Scale invariant Strichartz estimates on compact globally symmetric spaces

Yunfeng Zhang

Abstract. In the first part of this paper, we prove scale-invariant Strichartz estimates

$$\|e^{it\Delta}f\|_{L^p(I \times M)} \lesssim \|f\|_{H^{\frac{d}{2} - (d+2)/2p}(M)}$$

on any compact globally symmetric space $M$ equipped with a rational metric, for all $p \geq 2 + 8/r$, $d, r$ being respectively the dimension and rank of $M$. This generalizes the previous results by the author on compact Lie groups. The proof relies on an integral formula discovered by J.-L. Clerc for spherical functions on symmetric spaces of compact type. In the second part of the paper, we improve the range of $p$ to $p > 2 + 4/(r - 1)$ for class functions on compact Lie groups, and proposes an approach toward further improvement.

1. Introduction

On the Euclidean spaces $\mathbb{R}^d$, the classical scale-invariant Strichartz estimates for the Schrödinger equation read

$$\|e^{it\Delta}f\|_{L^p(I, L^q(\mathbb{R}^d))} \lesssim \|f\|_{H^s(\mathbb{R}^d)}, \ s = \frac{d}{2} - \frac{2}{p} - \frac{d}{q} \geq 0, \ p, q \geq 2, \ (p, q, d) \neq (2, \infty, 2).$$

Here $I$ is a finite interval. These were established by Ginibre and Velo [13] except for the endpoint case which was obtained by Keel and Tao [18]. They have applications to well-posedness and scattering theory for nonlinear Schrödinger equations (NLS) especially with critical exponents, and have been generalized to other manifolds. For example, Anker and Pierfelice [1], Banica [3], Ionescu and Staffilani [17], studied NLS on the real hyperbolic spaces, Anker, Pierfelice and Vallarino [2], and Pierfelice [21] studied NLS on Damek-Ricci spaces which include all rank one symmetric spaces of noncompact type. For compact manifolds, scale-invariant Strichartz estimates (and their $\varepsilon$-loss versions) are mostly established on compact symmetric spaces, usually using both harmonic analytic and number theoretic techniques. We list known results here:

- On rectangular tori $M = T^d$, scale-invariant estimates

$$\|e^{it\Delta}f\|_{L^p(I \times M)} \lesssim \|f\|_{H^{\frac{d}{2} - \frac{2}{p} + \frac{d}{q}}(M)}$$

were obtained by Bourgain [7, 8] for a limited range of $p$, and then by Bourgain and Demeter [9] for the optimal range $p > 2 + \frac{4}{q}$ with a possible $\varepsilon$-loss for irrational tori. Killip and Visan [19] then showed that this loss can be eliminated.

- On spheres and more generally Zoll manifolds which include all symmetric spaces of compact type of rank one, the same estimates as (1.1) were first proved by Burq, Gérard and Tzvetkov [11] on the standard three-sphere for $p = 6$, then by Herr [16] for $p > 4$ on any three-dimensional Zoll manifold. An inspection of the proof of Herr [16] tells that (1.1) also holds for $p > 4$ on any higher dimensional Zoll manifolds and for $p \geq 6$ on any Zoll surfaces. $p > 4$ is also the optimal range for three and higher dimensional spheres ([10]).

- On compact symmetric spaces (global or local), Marshall [20] established the $L^p$ bounds of joint eigenfunctions of all invariant differential operators at least when the spectral parameter of the eigenfunction lies in a fixed cone away from the walls of the Weyl chamber, and these bounds are
sharp up to a logarithmic factor for spaces of compact type. By an argument similar to how Herr [16] obtained Strichartz estimates using Sogge’s $L^p$ bounds of spectral clusters [22], Marshall’s result implies that (1.1) holds for a symmetric space $M$ of compact type with an $\varepsilon$ loss for any $p \geq 2 + \frac{4r_0}{d_0 - r_0}$ where $d_0, r_0$ are respectively the dimension and rank of the irreducible factor of $M$ such that $\frac{d_0}{r_0}$ is smallest among all the irreducible factors, at least when the function $f$ has its spectral parameter lying in a fixed cone away from the walls of the Weyl chamber.

• On compact Lie groups, the estimates (1.1) were established by the author [25] for $p \geq 2 + \frac{8}{3}$, $r$ the rank of the group, combining harmonic analytic arguments from Bourgain [7] and Lie theoretic techniques. In a subsequent work [24], the same range of estimates but with an $\varepsilon$-loss were obtained for rational products of compact Lie groups and odd-dimensional spheres. The proofs of these results are independent of the above result of Marshall’s.

In this paper, we stretch the above results further and obtain scale-invariant estimates (1.1) for any $p \geq 2 + \frac{8}{3}$ on a general compact globally symmetric space of any rank. As in [25] and [24], we will reduce these estimates to $L^\infty(M)$ kernel bounds on major arcs of the time variable. The key ingredient is an integral formula for spherical functions on symmetric spaces of compact type discovered by Clerc [12] similar to those of Harish-Chandra on symmetric spaces of noncompact type. Using Clerc’s formula, we are able to avoid discussing different scenarios determined by how close the points are to the faces of an alcove in a maximal torus (as done in [25] and [24]), and obtain the kernel bounds in a uniform way.

Compared with the above mentioned consequence of Marshall’s result, our results provide larger range of estimates if $\frac{8}{3} \leq \frac{4r_0}{d_0 - r_0}$. This condition typically holds when the symmetric space has a large number of irreducible factors. This parallels with how one cannot use Sogge’s $L^p$ bounds [22] to derive nontrivial scale-invariant Strichartz estimates on tori.

None of the ranges $p \geq 2 + \frac{8}{3}$ or $p > 2 + \frac{4r_0}{d_0 - r_0}$ are expected to be optimal. The second part of this paper is to improve the range of $p$ to $p > 2 + \frac{4}{r-1}$ ($r \geq 2$) for which hold Strichartz estimates for class functions on compact Lie groups. To this end, we go back to the decomposition of an alcove in the maximal torus depending on how close the points are to the faces as done in [25], and apply tools such as the optimal Strichartz estimates on tori, and rather standard estimates on number of integral solutions to a positive definite quadratic form. We also show how the largest possible range $p > 2 + \frac{4}{3}$ can be achieved for the special unitary groups, conditional on some conjectured mixed norm Strichartz type estimates on tori.

Throughout this paper, we use $A \lesssim B$ to mean $A \leq CB$ for some positive constant $C$, $A \lesssim x$ $B$ to mean $A \leq C(x)B$ for some positive constant $C(x)$ depending on $x$, and $A \asymp B$ to mean $|A| \lesssim |B| \lesssim |A|$.

2. Part I

2.1. Clerc’s Formula. Let $\mathfrak{g}$ be a real semisimple Lie algebra with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace. Let $\mathfrak{g}^c$ be the complexification of $\mathfrak{g}$ and $\mathfrak{a}^c$ the subspace of $\mathfrak{g}^c$ generated by $\mathfrak{a}$. Let $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$. Let $G^c$ be the simply connected complex Lie group with Lie algebra $\mathfrak{g}^c$ and let $G, K, U$ be the analytic subgroups of $G^c$ with Lie algebras $\mathfrak{g}, \mathfrak{k}, \mathfrak{u}$ respectively. Let $\Sigma \subset \mathfrak{a}^*$ denote the restricted root system with respect to $(\mathfrak{g}, \mathfrak{a})$ and $\Sigma^+$ the set of positive roots. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}^c$ containing $\mathfrak{a}$, $(\cdot, \cdot)$ denoting the Killing form, we have the weight lattice $\Lambda = \{\mu \in \mathfrak{a}^* : \frac{\mu(\alpha)}{\alpha(\alpha) \mathfrak{g}} \in \mathbb{Z}, \forall \alpha \in \Sigma\}$ and the subset of dominant weights $\Lambda^+ = \{\mu \in \mathfrak{a}^* : \frac{\mu(\alpha)}{\alpha(\alpha) \mathfrak{g}} \in \mathbb{Z}_{\geq 0}, \forall \alpha \in \Sigma^+\}$. By a theorem of Helgason [15, Theorem 4.1 of Chapter V], $\Lambda^+$ indexes the equivalence classes of irreducible spherical representations, as restrictions on $\mathfrak{a}$ of the highest weights which vanish on the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{h}$. For $\mu \in \Lambda^+$ and $\pi_\mu$ the corresponding irreducible spherical representation and $e_\mu$ a $K$-invariant unit vector in the representation
space, let $\varphi_\mu(g) = (\pi_\mu(g)e_\mu, e_\mu)$ be the corresponding spherical function on $U/K$. $\varphi_\mu$ can be holomorphically extended to $G^\mathbb{C}$. Let $dk$ denote the Haar measure on $K$ normalized such that $\int_K dk = 1$.

