\det s_H\ e^{i\langle s_H(\mu^{\parallel}+\lambda_1),H^{\parallel}\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^{\parallel}\rangle}-1)}.$$

Then

$$|K^{\mu}_{N,\mathbb{1}}(t,[iH])| \lesssim \frac{N^{d-|P\setminus P_H|}}{\left(\sqrt{q}(1+N|\frac{t}{2\pi D}-\frac{a}{q}|^{1/2})\right)^r} \tag{4.17}$$

for $\frac{t}{2\pi D}\in\mathcal{M}_{a,q}$, uniformly in $H\in\mathfrak{a}$.

PROOF. We apply Lemma 30 to the lattice $\mathbb{Z}\alpha'_1+\cdots+\mathbb{Z}\alpha'_{r_H}+\mathbb{Z}u_{r_H+1}+\cdots+\mathbb{Z}u_r$. Let

$$\chi(\lambda_1,H^{\parallel}) = \frac{\sum_{s_H\in W_H}\det s_H\ e^{i\langle s_H(\mu^{\parallel}+\lambda_1),H^{\parallel}\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^{\parallel}\rangle}-1)}.$$

Viewing $\prod_{\alpha\in P}\langle\alpha,\mu+\lambda'_1+\lambda_2\rangle\chi(\lambda_1,H^{\parallel})$ as a function on the lattice $(n_1,\cdots,n_r)\in\mathbb{Z}^r$, where $\lambda'_1=n_1\alpha'_1+\cdots+n_{r_H}\alpha'_{r_H}$, $\lambda_1=n_1\alpha_1+\cdots+n_{r_H}\alpha_{r_H}$, $\lambda_2=n_{r_H+1}u_{r_H+1}+\cdots+n_r u_r$, then it suffices to show that it satisfies estimate of the form (1.3)

$$\left|D_{i_1}\cdots D_{i_k}\left(\prod_{\alpha\in P}\langle\alpha,\mu+\lambda'_1+\lambda_2\rangle\chi(\lambda_1,H^{\parallel})\right)\right| \lesssim N^{d-|P\setminus P_H|-r-k},$$

uniformly for $|n_i|\lesssim N$, $i=1,\cdots,r$. Since $\prod_{\alpha\in P}\langle\alpha,\mu+\lambda'_1+\lambda_2\rangle$ is a polynomial of degree $|P|$,

$$\left|D_{i_1}\cdots D_{i_k}\left(\prod_{\alpha\in P}\langle\alpha,\mu+\lambda'_1+\lambda_2\rangle\right)\right| \lesssim N^{|P|-k}.$$

Thus by the Leibniz rule (1.2) for the D_i's, it suffices to show that

$$\left|D_{i_1}\cdots D_{i_k}(\chi(\lambda_1,H^{\parallel}))\right| \lesssim N^{d-|P\setminus P_H|-r-|P|-k} = N^{|P_H|-k}. \tag{4.18}$$

Since $\chi(\lambda_1)$ does not involve the variables $n_{r_H+1},\cdots,n_r$, it suffices to prove (4.18) for $1\le i_1,\cdots,i_k\le r_H$. Recall (4.6), then (4.18) follows by noting Remark 58 and applying Lemma **??** to the reduced root system Δ_H and the proof is finished. ☐

5. L^p Estimates

We prove in this section $L^p(M)$ estimates of the Schrödinger kernel for p not necessarily equal to infinity. Though they are not used in the proof of the main theorem, they encapsulate the essential results in the proof of the $L^\infty(M)$ estimates and are of independent interest.

PROPOSITION 60. *Let K_N be the Schrödinger kernel as in* (3.11). *Then for any $p>3$, we have*

$$\|K_N(t,\cdot)\|_{L^p(M)} \lesssim \frac{N^{d-\frac{d}{p}}}{(\sqrt{q}(1+N\|\frac{t}{2\pi D}-\frac{a}{q}\|^{1/2}))^r} \tag{5.1}$$

for $\frac{t}{2\pi D}\in\mathcal{M}_{a,q}$.