**Theorem 1** ([12]). There exists a dense open subset $\omega$ of $G^\mathbb{C}$ and a multivalued holomorphic function $\mathcal{H} : \omega \to \mathfrak{a}^\mathbb{C}$ (which coincides with the component function in the Iwasawa decomposition $g = k(g) \exp \mathcal{H}(g)n(g)$, $\forall g \in G$), such that for any $g \in G^\mathbb{C}$,

$$K_g = \{k \in K : gk \in \omega\}$$

is open in $K$ and $K \setminus K_g$ is of zero measure, and

$$\varphi_\mu(g) = \int_{K_g} e^{\mu(\mathcal{H}(gk))} dk, \ \forall g \in G^\mathbb{C}.$$  \hspace{1cm} (2.1)

Moreover, for $u \in U$,

$$Re \mu(\mathcal{H}(uk)) \leq 0, \ \forall k \in K.$$  \hspace{1cm} (2.2)

Using this integral formula, we can rewrite sums of the form $\sum_{\mu \in \Lambda^+} f_\mu \varphi_\mu(g)$ into

$$\sum_{\mu \in \Lambda^+} f_\mu \varphi_\mu(g) = \int_{K_g} \left( \sum_{\mu \in \Lambda^+} f_\mu e^{\mu(\mathcal{H}(gk))} \right) dk.$$  \hspace{1cm} (2.3)

The sum on the right could become handy for estimates. We will follow this method to derive estimates for the kernel function associated to the Schrödinger flow $e^{it\Delta}$, to prove the following theorem.

**Theorem 2** (Main Theorem of Part I). Let $M = U/K$ be a simply connected symmetric space of compact type, of dimension $d$ and rank $r$. Then

$$\|e^{it\Delta} f\|_{L^p(I \times M)} \lesssim \|f\|_{H^{\frac{d}{2} - \frac{4}{r} - \frac{r}{p}}(M)}$$  \hspace{1cm} (2.4)

for any $p \geq 2 + \frac{4}{r}$. Moreover, let $m_\lambda$ denote the measure of the level set $\{(t, x) \in I \times M : |P_N e^{it\Delta} f(x)| > \lambda\}$, then

$$m_\lambda \lesssim N^{\frac{d}{2} - (d+2)\frac{r}{p}} \|f\|_{L^\infty(M)}^p, \ \forall \lambda \gtrsim N^{\frac{d}{2} - \frac{4}{r}}, \ \forall p > 2 + \frac{4}{r}.$$  \hspace{1cm} (2.5)

2.2. **Reduction to an Exponential Sum Estimate.** Let $M = U/K$ be a simply connected symmetric space, equipped with canonical metrics induced from the Killing form. Let $| \cdot |$ denote the corresponding norm on $\mathfrak{a}$ as well as on $\mathfrak{a}^\ast$. Let $d_\mu$ denote the dimension of the irreducible spherical representation with highest weight $\mu \in \Lambda^+$. The kernel function (distribution) for the Schrödinger flow reads

$$\mathcal{K}(t, x) = \sum_{\mu \in \Lambda^+} e^{-it(\mu + |\rho|^2 - |\rho|^2)} d_\mu \varphi_\mu(x),$$

where $\rho = \frac{1}{2} \sum_{\lambda \in \Sigma^+} m_\lambda \lambda$, for which

$$e^{it\Delta} f(x) = f(x) \ast \mathcal{K}(t, x) = \int_U f(uK) \mathcal{K}(t, u^{-1}x) \ du, \ \forall x \in M.$$  \hspace{1cm} (2.6)

We may consider the mollified Schrödinger flow

$$P_N e^{it\Delta} := \phi \left( \frac{-\Delta}{N^2} \right) e^{it\Delta}, \ N \geq 1,$$  \hspace{1cm} (2.7)

for a bump function $\phi$, so that

$$P_N e^{it\Delta} f(x) = f(x) \ast \mathcal{K}_N(t, x)$$  \hspace{1cm} (2.8)
where
\[ \mathcal{K}(t, x) = \sum_{\mu \in \Lambda^+} \phi \left( \frac{[\mu + \rho]^2 - |\rho|^2}{N^2} \right) e^{-it([\mu + \rho]^2 - |\rho|^2)} d_{\mu} \varphi_\mu. \]

By rationality of the weight lattice (cf. [14, Lemma 8.4 of Chapter VII]), there exists \( D \in \mathbb{R} \), such that
\[ |\mu + \rho|^2 - |\rho|^2 \in D^{-1} \mathbb{Z}, \ \forall \mu \in \Lambda^+, \]
thus both \( \mathcal{K} \) and \( \mathcal{K}_N \) are actually periodic in \( t \) with a period \( 2\pi D \). Thus we may think of the variable \( t \) as lying on the torus \( T = \mathbb{R}/T \mathbb{Z} \), where \( T = 2\pi D \). By Littlewood-Paley theory (\( L^p \) theory for spectral localization) on compact manifolds ([10, Corollary 2.3]), Strichartz estimates (1.1) may be reduced into
\[ \| \mathcal{P}_N e^{it\Delta} f \|_{L^p(T \times M)} \lesssim N^{\frac{d}{2} - \frac{d+2}{p}} \| f \|_{L^2(M)} \]
and further by a \( TT^* \) argument into
\[ \| \mathcal{K}_N \times F \|_{L^p(T \times M)} \lesssim N^{d - \frac{2d+2}{p}} \| F \|_{L^{p'}(T \times M)}. \]
Here \( 1/p + 1/p' = 1 \) and \( \times \) stands for the convolution on the product symmetric space \( T \times M \), and the above estimate can be thought of as a Fourier restriction estimate on such product spaces. The following key \( L^\infty(M) \) estimates on \( \mathcal{K}_N(t, \cdot) \) for \( t \) lying on major arcs of \( T \), was conjectured in [24] and will be proved in this paper.

**Theorem 3.** Let \( \| \cdot \| \) stand for the distance from the nearest integer. Define the major arcs
\[ \mathcal{M}_{a,q} = \left\{ s \in \mathbb{R}/\mathbb{Z} : \left\| s - \frac{a}{q} \right\| < \frac{1}{qN} \right\} \]
where
\[ a \in \mathbb{Z}_{\geq 0}, \ q \in \mathbb{N}, \ a < q, \ (a, q) = 1, \ q < N. \]
Then
\[ |\mathcal{K}_N(t, x)| \lesssim \frac{N^d}{\sqrt{q(1 + N\| \frac{T}{T} - \frac{a}{q}\|^{1/2})}} \]
for \( \frac{T}{T} \in \mathcal{M}_{a,q} \), uniformly for \( x \in M \).

Generalizing the harmonic analytic framework developed by Bourgain [7] on tori to that on compact symmetric spaces as done in [24, Theorem 15], we may use the above theorem to establish estimates of the form (2.6), which imply Theorem 2. The key modification needed is to replace the standard Fourier series on tori by the convolutional spherical Fourier series on \( M \) and the space-time version on \( T \times M \)
\[ f = \sum_{\mu \in \Lambda^+} d_{\mu} f \ast \varphi_\mu, \ f \in L^2(M), \]
\[ F = \sum_{\mu \in \Lambda^+} d_{\mu} F \ast [e^{2\pi i t/T} \varphi_\mu], \ F \in L^2(T \times M), \]
and apply the Plancherel identities in these settings
\[
\|f\|_{L^2(M)}^2 = \sum_{\mu \in \Lambda^+} d^2_{\mu} \|f \ast \varphi_{\mu}\|_{L^2(M)}^2,
\]
\[
\|F\|_{L^2(\mathbb{T} \times M)}^2 = \sum_{n \in \mathbb{Z}, \mu \in \Lambda^+} d^2_{\mu} \|F \times [e^{2\pi int/T} \varphi_{\mu}]\|_{L^2(\mathbb{T} \times M)}^2.
\]

We refer to [24, Section 3] for details.

2.3. **Proof of Theorem 3.** First, we record some facts concerning the \(d_{\mu}\)’s (\(\mu \in \Lambda^+\)).