PROOF. As a linear combination of characters, the Schrödinger kernel $K_N(t,\cdot)$ is invariant under the adjoint action. Then we can apply to it the Weyl integration formula (2.14)

$$\|K_N(t,\cdot)\|_{L^p(M)}^p = \frac{1}{|W|}\int_A |K_N(t,a)|^p |D_P(a)|^2\, da. \tag{5.2}$$

We have shown in Section 4 that each $H \in \mathfrak{a}$ is associated to a root subsystem Δ_H such that (4.2) and (4.3) hold. Note that there are finitely many root subsystems of a given root system, thus A is covered by finitely many subsets R of the form

$$R = \{[iH] \in A : \|\frac{1}{2\pi}\langle\alpha, H\rangle\| \lesssim N^{-1}, \forall \alpha \in \Psi; \|\frac{1}{2\pi}\langle\alpha, H\rangle\| > N^{-1}, \forall \alpha \in \Delta \setminus \Psi\} \tag{5.3}$$

where Ψ is a root subsystem of Δ. Thus to prove (5.1), using (5.2), it suffices to show

$$\int_R |K_N(t,[iH])|^p |D_p(H)|^2\, dH \lesssim \left(\frac{N^d}{(\sqrt{q}(1+N\|\frac{t}{2\pi D}-\frac{a}{q}\|^{1/2}))^r}\right)^p N^{-d}. \tag{5.4}$$

By (4.5), (4.8) and (4.10), we have

$$K_N(t,[iH]) \lesssim \frac{1}{\prod_{\alpha\in P\setminus Q}(e^{i\langle\alpha,H\rangle}-1)} \cdot \frac{N^{d-|P\setminus Q|}}{\left(\sqrt{q}(1+N|\frac{t}{2\pi D}-\frac{a}{q}|^{1/2})\right)^r}$$

where P, Q are respectively the sets of positive roots of Δ and Ψ with $P \supset Q$. Recall $D_P(H) = \prod_{\alpha\in P}(e^{i\langle\alpha,H\rangle}-1)$, (5.4) is then reduced to

$$\int_R \left|\frac{1}{\prod_{\alpha\in P\setminus Q}(e^{i\langle\alpha,H\rangle}-1)}\right|^{p-2} \left|\prod_{\alpha\in Q}(e^{i\langle\alpha,H\rangle}-1)\right|^2\, dH \lesssim N^{p|P\setminus Q|-d}.$$

Using

$$|e^{i\langle\alpha,H\rangle}-1| \approx \|\frac{1}{2\pi}\langle\alpha,H\rangle\|,$$

it suffices to show

$$\int_R \left|\frac{1}{\prod_{\alpha\in P\setminus Q}\|\frac{1}{2\pi}\langle\alpha,H\rangle\|}\right|^{p-2} \left|\prod_{\alpha\in Q}\|\frac{1}{2\pi}\langle\alpha,H\rangle\|\right|^2\, dH \lesssim N^{p|P\setminus Q|-d}. \tag{5.5}$$

For each $H \in \mathfrak{a}$, write

$$H = H' + H_0$$

such that

$$\|\frac{1}{2\pi}\langle\alpha,H\rangle\| = |\frac{1}{2\pi}\langle\alpha,H'\rangle|, \quad \langle\alpha,H_0\rangle \in 2\pi\mathbb{Z}, \quad \forall\alpha\in P.$$

Then write

$$R \subset \bigcup_{[iH_0] \text{ is a corner}} R' + [iH_0] \tag{5.6}$$

where

$$R' = \{[iH] \in A : |\frac{1}{2\pi}\langle\alpha,H\rangle| \lesssim N^{-1}, \forall\alpha\in Q; |\frac{1}{2\pi}\langle\alpha,H\rangle| > N^{-1}, \forall\alpha\in P\setminus Q\}. \tag{5.7}$$

Recall that there are only finitely many corners. Thus using (5.6), (5.5) is further reduced to

$$\int_{R'} \left| \frac{1}{\prod_{\alpha \in P \setminus Q} |\frac{1}{2\pi}\langle \alpha, H\rangle|} \right|^{p-2} \left| \prod_{\alpha \in Q} |\frac{1}{2\pi}\langle \alpha, H\rangle| \right|^2 dH \lesssim N^{p|P\setminus Q|-d}. \tag{5.8}$$

Now we reparametrize the maximal torus A by

$$H = \sum_{i=1}^{r} t_i H_{w_i}, \quad (t_1, \cdots, t_r) \in D$$

where $\{w_1, \cdots, w_r\}$ is the set of fundamental weights associated to a set $\{\alpha_1, \cdots, \alpha_r\}$ of simple roots and H_{w_i} corresponds to w_i by $\mathfrak{a} \xrightarrow{\sim} \mathfrak{a}^*$, and D is a bounded domain in $\mathbb{R}^r$. Then the normalized Haar measure dH equals

$$dH = C dt_1 \cdots dt_r$$

for some constant C. Let $s \leq r$ such that

$$\{\alpha_1, \cdots, \alpha_s\} \subset P \setminus Q,$$
$$\{\alpha_{s+1}, \cdots, \alpha_r\} \subset Q.$$

Using (5.7), we estimate

$$\begin{aligned} &\int_{R'} \left| \frac{1}{\prod_{\alpha \in P \setminus Q} |\frac{1}{2\pi}\langle \alpha, H\rangle|} \right|^{p-2} \left| \prod_{\alpha \in Q} |\frac{1}{2\pi}\langle \alpha, H\rangle| \right|^2 dH \\ \lesssim &\int_{\substack{|t_1|,\cdots,|t_s| \gtrsim N^{-1}, \\ |t_{s+1}|,\cdots,|t_r| \lesssim N^{-1}}} \frac{1}{|t_1 \cdots t_s|^{p-2}} N^{(p-2)(|P\setminus Q|-s)} N^{-2|Q|} \, dt_1 \cdots dt_r. \end{aligned} \tag{5.9}$$

If $p > 3$, the above is bounded by

$$\lesssim N^{(p-2)(|P\setminus Q|-s)} N^{-2|Q|} N^{s(p-3)-(r-s)} = N^{p|P\setminus Q|-d},$$

noting that $2|P \setminus Q| + 2|Q| + r = 2|P| + r = d$. □

REMARK 61. *The requirement $p > 3$ is by no means optimal. The estimate in* (5.9) *may be improved to lower the exponent p. We conjecture that* (5.1) *holds for all $p > p_r$ such that $\lim_{r\to\infty} p_r = 2$, r being the rank of M.*

APPENDIX

A. Proof of an Interpolation Lemma

LEMMA A.1. *Let (X,μ) and (Y,ν) be σ-finite measure spaces. Let $p_0,p_1,q_0,q_1\in[1,\infty]$, $p_0\neq p_1$. Suppose that T is a linear operator from $L^{p_0}(X,\mu)+L^{p_1}(X,\mu)$ to $L^{q_0}(Y,\nu)+L^{q_1}(Y,\nu)$ such that*

$$\|Tf\|_{L^{q_0}}\leq A\|f\|_{L^{p_0}},\ \forall f\in L^{p_0}, \tag{A.1}$$

$$\|Tf\|_{L^{q_1}}\leq B\|f\|_{L^{p_1}}+D\|f\|_{L^{p_0}},\ \forall f\in L^{p_1}, \tag{A.2}$$

for some positive constants A,B,D. Let $0<\theta<1$ and

$$\frac{1}{p_\theta}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\ \frac{1}{q_\theta}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}.$$

Then for some universal constant C,

$$\|Tf\|_{L^{q_\theta}}\leq C(A^{1-\theta}B^{\theta}\|f\|_{p_\theta}+A^{1-\theta}D^{\theta}\|f\|_{p_0}),\ \forall f\in L^{p_\theta}\cap L^{p_0}. \tag{A.3}$$

PROOF. [1] By scaling the measure ν, noting the assumption $p_0\neq p_1$, we can assume that

$$B=D.$$

We now use complex interpolation theory (see Chapter 4 and 5 in [**BL76**] as a reference) to prove the lemma. Let $(X_0,X_1)_\theta$ denote the *complex interpolation space* between compatible complex Banach spaces X_0 and X_1 of parameter θ. By Theorem 4.1.2 in [**BL76**], it suffices to prove

$$(L^{q_0},L^{q_1})_\theta=L^{q_\theta},$$
$$(L^{p_0},L^{p_1}\cap L^{p_0})_\theta=L^{p_\theta}\cap L^{p_0},$$

in the sense that the norm on either side of the equation is bounded by the norm on the other side multiplied by a **universal** positive constant. The first equation is given by Theorem 5.1.1 in [**BL76**]. The second equation follows by the same line of proof of Theorem 3 in [**Rie12**]. In fact, we can generalize it to

$$(L^{p_0}\cap L^{p},L^{p_1}\cap L^{p})=L^{p_\theta}\cap L^{p} \tag{A.4}$$

for either $1\leq p\leq p_0,p_1\leq\infty$, or $1\leq p_0,p_1\leq p\leq\infty$. For the sake of completeness, we sketch the proof here. We prove the case when $1\leq p\leq p_0,p_1\leq\infty$, and the other case can be proved similarly. By Theorem 4 in [**Rie12**], given any $f\in L^1(M)+L^\infty(M)$ (originally stated with respect to a domain of $\mathbb{R}^n$, but can be generalized to any σ-measure space M by its proof), there exist linear maps

$$S_1:L^1(M)+L^\infty\to L^1(0,1),\quad S_2:L^1(M)+L^\infty(M)\to l^\infty$$
$$T_1:L^1(0,1)\to L^1(M)+L^\infty(M),\quad T_2:l^\infty\to L^1(M)+L^\infty(M),$$

such that

$$f=T_1S_1f+T_2S_2f \tag{A.5}$$

[1]The author thanks `mathoverflow.net` for providing a forum where he could ask about the proof and be provided with an authoritative reference.