**Lemma 4.** Let \(\Phi\) denote the root system associated to \((\mathfrak{g}^C, \mathfrak{h})\) and let \(\Phi^+ \subset \Phi\) be the set of positive roots with respect to an ordering compatible with that on \(\mathfrak{a}^*\). For each \(\mu \in \Lambda^+\), extend it to be zero on the orthogonal complement of \(\mathfrak{a}\) in \(\mathfrak{h}\), and denote the extension also by \(\mu\). Then
\[
d_{\mu} = \frac{\prod_{\alpha \in \Phi^+, \alpha |a| \neq 0} (\mu + \rho', \alpha)}{\prod_{\alpha \in \Phi^+, \alpha |a| \neq 0} (\rho', \alpha)}, \quad \text{for } \rho' = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.
\]

This is a direct consequence of the Weyl dimension formula applied to the irreducible representation of \(\mathfrak{g}^C\) of highest weight \(\mu\). It has the following consequence.

**Corollary 5.** Let \(\{\gamma_1, \ldots, \gamma_r\}\) be the set of simple roots in \(\Sigma^+\). Let \(w_1, \ldots, w_r \in \mathfrak{a}^*\) be the fundamental weights such that \(\frac{w_i(\gamma_j)}{(\gamma_j, \gamma_j)} = \delta_{ij}\) (\(1 \leq i, j \leq r\)). Then for \(\mu = n_1 w_1 + \cdots + n_r w_r \in \Lambda^+\), \(d_{\mu}\) is a polynomial in \(n_1, \ldots, n_r\) of degree \(d - r\) (\(d\) is the dimension of the symmetric space \(U/K\)). Furthermore, \(d_{\mu}\) has at least one linear factor of the form \(a_j n_j + b_j\) for each \(j = 1, \ldots, r\), with \(a_j, b_j\) constant.

**Proof.** The degree equals the number of restricted roots counted with multiplicities, which is \(d - r\). By the above lemma,
\[
d_{\mu} = \frac{\prod_{\alpha \in \Phi^+, \alpha |a| \neq 0} (\mu + \rho', \alpha)}{\prod_{\alpha \in \Phi^+, \alpha |a| \neq 0} (\rho', \alpha)} \cdot \prod_{1 \leq j \leq r} \prod_{\alpha \in \Phi^+, \alpha |a| = \gamma_j} (n_j(\gamma_j, \gamma_j) + (\rho', \alpha)).
\]

Then the second claim is clear. \(\square\)

Using Clerc’s formula (2.1), we rewrite
\[
\mathcal{X}_N(t, uK) = \int_{K_u} \kappa_N(t, u, k) \, dk,
\]
where
\[
\kappa_N(t, u, k) := \sum_{\mu \in \Lambda^+} \phi \left( \frac{|\mu + \rho|^2 - |\rho|^2}{N^2} \right) e^{-it(\rho + |\rho|^2)} e^{it(\mu) + \mu \mathcal{H}(u)} d_{\mu}.
\]

By Theorem 1, \(\int_{K_u} dk = 1\) for any \(u \in U\), thus we have
\[
|\mathcal{X}_N(t, uK)| \leq \sup_{k \in K} |\kappa_N(t, u, k)|.
\]

It then suffices to derive estimates as (2.7) for \(\kappa_N(t, u, k)\) uniformly in \(u \in U\) and \(k \in K\).

For \(k \in K\) and \(u \in U\), write
\[\mathcal{H}(u) = H_0 + iH, \text{ for } H_0, H \in \mathfrak{a}.\]

By (2.2),
\[
\lambda(H_0) \leq 0, \forall \lambda \in \Lambda^+.
\]
Denote
\[|\mu|^2 = |\mu + \rho|^2 - |\rho|^2\]
and perform Weyl’s differencing technique, we have
\[
|\kappa_N|^2 = \sum_{\mu_1, \mu_2 \in \Lambda^+} \phi \left( \frac{|\mu_1|^2}{N^2} \right) \phi \left( \frac{|\mu_2|^2}{N^2} \right) e^{-i(\mu_1 \rho - \mu_2 \rho) + (\mu_1 + \mu_2)(\mu_2 - \mu_1)(H_0) + i(\mu_1 - \mu_2)H} d_{\mu_1} d_{\mu_2}
\]
\[= \sum_{\lambda \in \Lambda^+} e^{i|\lambda|^2 + \lambda(H_0) - i\lambda(H)} \sum_{\mu \in \Lambda^+ \cap (\lambda - \Lambda^+)} \phi \left( \frac{|\mu|^2}{N^2} \right) \phi \left( \frac{|\lambda - \mu|^2}{N^2} \right) e^{2i[\mu(H) - t(\mu, \lambda + 2\rho)]} d_{\mu} d_{\lambda - \mu}
\]
(2.12) \[\lesssim \sum_{\lambda \in \Lambda^+, |\lambda| \leq N} e^{i|\lambda|^2} \sum_{\mu \in \Lambda^+ \cap (\lambda - \Lambda^+)} \phi \left( \frac{|\mu|^2}{N^2} \right) \phi \left( \frac{|\lambda - \mu|^2}{N^2} \right) e^{2i[\mu(H) - t(\mu, \lambda + 2\rho)]} d_{\mu} d_{\lambda - \mu}.
\]
Here, we have crucially used (2.11). Recall that \(w_1, w_2, \ldots, w_r\) denote the fundamental dominant weights so that \(\Lambda^+ = \mathbb{Z}_{\geq 0}w_1 + \cdots + \mathbb{Z}_{\geq 0}w_r\) and \(\Lambda = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_r\). For \(\lambda \in \Lambda^+\), write
\[\lambda = n_1^1 w_1 + \cdots + n_r^r w_r.
\]
Then
\[\mu \in \Lambda^+ \cap (\lambda - \Lambda^+)\]
if and only if \(\mu = n_1 w_1 + \cdots + n_r w_r, 0 \leq n_j \leq n_j^\lambda, j = 1, \ldots, r\).

For \(\mu = n_1 w_1 + \cdots + n_r w_r\), let
\[f(n_1, \ldots, n_r) = f(\mu) = f(\mu, N, \lambda) := \phi \left( \frac{|\mu|^2}{N^2} \right) \phi \left( \frac{|\lambda - \mu|^2}{N^2} \right) d_{\mu} d_{\lambda - \mu}.
\]

Define the difference operators with respect to \(\mu\)
\[D_j f(\mu) := f(\mu + w_j) - f(\mu), j = 1, \ldots, r.
\]

**Lemma 6.** Let \(n \in \mathbb{Z}_{\geq 0}, j_1, \ldots, j_n \in \{1, \ldots, r\}\).

1. \(|D_{j_1} D_{j_2} \cdots D_{j_n} f(\mu)| \lesssim_n N^{2d-2r-n}, \forall \mu \in \Lambda^+, \forall \lambda \in \Lambda\).

2. Fix \(J \subset \{1, \ldots, r\}\). Suppose \(j_k \notin J\), for any \(k = 1, \ldots, n\). Suppose for each \(j \in J\), either \(n_j = 0\) or \(n_j = n_j^\lambda + 1\). Then
\[|D_{j_1} D_{j_2} \cdots D_{j_n} f(n_1, \ldots, n_r)| \lesssim_n N^{2d-2r-n-|J|}, \forall n_j \in \mathbb{Z}_{\geq 0}, j \notin J, \forall \lambda \in \Lambda.
\]

**Proof.** Since \(\phi\) is a bump function, we may assume \(|\mu|, |\lambda| \leq N\). Noting the Leibniz rule for taking difference operators
\[D_j \left( \prod_{i=1}^n f_i \right) = \sum_{l=1}^n \sum_{1 \leq k_1 < \cdots < k_l \leq n} (D_j f_{k_1}) \cdots (D_j f_{k_l}) \cdot \prod_{i \neq k_1, \ldots, k_l} f_i,
\]
then the inequality of (1) is a consequence of the fact from Corollary 5 that \(d_{\mu}\) (as well as \(d_{\lambda - \mu}\)) is a polynomial function in \(\mu\) of degree \(d - r\). For (2), compared with (1), the extra decay factor of \(N^{-|J|}\) results from the fact from Corollary 5 that \(d_{\mu}\) (resp. \(d_{\lambda - \mu}\)) has a linear factor of the form \(a_j n_j + b_j\) (resp. \(a_j(n_j^\lambda - n_j) + b_j\)) for each \(j = 1, \ldots, r\). In fact, for each \(j \in J\), as \(n_j = 0\) (resp. \(n_j = n_j^\lambda + 1\)), this linear factor is bounded \(\lesssim 1\); while in (1), the linear factor gives a contribution of \(\lesssim N\) instead of \(\lesssim 1\). Noting the assumption that no \(D_j (j \in J)\) appears in the inequality in (2), for each \(j \in J\), this linear factor provides an extra decay factor of \(N^{-1}\) compared to (1), and together yielding an extra decay factor of \(N^{-|J|}\). \(\square\)
Now let $\kappa^\lambda$ be the sum in (2.12) inside of the absolute value. Then

$$\kappa^\lambda = \sum_{0 \leq n_1 \leq n_1^\lambda} \sum_{0 \leq m_1 \leq n_1^\lambda} e^{in_1 \theta_1} \cdots \sum_{0 \leq n_r \leq n_r^\lambda} e^{in_r \theta_r} f(\mu)$$

where $\theta_j = \theta_j(t, \lambda) := 2[w_j(H) - t(w_j, \lambda + 2\rho)], \ j = 1, \ldots, r$. We perform summation by parts on $\kappa^\lambda$ choicefully:

- If $|1 - e^{i\theta_1}| \geq N^{-1}$, we write

$$\sum_{0 \leq n_1 \leq n_1^\lambda} e^{in_1 \theta_1} f(\mu) = \frac{1}{1 - e^{i\theta_1}} \sum_{0 \leq n_1 \leq n_1^\lambda} e^{i(n_1 + 1) \theta_1} D_1 f(\mu)$$

$$+ \frac{e^{i\theta_1}}{(1 - e^{i\theta_1})^2} D_1 f(0, n_2, \ldots, n_r) - \frac{e^{i(n_1 + 2) \theta_1}}{(1 - e^{i\theta_1})^2} D_1 f(n_1^\lambda + 1, n_2, \ldots, n_r)$$

$$= \frac{1}{1 - e^{i\theta_1}} \sum_{0 \leq n_1 \leq n_1^\lambda} e^{i(n_1 + 2) \theta_1} D_1^2 f(\mu)$$

$$+ \frac{e^{i\theta_1}}{(1 - e^{i\theta_1})^2} D_1 f(0, n_2, \ldots, n_r) - \frac{e^{i(n_1 + 2) \theta_1}}{(1 - e^{i\theta_1})^2} D_1 f(n_1^\lambda + 1, n_2, \ldots, n_r)$$

(2.13)

$$+ \frac{1}{1 - e^{i\theta_1}} f(0, n_2, \ldots, n_r) - \frac{e^{i(n_1 + 1) \theta_1}}{1 - e^{i\theta_1}} f(n_1^\lambda + 1, n_2, \ldots, n_r),$$

- Otherwise, no operation is performed on

$$\sum_{0 \leq n_1 \leq n_1^\lambda} e^{in_1 \theta_1} f(\mu).$$

Next similarly we perform summation by parts choicefully for each of the layers of summation $\sum_{0 \leq n_j \leq n_j^\lambda} e^{in_j \theta_j}, \ldots, j = 2, \ldots, r$. What we end up with is a sum of $\leq 5^r$ terms, each term of the form

$$\kappa^\lambda_{\text{term}} = \sum_{\sum_{k=1}^r 0 \leq j_k \leq n_j^\lambda} z \cdot D_{j_1}^2 \cdots D_{j_i}^2 \cdots D_{j_{i+1}} \cdots D_{j_{r+1}} f(\mu_*)$$

$$\prod_{1 \leq k \leq l + m + s + t + u + v + r} (1 - e^{i\theta_{j_k}})^2 \prod_{l + m + s + t + u + v + r} (1 - e^{i\theta_{j_k}})^2,$$

where $l + m + s + t + u + v = r$ is a partition of $r$ into six nonnegative integers, and

(2.14)

$$\mu_* = \sum_{k=1}^l n_{j_k} w_{j_k} + \sum_{k=l+1}^{l+m+s} (n_{j_k}^\lambda + 1) w_{j_k},$$

with

$$|z| = 1.$$

Using (2.14), apply (2) of Lemma 6 for $J = \{j_k, l + m + s + 1 \leq k \leq l + m + s + t + u\}$, we get
Theorem 8. Let $M = M_0 \times M_1 \cdots \times M_n$ be the product space where the $M_j$'s are irreducible symmetric spaces of compact type, with $d, r, d_j, r_j$ denoting respectively the dimension and rank of $M$ and $M_j$. Suppose

\begin{align*}
|\kappa_\lambda|^2 &\lesssim N^{t+v+2d-2r-2l-m-s-u} \prod_{1 \leq k \leq l+m+s} |1 - e^{i\theta_j}|^2 \prod_{l+m+s+1 \leq k \leq l+m+s+t+u} |1 - e^{i\theta_j}| \\
&\lesssim N^{t+v+2d-2r-2l-m-s-u} \prod_{1 \leq k \leq l+m+s} |1 - e^{i\theta_j}|^2 \prod_{l+m+s+1 \leq k \leq l+m+s+t+u} |1 - e^{i\theta_j}|^2 \\
&\lesssim N^{2d-3r} \prod_{j=1}^r \frac{1}{\max\{\frac{1}{N}, |\theta_j|/2\pi\}} \\
&\lesssim N^{2d-3r} \prod_{j=1}^r \frac{1}{\max\{\frac{1}{N}, |\theta_j|/2\pi\}}^2.
\end{align*}

The same estimates hold for $|\kappa_\lambda|$. Now a good choice of $D$ in (2.5) also makes $-2(w_j, \lambda + 2\rho) \in D^{-1}\mathbb{Z}$, $j = 1, \ldots, r$. Let

$$m_j = -2(w_j, \lambda + 2\rho) \cdot D, \quad j = 1, \ldots, r.$$ 

Since the map $\Lambda \ni \lambda \mapsto (m_1, \ldots, m_r) \in \mathbb{Z}^r$ is one-one, we can write (2.12) into

$$|\kappa_N|^2 \lesssim N^{2d-3r} \prod_{j=1}^r \left( \sum_{|m_j| \leq N} \frac{1}{\max\{\frac{1}{N}, \|m_j t/T + w_j(H)/\pi\}\}} \right)^{1/2}.$$

By a standard estimate as in deriving the classical Weyl inequality in one dimension, we get

$$\sum_{|m_j| \leq N} \frac{1}{\max\{\frac{1}{N}, \|m_j t/T + w_j(H)/\pi\}\}} \lesssim \frac{N^3}{\sqrt{q}(1 + N\|\frac{t}{T} - \frac{a}{q}\|^{1/2})^2}$$

for $\frac{t}{T}$ lying on the major arc $\mathcal{M}_{a,q}$. Hence

$$|\kappa_N(t, u, k)|^2 \lesssim \frac{N^{2d}}{\sqrt{q}(1 + N\|\frac{t}{T} - \frac{a}{q}\|^{1/2})^{2r}}.$$

An inspection of the above argument shows that this estimate holds uniformly for $u \in U$ and $k \in K$. By (2.10), (2.7) now holds uniformly for $uK \in U/K$.

Remark 7. The last piece of information in Corollary 5 about linear factors of $d_\mu$ is used to deal with the last two terms in (2.13). Without it, one could only get the bound $|\kappa_\lambda| \lesssim N^{2d-2r} \prod_{j=1}^r \frac{1}{\max\{\frac{1}{N}, |\theta_j|/2\pi\}}^{1/2}$, which would yield (2.7) with an $N^2$ loss.

2.4. On a General Compact Globally Symmetric Space with Rational Metrics. Theorem 2 can be generalized to all compact globally symmetric spaces in the sense of Takeuchi [23] where one allows a nontrivial center of $u$, equipped with rational metrics (that is, a product of rational multiples of Killing forms of each irreducible factor of compact type and a rational metric on the toric factor). This can be done either by proving the bound (2.7) in a similar way for the slightly generalized kernel, or using the product version of Littlewood-Paley projections to reduce the kernel bound to that on each irreducible factor of $M$. For details, we refer to [24, Theorem 15].


Theorem 8. Let $M = M_0 \times M_1 \cdots \times M_n$ be the product space where the $M_j$’s are irreducible symmetric spaces of compact type, with $d, r, d_j, r_j$ denoting respectively the dimension and rank of $M$ and $M_j$. Suppose
\(d_0/r_0\) is smallest among \(d_j/r_j\) \((0 \leq j \leq n)\). Suppose \(r \geq 2\). Then (2.3) holds for any \(p \geq \frac{2(d_0 + r_0)}{d_0 - r_0}\) with an \(\varepsilon\) loss, provided the spectral parameter of the function \(f\) lies in a fixed cone centered at the origin and away from the walls of the Weyl chamber.