holds almost everywhere, and

$$\|S_1 u\|_{L^r(0,1)} \le \|u\|_{L^r(M)}, \quad \|S_2 u\|_{l^r} \le \|u\|_{L^r(M)},$$
$$\|T_1 u\|_{L^r(M)} \le \|u\|_{L^r(0,1)}, \quad \|T_2 u\|_{L^r(M)} \le \|u\|_{l^r}$$

for all $1 \le r \le \infty$ and all u in the respective Lebesgue spaces. Note that for all $p \le r$,

$$\|u\|_{L^p(0,1)} \le \|u\|_{L^r(0,1)}, \quad \|u\|_{l^r} \le \|u\|_{l^p}$$

for u in the respective Lebesgue spaces, whence we have for all $p \le r$,

$$\|S_1 u\|_{L^r(0,1)} \le \|u\|_{L^r(M)\cap L^p(M)}, \tag{A.6}$$
$$\|S_2 u\|_{l^p} \le \|u\|_{L^r(M)\cap L^p(M)}, \tag{A.7}$$
$$\|T_1 u\|_{L^r(M)\cap L^p(M)} \le \|u\|_{L^r(0,1)}, \tag{A.8}$$
$$\|T_2 u\|_{L^r(M)\cap L^p(M)} \le \|u\|_{l^p}. \tag{A.9}$$

Then by Theorem 4.1.2 and 5.1.1 in [**BL76**], the above inequalities imply

$$\|S_1 u\|_{L^{p_\theta}(0,1)} \le \|u\|_{(L^{p_0}(M)\cap L^p(M), L^{p_1}(M)\cap L^p(M))_\theta}, \tag{A.10}$$
$$\|S_2 u\|_{l^{p_\theta}} \le \|u\|_{(L^{p_0}(M)\cap L^p(M), L^{p_1}(M)\cap L^p(M))_\theta}, \tag{A.11}$$
$$\|T_1 u\|_{(L^{p_0}(M)\cap L^p(M), L^{p_1}(M)\cap L^p(M))_\theta} \le \|u\|_{L^{p_\theta}(0,1)}, \tag{A.12}$$
$$\|T_2 u\|_{(L^{p_0}(M)\cap L^p(M), L^{p_1}(M)\cap L^p(M))_\theta} \le \|u\|_{l^{p_\theta}}. \tag{A.13}$$

Now let $f \in L^{p_\theta}(M) \cap L^p(M)$ and let the linear maps S_1, S_2, T_1, T_2 be the maps defined as above for f. Now (A.6), (A.12), (A.7), (A.13) imply

$$\|T_1 S_1 f\|_{(L^{p_0}(M)\cap L^p(M), L^{p_1}(M)\cap L^p(M))_\theta} \le \|f\|_{L^{p_\theta}(M)\cap L^p(M)},$$
$$\|T_2 S_2 f\|_{(L^{p_0}(M)\cap L^p(M), L^{p_1}(M)\cap L^p(M))_\theta} \le \|f\|_{L^{p_\theta}(M)\cap L^p(M)},$$

then by (A.5),

$$\|f\|_{(L^{p_0}(M)\cap L^p(M), L^{p_1}(M)\cap L^p(M))_\theta} \le 2\|f\|_{L^{p_\theta}(M)\cap L^p(M)}.$$

On the other hand, let $f \in (L^{p_0}(M) \cap L^p(M), L^{p_1}(M) \cap L^p(M))_\theta$ and let the linear maps S_1, S_2, T_1, T_2 be the maps defined as above for this f. Then (A.10), (A.8), (A.11), (A.9) imply

$$\|T_1 S_1 f\|_{L^{p_\theta}(M)\cap L^p(M)} \le \|f\|_{(L^{p_0}(M)\cap L^p(M), L^{p_1}(M)\cap L^p(M))_\theta},$$
$$\|T_2 S_2 f\|_{L^{p_\theta}(M)\cap L^p(M)} \le \|f\|_{(L^{p_0}(M)\cap L^p(M), L^{p_1}(M)\cap L^p(M))_\theta},$$

which imply by (A.5)

$$\|f\|_{L^{p_\theta}(M)\cap L^p(M)} \le 2\|f\|_{(L^{p_0}(M)\cap L^p(M), L^{p_1}(M)\cap L^p(M))_\theta}.$$