**Proof.** For any \(f \in L^2(M)\), we may write its spherical Fourier series \(f = \sum_{\mu \in \Lambda^+} a_\mu f_\mu\), where \(f_\mu(g) = \langle \pi_\mu(g)e_\mu, v_\mu \rangle\) is a matrix coefficient (\(\pi_\mu\) the irreducible representation corresponding to \(\mu \in \Lambda^+\), \(e_\mu\) a \(K\)-invariant unit vector, and \(v_\mu\) any vector in the representation space), such that \(\|f\|_{L^2(M)} = \|a_\mu\|_{l^2(\Lambda^+)}\). As \(f_\mu\) is a joint eigenfunction of all invariant differential operators with spectral parameter \(\mu\), and we assume that \(\mu\) lies in a fixed cone centered at the origin and away from the walls of the Weyl chamber, by [20, Theorem 1.1 and the remarks right below it], we have

\[
\|f_\mu\|_{L^p(M)} \lesssim |\mu|^{d(\frac{1}{2} - \frac{1}{p}) - \frac{d}{2}}, \quad \text{for all } p > \frac{2(d_0 + r_0)}{d_0 - r_0}.
\]

For \(e^{itA}f = \sum_{\mu \in \Lambda^+} e^{-it|\mu|^2} a_\mu f_\mu\), we estimate for any \(p > \frac{2(d_0 + r_0)}{d_0 - r_0}\)

\[
\left\| \sum_{|\mu| \leq N, \mu \in \Lambda^+} e^{-it|\mu|^2} a_\mu f_\mu \right\|_{L^p(I \times M)} \lesssim \left( \sum_{|\mu| \leq N^2, \mu \in \frac{2\pi}{r} \mathbb{Z}} |a_\mu f_\mu|^2 \right)^{\frac{1}{2}} \lesssim N^{1 - \frac{d}{2} + \varepsilon} \left( \sum_{\mu \in \Lambda^+} |a_\mu f_\mu|^2 \right)^{\frac{1}{2}} \lesssim \|f_\mu\|_{L^p(M)} \lesssim N^{\frac{d}{2} - \frac{d+2}{4} + \varepsilon} \|a_\mu\|_{l^2(\Lambda^+)},
\]

Suppose \(M\) is itself irreducible. Note that \(\frac{2(d+1)}{d-1} = 2 + \frac{4}{d-1} \leq 2 + \frac{8}{r+1}\) (equality holds when \(M = SU(r+1)/SO(r+1)\)), thus the range \(p > \frac{2(d+1)}{d-1}\) in the above theorem is larger than \(p \geq 2 + \frac{8}{r+1}\) in Theorem 2. However, Theorem 2 provides much better range if \(M\) has a large number of irreducible factors. Also, if \(M\) has any toric factor, then the above argument using eigenfunction bounds provides no nontrivial scale-invariant Strichartz estimates.

### 3. Part II

#### 3.1. Improvement for Class Functions on Compact Lie Groups

In this section, we specialize to compact Lie groups, and provide better range of \(p\) for (2.3) for class functions \(f\). For the sake of simplicity of exposition, we first assume that \(U\) is a compact simply connected simple Lie group, of dimension \(d\) and rank
$r \geq 2$. At the end of this section, we will indicate the necessary changes in the arguments for the general case where $U$ is any compact Lie group equipped with rational metrics.

Let $T$ be a maximal torus of $U$ with Lie algebra $\mathfrak{t}$ which is a Cartan subalgebra of the Lie algebra $\mathfrak{u}$ of $U$. Let $\Sigma \subset \mathfrak{t}^*$ denote the root system of $(\mathfrak{u}, \mathfrak{t})$, and $\Sigma^+ \subset \Sigma$ a positive system. Let $\Lambda = \{ \mu \in \mathfrak{t}^* : \frac{2 \langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}, \ \forall \alpha \in \Sigma \}$ be the associated weight lattice, and $\Lambda^+ = \{ \mu \in \Lambda : \frac{2 \langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \geq 1, \ \forall \alpha \in \Sigma^+ \}$ the subset of strictly dominant weights. (We have chosen here the strictly dominant weights instead of the larger set of dominant weights to slightly improve simplicity of notation.) Let $W$ denote the Weyl group, and $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha$ the Weyl vector. For each $\mu \in \Lambda^+$, there associates a character given by Weyl's formula

$$\chi_\mu(H) = \frac{\sum_{s \in W} \det s \ e^{(sp)(H)}}{\sum_{s \in W} \det s \ e^{(sp)(H)}}, \ H \in \mathfrak{t}.$$ 

By Schur's orthogonality relations, it is well known that with respect to the Haar measure on $U$ normalized such that $\int_U du = 1$,

$$\|\chi_\mu\|_{L^2(U)} = 1, \ \forall \mu \in \Lambda^+.$$ 

Let $L^2(U)$ denote the set of class functions in $L^2(U)$. Then $L^2(U) \cong l^2(\Lambda^+)$, by

$$L^2(U) \ni f = \sum_{\mu \in \Lambda^+} a_\mu \chi_\mu \mapsto (a_\mu)_{\mu \in \Lambda^+} \in l^2(\Lambda^+).$$

The mollified Schrödinger flow reads

$$P_N e^{it\Delta} f = \sum_{\mu \in \Lambda^+} \varphi \left( \frac{|\mu|^2 - |\rho|^2}{N^2} \right) e^{-it(|\mu|^2 - |\rho|^2)} a_\mu \chi_\mu.$$ 

We will establish the following theorem on Strichartz estimates for class functions.

**Theorem 9** (Main Theorem of Part II). For any $p > 2 + \frac{4}{r-1}$,

$$\left\| \sum_{\mu \in \Lambda^+, |\mu| \leq N} \varphi \left( \frac{|\mu|^2 - |\rho|^2}{N^2} \right) e^{-it(|\mu|^2 - |\rho|^2)} a_\mu \chi_\mu \right\|_{L^p(I \times U)} \lesssim N^\frac{d}{2} - \frac{d+2}{r} \|a_\mu\|_{l^2(\Lambda^+)}.$$ 

The above inequality is also equivalently rewritten as

$$\left(3.1\right) \left\| \sum_{|\mu| \leq N} e^{-it|\mu|^2} a_\mu \chi_\mu \right\|_{L^p(I \times U)} \lesssim N^\frac{d}{2} - \frac{d+2}{r} \|a_\mu\|_{l^2(\Lambda^+)}.$$ 

The exponent $2 + \frac{1}{r-1}$ is smaller than $2 + \frac{3}{r}$ for $r \geq 3$ (while equal for $r = 2$). Thus it provides an improvement on Theorem 2 for this specialized scenario. This new range is still not better than that covered by Theorem 8 when $U$ is simple (i.e., irreducible), but it is much better if $U$ has a large number of irreducible factors or has a toric factor.

To prove this theorem, we first reduce the integration on $U$ to that on a fundamental alcove in $T$. By Weyl’s integration formula, for any continuous class function $f$,

$$\int_U f(u) \ du = \frac{1}{|W|} \int_T f(t) |\delta(t)|^2 \ dt$$

with $\delta(t)$ the Weyl denominator

$$\delta(\exp H) = \prod_{\alpha \in \Sigma^+} \left( e^{\alpha(H)} - e^{-\alpha(H)} \right) = \prod_{\alpha \in \Sigma^+} \left( 2i \sin \frac{\alpha(H)}{2i} \right) = \sum_{s \in W} \det s \ e^{(sp)(H)}.$$ 

Here the Haar measures are normalized so that $\int_U du = 1 = \int_T dt$. 

The alcoves are connected components of the space of regular elements \( \{H \in t : \alpha(H)/i \notin 2\pi \mathbb{Z} \forall \alpha \in \Sigma \} \). Let \( \{\alpha_1, \ldots, \alpha_r\} \subset \Sigma^+ \) be the set of simple roots, and \( \alpha_0 \in -\Sigma^+ \) the lowest root. The fundamental alcove reads
\[
A = \{H \in t : \alpha_j(H)/i + 2\pi \delta_{0j} > 0 \forall i = 0, \ldots, r\}.
\]
It is known that \( A \) naturally embeds isometrically into the maximal torus \( T \), and the Weyl group \( W \) acts on simply transitively on the set of all the \( |W| \) embedded alcoves in \( T \) so that \( T - \bigsqcup_{a \in W} s_A \) is a set of measure zero in \( T \) ([14, Section 7 of Chapter VII]). Note that \( |\delta(t)|^2 \) is also a class function so that \( f(t)|\delta(t)|^2 \) is invariant under the Weyl group, thus we may rewrite Weyl’s integration formula into
\[
\int_{\mathcal{U}} f(u) \, du = \int_A f(t)|\delta(t)|^2 \, dt.
\]
Hence we reduce (3.1) into
\[
(3.3) \quad \left\| \sum_{\mu \in \Lambda^+} e^{-it|\mu|^2}a_{\mu}\chi_{\mu}|\delta|^2 \right\|_{L^p(I \times A)} \lesssim N^{\frac{d}{2} - \frac{d+2}{r}} \|a_{\mu}\|_{L^2(\Lambda^+)}.
\]
We prove the above estimate for \( p > 2 + \frac{4}{r-1} \), in essentially two steps. First, we decompose the alcove \( A \) (which is a simplex) into polytopes according to how close the points in \( A \) are from the faces of \( A \), so to exploit the size variation of the character \( \chi_{\mu}'s \) in the \( |\delta|^2 \) inside of \( A \). This has been done in [25], but it will be presented in this paper in a more streamlined way. Second, each of the polytopes is geometrically a product of a \( k \)-dimensional facet \( A^k \) of \( A \) and a \( (r-k) \)-dimensional cube \( (0 \leq k \leq r) \), and we apply for \( k = r, r-1 \) the optimal Strichartz estimates for tori on \( A^k \) thought of as a subset of the \( k \)-dimensional torus \( \mathbb{T}^k \), and apply for \( k \leq r-2 \) a standard estimate on number of integral solutions to a positive definite quadratic polynomial, combined with Bernstein’s inequalities.

We refer to [6] and review some standard facts concerning the affine Weyl group and its parabolic subgroups relevant to the structure of alcoves. For \( j = 0, 1, \ldots, r \), Let \( s_{\alpha_j} \) denote the reflection on \( t \) across the hyperplane
\[
H_j = \{H \in t : \alpha_j(H)/i + 2\pi \delta_{0j} = 0\}.
\]
Let \( s_{\alpha_0} = s_{\alpha_0} - s_{\alpha_0}(0) \) be the reflection across \( \{H \in t : \alpha_0(H) = 0\} \), and denote \( s_{\alpha_j} = s_{\alpha_j} \) for \( j = 1, \ldots, r \). Let \( \tilde{W} \) be the affine Weyl group generated by the \( s_{\alpha_j} \)'s \( (j = 0, \ldots, r) \). The proper parabolic subgroups of \( \tilde{W} \) is in one-to-one correspondence to the proper subsets of \( \{0, 1, \ldots, r\} \). For \( J \subsetneq \{0, 1, \ldots, r\} \), let \( \tilde{W}_J \) denote the corresponding parabolic subgroup of \( \tilde{W} \) generated by the \( s_{\alpha_j} \)'s \( (j \in J) \). Under the map \( \tilde{W} \ni s \mapsto s - s(0) \), \( \tilde{W}_J \) is isomorphic to the subgroup \( W_J \) of the Weyl group \( W \) generated by the \( s_{\alpha_j} \)'s \( (j \in J) \), and in particular, \( W \) equals \( \tilde{W}_{\{1, \ldots, r\}} \). \( W_J \) is also the Weyl group of the root subsystem \( \Sigma_J \) of \( \Sigma \) having the set of simple roots \( \{\alpha_j, j \in J\} \). The facets of the simplicial alcove \( A \) are given by \( (J \subsetneq \{0, \ldots, r\}) \)
\[
A_J = \{H \in t : \alpha_j(H)/i + 2\pi \delta_{0j} = 0 \forall j \in J, \text{ and } \alpha_j(H)/i + 2\pi \delta_{0j} > 0 \forall j \notin J\}
\]
and the stabilizer in \( \tilde{W} \) of any element in \( A_J \) coincides with \( \tilde{W}_J \).

We define some orthogonal projections with respect to each \( A_J \). For \( H \in t \), write
\[
H = H_J^\perp + H_J
\]
where \( H_J^\perp \) is the orthogonal projection of \( H \) on the \( (r - |J|) \)-dimensional plane
\[
(3.4) \quad \mathcal{P}_J = \{H \in t : \alpha_j(H) = 0 \forall j \in J\},
\]
and $H_J$ lies on $\bigoplus_{j \in J} \mathbb{R}H_{\alpha_j}$, where $H_{\alpha_j} \in \mathfrak{t}$ is defined such that $(H_{\alpha_j}, H) = \alpha_j(H)/i$ for all $H \in \mathfrak{t}$. Let $\Lambda^J$ denote the orthogonal projection of the weight lattice $\Lambda$ on the $|J|$-dimensional subspace $V_J$ of $i^*$ spanned by $\Sigma_J$. We record some facts about $\Lambda^J$ as a lemma, which can be checked straightforwardly by definition. We say an element $\mu \in i^*$ is regular with respect to a given root system $\Sigma \subset i^*$, provided $(\mu, \alpha) \neq 0$ for any $\alpha \in \Sigma$.

**Lemma 10.** (cf. [25, Lemma 7.25]) $\Lambda^J$ is a $|J|$-dimensional lattice contained in the weight lattice $\Lambda_{\Sigma_J}$ of $\Sigma_J$, and we can pick a basis $\{w_1^J, \ldots, w_r^J\}$ of $\Lambda$ such that the projections of the $w_i^J$’s ($i \leq |J|$) on $V_J$ are a basis of $\Lambda^J$, and the projections of the other $w_i^J$’s ($i > |J|$) are zero. Furthermore, any regular element in $\Lambda$ with respect to $\Sigma$ projects to a regular element in $\Lambda_{\Sigma_J}$ with respect to $\Sigma_J$.

For $\mu \in \Lambda$, write

$$\mu = \mu_J + \mu_J^\perp$$

in which $\mu_J \in \Lambda_J = \mathbb{Z}w_1^J + \cdots + \mathbb{Z}w_{|J|}^J$ and $\mu_J^\perp \in \Lambda_J^\perp := \mathbb{Z}w_{|J|+1}^J + \cdots + \mathbb{Z}w_r^J$ according to the above lemma. Also let

$$\mu^J = \text{Proj}_{V_J} \mu = \text{Proj}_{V_J} \mu_J \in \Lambda^J$$

be the projection of $\mu$ on $V_J$.

Let $\Sigma^+_J = \Sigma_J \cap \Sigma^+$ to be the positive system of $\Sigma_J$ and let $\rho_J = \frac{1}{2} \sum_{\alpha \in \Sigma_J^+} \alpha_J$. For each regular element $\gamma$ in $\Lambda_{\Sigma_J}$, let

$$\chi^J_{\gamma} = \frac{\sum_{s \in W_J} \det s e^{s^J \gamma}}{\sum_{s \in W_J} \det s e^{s^J \rho}}$$

be the associated character. We will need the following lemma to describe the behavior of the characters near facets of the alcove.

**Lemma 11.** For any regular element $\mu$ in $\Lambda$, Weyl’s character formula can be rewritten into

$$\chi_{\mu}(H) = \frac{1}{|W_J| \prod_{\alpha \in \Sigma^+ \setminus \Sigma_J^+} \left( e^{\alpha(H)} - e^{-\alpha(H)} \right)} \cdot \sum_{s \in W} \det s e^{(s\mu)(H^J)} \chi_{(s\mu)^J}(H_J).$$

**Proof.** This is essentially equation (7-56) of [25], and its proof is a computation out of the above definitions, based on the Weyl denominator formula (3.2) and rewriting the numerator of $\chi_{\mu}$ as

$$\sum_{s \in W} \det s e^{s\mu} = \frac{1}{|W_J|} \sum_{s \in W} \sum_{s \in W_J} \det s \det s_J e^{s_J s \mu}. \qed$$

We will need the following uniform estimate on characters.

**Lemma 12.** $|\chi_{\mu}(H)| \lesssim |\mu|^{|\Sigma^+|}$, for all regular elements $\mu$ in $\Lambda$, uniformly in $H$.

**Proof.** For any given root system, let

$$d_\mu = \lim_{H \to 0} \chi_{\mu}(H) = \prod_{\alpha \in \Sigma^+} s(\mu, \alpha) \prod_{\alpha \in \Sigma^+} s(\mu, \alpha)$$

be the dimension of the irreducible representation associated with $\chi_{\mu}$ ($\mu \in \Lambda^+$). Since the character $\chi_{\mu}$ is a sum of $d_\mu$ exponentials, and $d_\mu$ is a polynomial in $\mu$ of dimension $|\Sigma^+|$ by the above formula of Weyl, we have the desired estimate for $\mu \in \Lambda^+$. The general result follows from (1) for any $s \in W, \mu \in \Lambda^+$, $\chi_{s\mu} = \det s \chi_{\mu}$, (2) regular elements in $\Lambda$ compose the set $\bigcup_{s \in W} s\Lambda^+$. \qed
Now fix $N \gg 1$. For $J \subsetneq \{0, \ldots, r\}$, define the polytopes

$$P_J = \{ H \in A : \alpha_j(H)/i + 2\pi\delta_{0j} \leq N^{-1} \forall j \in J, \text{ and } \alpha_j(H)/i + 2\pi\delta_{0j} \geq N^{-1} \forall j \notin J \}. $$

Geometrically, $P_J$ looks like a product of the $(r - |J|)$-dimensional facet $A_J$ of $A$ and a cube of dimension $|J|$ (see Figure 1). Observe that $A = \bigcup_{J \subseteq \{0, \ldots, r\}} P_J$, hence it suffices to prove (3.3) uniformly on each $P_J$. In the following, for $a_\mu$ initially defined for $\mu \in \Lambda^+$, we let $a_{s\mu} = a_\mu$, $\forall \mu \in \Lambda^+$, $s \in W$.