Thus (A.4) is proved, and the lemma follows. □

B. Proof of a Major Arc Lemma

LEMMA B.1. *Let $N \in \mathbb{N}$, $a \in \mathbb{Z}_{\geq 0}$, $q \in \mathbb{N}$, $a < q$, $(a,q) = 1$, and $q < N$. Let $\|\cdot\|$ denote the distance from 0 on the standard unit length circle. Suppose $\|t - \frac{a}{q}\| \leq \frac{1}{qN}$. Then we have*

$$\sum_{|n|\lesssim N} \frac{1}{(\max\{\|nt\|, \frac{1}{N}\})^2} \lesssim \frac{N^3}{(\sqrt{q}(1 + N\|t - \frac{a}{q}\|^{1/2}))^2}. \tag{B.1}$$

PROOF. Let $\tau = t - \frac{a}{q}$, then $\|\tau\| < \frac{1}{qN}$, $\|nt\| = \|n\frac{a}{q} + n\tau\|$. We see that for each q consecutive numbers of n, say $n \in A = \{0, 1, \cdots, q-1\}$, the distribution of $S(A) = \{\|n\frac{a}{q} + n\tau \mid n \in A\}$ on the unit circle follows the patern that apart from the closest point to 0, the other $q - 1$ points out of $S(A)$ stays away from 0 by the distances of about $\frac{m}{q}$, $m = 1, 2, \cdots, q-1$. The set $\{n \mid |n| \lesssim N\}$ lies in the disjoint union of $A + lq$, for $l \in \mathbb{Z}$, $|l| \lesssim \frac{N}{q}$. So first we have that the contribution to the left of (B.1) from the points away from 0 out of $A + lq$ for all $l \in \mathbb{Z}$, $|l| \lesssim \frac{N}{q}$, is

$$\lesssim \sum_{|l|\lesssim \frac{N}{q}} \sum_{m=1}^{q-1} \frac{1}{(\frac{m}{q})^2} \lesssim Nq. \tag{B.2}$$

Now let $p(A)$ denote the point out of $S(A)$ that is closest to 0. Then compared with $p(A)$, $p(A \pm q)$ moves away or towards 0 by a distance of $q\|\tau\|$. We consider two separate cases.

Case I. Suppose that $\frac{1}{q\|\tau\|} \geq \frac{N^2}{q}$. Then we simply estimate the contribution from the points closest to 0 out of $p(A + lq)$ for all l to the left of (B.1) to be

$$\lesssim \sum_{|l|\lesssim \frac{N}{q}} \frac{1}{\frac{1}{N^2}} \lesssim \frac{N^3}{q}. \tag{B.3}$$

Case II. Suppose on the contrary that $\frac{1}{q\|\tau\|} \leq \frac{N^2}{q}$. Then if the closest point to 0 out of some $A + lq$ say for $l = l_0$ is ever within the distance of $\frac{1}{N}$ from 0, the closest point out of $A + lq$ will stay the distance of $\frac{1}{N}$ away from 0 when $|l - l_0| \geq \frac{2}{Nq\|\tau\|}$. This implies that the contribution to the left of (B.1) out of the closest points from 0 is

$$\lesssim \frac{1}{Nq\|\tau\|} \cdot \frac{1}{\frac{1}{N^2}} + \sum_{\frac{1}{Nq\|\tau\|} \lesssim l \lesssim \frac{N}{q}} \frac{1}{(lq\|\tau\|)^2} \lesssim \frac{N}{q\|\tau\|}. \tag{B.4}$$

In summary, we have

$$\begin{aligned}
\sum_{|n|\lesssim N} \frac{1}{(\max\{\|nt\|, \frac{1}{N}\})^2} &\lesssim Nq + N\min\{\frac{N^2}{q}, \frac{1}{q\|\tau\|}\} \\
&\lesssim N\min\{\frac{N^2}{q}, \frac{1}{q\|\tau\|}\} \\
&\lesssim \frac{N^3}{q(1 + N\|\tau\|^{1/2})^2}.
\end{aligned} \tag{B.5}$$

□

REMARK B.2. *With the same notation as in the previous lemma, the proof can be slightly modified to show that*

$$\sum_{|n|\lesssim N} \frac{1}{\max\{\|nt\|, \frac{1}{N}\}} \lesssim \frac{N^2 \log N}{(\sqrt{q}(1 + N\|t - \frac{a}{q}\|^{1/2}))^2}. \tag{B.6}$$

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