**Case 1.** $J = \emptyset$. For any $H \in P_\emptyset$, $\|a(H)\|_2 \gtrsim N^{-1}$ for any $a \in \Sigma$ ($\| \cdot \|$ denoting the distance from the nearest integer), hence

$$|\delta(\exp H)| \gtrsim N^{-|\Sigma^+|}. $$

Using Weyl's denominator formula (3.2) and character formula, we have

$$\left\| \sum_{\mu \in \Lambda^+, |\mu| \leq N} e^{-it|\mu|^2} a_\mu \chi_\mu |\delta|^{-\frac{3}{2}} \right\|_{L^p(I \times P_\emptyset)} \lesssim N^{|\Sigma^+|(1 - \frac{3}{2})} \sum_{s \in W} \left\| \sum_{\mu \in s\Lambda^+, |\mu| \leq N} e^{-it|\mu|^2 + s(H)} a_\mu \right\|_{L^p(I \times P_\emptyset)}. $$

Here we have used $|s\mu| = |\mu|$, $\forall s \in W$, $\mu \in \Lambda$. Now we can apply Strichartz estimates on tori ([9, Theorem 2.4 and Remark 2.5]) to the sum on the right inside of $\| \cdot \|_{L^p}$, and conclude that for any $p > 2 + \frac{4}{r}$

$$\left\| \sum_{\mu \in \Lambda^+, |\mu| \leq N} e^{-it|\mu|^2} a_\mu \chi_\mu |\delta|^{-\frac{3}{2}} \right\|_{L^p(I \times P_\emptyset)} \lesssim \varepsilon^{\frac{1}{2}} N^{|\Sigma^+|(1 - \frac{3}{2}) + \frac{3}{2} - \frac{d-1}{p} + \varepsilon} \|a_\mu\|_{l^2(\Lambda^+)}$$

$$\lesssim \varepsilon \frac{N^2 - \frac{d-1}{p} + \varepsilon}{2} \|a_\mu\|_{l^2(\Lambda^+)}, $$

since $|\Sigma^+| = \frac{d-1}{2}$.

**Case 2.** $|J| = 1$. Let

$$\delta_\mu = \prod_{\alpha \in \Sigma^+ \backslash \Sigma^+_J} \left( e^{\frac{a(H)}{2} - e^{-\frac{a(H)}{2}}} \right), \quad \delta'_\mu = \prod_{\alpha \in \Sigma^+_J} \left( e^{\frac{a(H)}{2} - e^{-\frac{a(H)}{2}}} \right).$$

Figure 1. Decomposition of the alcove (the example of SU(3))
Apply Lemma 11, we have

\[
(3.6) \quad \left| \sum_{\mu \in \Lambda^+, |\mu| \leq N} e^{-it|\mu|^2} a_{\mu} \chi_{\mu} J(\delta J) \right| = \frac{|\delta J|^2}{|W_j|} \sum_{s \in W} \sum_{\mu \in \Lambda^+, |\mu| \leq N} e^{-it|\mu|^2 + \mu J} a_{\mu} \chi_{\mu} J(\delta J) .
\]

By the definition of \( P_J \), for any \( H \in P_J, \) \( \| \alpha (H) \|_{2\pi} \geq N^{-1} \) for \( \alpha \in \Sigma_j \setminus \Sigma_j, \) and \( \| \alpha (H) \|_{2\pi} \lesssim N^{-1} \) for \( \alpha \in \Sigma_j. \) This implies that

\[
(3.7) \quad |\delta J| \gtrsim N^{-|\Sigma_j^+| + |\Sigma_j^+|}, \quad |\delta J| \lesssim N^{-|\Sigma_j^+|}.
\]

By Lemma 12,

\[
(3.8) \quad |\chi_{\mu} J(\delta J)| \lesssim |\mu J|^{\Sigma_j^+} \lesssim |\mu|^{\Sigma_j^+} \lesssim N^{\Sigma_j^+}, \text{ for } |\mu| \leq N.
\]

We estimate

\[
\left| \sum_{\mu \in \Lambda^+, |\mu| \leq N} e^{-it|\mu|^2 + \mu J} a_{\mu} \chi_{\mu} J(\delta J) \right| = \sum_{\mu J} e^{\mu J (H_j)} \sum_{\mu J} e^{-it|\mu J + \mu J|} a_{\mu} \chi_{\mu} J(\delta J)
\]

\[
\lesssim \left( \sum_{\mu J} \left| \sum_{\mu J} e^{-it|\mu J + \mu J|^2 + \mu J J} a_{\mu} \chi_{\mu} J(\delta J) \right|^2 \right)^{\frac{1}{2}}
\]

\[
\lesssim N^{\frac{1}{2J}} \left( \sum_{\mu J} \left| \sum_{\mu J} e^{-it|\mu J + \mu J|^2 + \mu J J} a_{\mu} \chi_{\mu} J(\delta J) \right|^2 \right)^{\frac{1}{2}}.
\]

Now note that \( |\mu J + \mu J|^2 \) is a positive definite quadratic form (with linear terms) in \( \mu J, \) and its second order terms are independent of \( \mu J \) thus its graph defines a hypersurface of constant positive principal curvatures independent of \( \mu J. \) Thus we may again apply Strichartz estimates on tori ([9, Theorem 2.4, Remark 2.5, and Section 7]) to bound the \( L^p \) norm of the above sum

\[
\left\| \sum_{\mu J} e^{-it|\mu J + \mu J|^2 + \mu J J} a_{\mu} \chi_{\mu} J(\delta J) \right\|_{L^p(\mathbb{R}^+ \times E_J, dt dH_J)} \lesssim N^{\frac{-|J|}{2} - \frac{|J+2|}{p} + \epsilon} \left( \sum_{\mu J} \left| a_{\mu} \chi_{\mu} J(\delta J) \right|^2 \right)^{\frac{1}{2}}
\]

\[
\lesssim N^{\frac{-|J|}{2} - \frac{|J+2|}{p} + |\Sigma_j^+| + \epsilon} \left( \sum_{\mu J} \left| a_{\mu} \right|^2 \right)^{\frac{1}{2}},
\]

uniformly for \( \mu J, \) for any

\[
(3.9) \quad p > 2 + \frac{4}{r - |J|}.
\]

The last step is by (3.8). Here \( E_J \) could be any bounded region in the plane \( Pl_j (3.4), \) and \( dH_J \) denote the canonical measure on \( Pl_j. \) For our application, let \( E_J = \{ H \in Pl_j : H + H_j \in P_j \}, \) and

\[
E_J = \{ H \in \bigoplus_{j \in J} \mathbb{R} H_{\alpha_j} : 0 \leq \alpha_j (H) / i + 2 \pi \delta_{\alpha_j} \leq N^{-1}, \forall j \in J \},
\]

so to express our polytope \( P_j \) as \( P_j = \{ H_j + H_j : H_j \in E_J (H_j), H_j \in E_J \}. \) We decompose the Lebesgue measure on \( t \) into the product of the measure on \( Pl_j \) and that on \( \bigoplus_{j \in J} \mathbb{R} H_{\alpha_j}. \) Note that the measure of
$E_J$ is $\asymp N^{-|J|}$. Apply Fubini’s theorem and Minkowski’s integral inequality, we get

$$
\| \sum_{\mu \in \Lambda^+, |\mu| \leq N} e^{-it|\mu|^2 + \mu(H_J^\perp)} a_\mu \chi_{\mu}^J (H_J) \|_{L^p(I \times P_J)} \lesssim N^{\frac{|J|}{2}} \left( \sum_{\mu_j} \left\| \sum_{\mu_j} e^{-it|\mu_j + \mu_j^J(H_J^\perp)} a_\mu \chi_{\mu}^J (H_J) \right\|_{L^p(I \times E_{\perp J}^J(H_J))} ^2 \right)^{\frac{1}{2}} \|_{L^p(E_J)}.
$$

By (3.7), this then implies that

$$
\left\| \sum_{\mu \in \Lambda^+, |\mu| \leq N} e^{-it|\mu|^2} a_\mu \chi_{\mu} |\delta| \right\|_{L^p(I \times P_J)} \lesssim N^{-|J|} \left( |\delta| + |(\Lambda^+ - \delta)| (1 - \frac{1}{p}) + \frac{1}{4} + \frac{|J|}{2} + |\delta| - \frac{|J|}{2} + |\delta| + \varepsilon \right) \|a_\mu\|_{L^2}.
$$

For $|J| = 1$, the range (3.9) of $p$ for the above estimate to hold is $p > 2 + \frac{4}{r+s}$. 

Case 3. $|J| \geq 2$. We will need the following Bernstein’s inequality on product tori.

**Lemma 13.** For $i = 1, 2$, let $\mathbb{R}^{d_i}$ be the $d_i$-dimensional Euclidean space equipped with an inner product $(\cdot, \cdot)$, and let $\Gamma_i$ be any lattice of rank $d_i$ in $\mathbb{R}^{d_i}$. Let $B_i$ be a bounded domain in $\mathbb{R}^{d_i}$, equipped with canonical Lebesgue measures. Let $N_i \geq 1$, $i = 1, 2$. Then for any $p \geq 2$,

$$
\left\| \sum_{\xi_i \in \Gamma_i, |\xi_i| \leq N_i, i = 1, 2} e^{i(x_1, \xi_1) + (x_2, \xi_2)} a_{\xi_1, \xi_2} \right\|_{L^p(B_1 \times B_2, dx_1, dx_2)} \lesssim N_1^{d_1 \left( \frac{1}{2} - \frac{1}{p} \right)} N_2^{d_2 \left( \frac{1}{2} - \frac{1}{p} \right)} \left( \sum_{\xi_1, \xi_2} |a_{\xi_1, \xi_2}|^2 \right)^{\frac{1}{2}} .
$$

**Proof.** This is classical and we outline the proof. For $i = 1, 2$, using a basis of $\Gamma_i$ and its dual basis, we may assume that $x_i = (x_i^1, \ldots, x_i^{d_i}) \in \mathbb{R}^{d_i}$, $\xi_i = (\xi_i^1, \ldots, \xi_i^{d_i}) \in \mathbb{Z}^{d_i}$ and $(x_i, \xi_i) = \sum_j x_i^j \xi_i^j$. Enlarging $B_i$ if necessary, we may assume that $B_i$ consists of finitely many translates of the torus $[0, 2\pi)^{d_i}$. Consider the product de la Vallée Poussin kernel

$$
K_N(x_1, x_2) = \prod_{j=1}^{d_1} K_{N_{1,j}}(x_1^j) \prod_{j=1}^{d_2} K_{N_{2,j}}(x_2^j)
$$

where

$$
K_{N_{i,j}}(x_i^j) = \frac{1}{N_i^{d_i}} \sum_{k=N_i+1}^{2N_i} \sum_{i=1-k}^{k} e^{ijx_i^j} = \frac{1}{N_i} \sin^2 N_i x_i^j - \frac{\sin^2 N_i x_i^j}{2} .
$$

Note that $\|K_{N_{i,j}}\|_{L^p} \lesssim N_i^{1-\frac{1}{p}}$, which implies

$$
\|K_N\|_{L^p(B_1 \times B_2)} \lesssim N_1^{d_1 \left( \frac{1}{2} - \frac{1}{p} \right)} N_2^{d_2 \left( \frac{1}{2} - \frac{1}{p} \right)}.
$$

By definition, $K_N(x_1, x_2)$ acts by convolution on the product tori $[0, 2\pi)^{d_1+d_2}$ as the identity on functions of the form $f = \sum_{\xi_i \in \Gamma_i, |\xi_i| \leq N_i, i = 1, 2} e^{i(x_1, \xi_1) + (x_2, \xi_2)} a_{\xi_1, \xi_2}$, thus we may apply Young’s inequality to conclude
that
\[ \| f \|_{L^p(B_1 \times B_2)} = \| f \ast K \|_{L^p(B_1 \times B_2)} \lesssim \| K \|_{L^q} \| f \|_{L^2} \lesssim N_1^{d_1} N_2^{d_2} \left( \sum_{\xi_1, \xi_2} |a_{\xi_1, \xi_2}|^2 \right)^{\frac{1}{2}}. \]

We also need to characterize the image \( J^\perp \) of the orthogonal projection of the weight lattice \( \Lambda \) on \( V_J := \{ \mu \in \mathbb{R}^r : (\mu, \alpha_j) = 0 \; \forall j \in J \} \).

**Lemma 14.** \( J^\perp \) is a lattice of rank \( r - |J| \). Thus we may decompose \( \mu \in \Lambda \) into
\[ \mu = \mu_J + \mu_J^\perp \]
where \( \mu_J \in J \Lambda := \mathbb{Z} \cdot J w_1 + \cdots + \mathbb{Z} \cdot J w_{|J|} \), \( \mu_J^\perp \in J^\perp \) is a lattice of rank \( r - |J| \), so that \( \Lambda = \mathbb{Z} \cdot J w_1 + \cdots + \mathbb{Z} \cdot J w_r \), and \( J^\perp = \mathbb{Z} \cdot \text{Proj}_{V_J} J w_{|J|+1} + \cdots + \mathbb{Z} \cdot \text{Proj}_{V_J} J w_r \).

**Proof.** By Lemma 10, \( J^\perp \) contains \( \Lambda^\perp \) which is a lattice of rank \( r - |J| \). On the other hand, pick \( I \subset \{0, \ldots, r\} \) such that \( J \subset I \) and \( |I| = r \). Consider the weight lattice \( \Lambda_{\Sigma_i} \) of the root system \( \Sigma_i \). Then \( \Lambda \subset \Lambda_{\Sigma_i} \). Let \( v_i \in \mathbb{R}^r \) (\( i \in I \)) be the fundamental weights such that \( \frac{\alpha_i}{\langle \alpha_i, \alpha_i \rangle} = \delta_{ij} \) (\( i, j \in I \)). Then the projection \( \Lambda_{\Sigma_i} = \bigoplus_{j \in I} \mathbb{Z} v_i \) on \( V_J^J = \bigoplus_{j \in I} \mathbb{R} v_i \) is contained in \( \frac{1}{N} \bigoplus_{j \in I} \mathbb{Z} v_i \) for some integer \( N \), by rationality of the weight vectors \( v_i \) (i.e., \( \frac{\alpha_i}{\langle \alpha_i, \alpha_i \rangle} \in \mathbb{Q}, \forall i \)). Since the \( \mathbb{Z} \)-linear space \( J^\perp \) contains and is also contained in a lattice of rank \( r - |J| \) respectively, it is itself a lattice of rank \( r - |J| \). The rest follows in a standard way. \( \square \)

We now write
\[ \sum_{\mu \in \Lambda^\perp, \| \mu \| \leq N} e^{-it|\mu|^2 + \mu(H_J)} a_\mu \chi_{\mu, J}^J(H_J) = \sum_{\mu_J^\perp} \sum_{m} e^{itm + \mu_J^\perp(H_J)} \sum_{|\mu_J, \mu_J^\perp|^2 = m} a_\mu \chi_{\mu_J, J}^J(H_J). \]

By rationality of the weight lattice \( \Lambda \), we can pick a universal constant \( M \in \mathbb{R} \) depending only on the root system \( \Sigma \), such that the quadratic form \( M|\mu|^2 \) in \( \mu \in \Lambda \) has integral coefficients, hence \( m = -|\mu|^2 \in M^{-1} \mathbb{Z} \) for all \( \mu \in \Lambda \). Also note that \( |\mu_J^\perp| \lesssim N \) and \( |m| \lesssim N^2 \) in the above sum. Using Lemma 13, we estimate for \( p \geq 2 \)
\[ \left\| \sum_{\mu \in \Lambda^\perp, \| \mu \| \leq N} e^{-it|\mu|^2 + \mu(H_J)} a_\mu \chi_{\mu, J}^J(H_J) \right\|_{L^p(I \times E_J^J(H_J))} \lesssim N^{(2+r-|J|)(\frac{1}{2} - \frac{1}{p})} \left( \sum_{\mu_J^\perp} \sum_{m} \sum_{|\mu_J, \mu_J^\perp|^2 = m} a_\mu \chi_{\mu_J, J}^J(H_J) \right)^{\frac{1}{2}}. \]

**Lemma 15.** For \( m \in M^{-1} \mathbb{Z} \), \( |m| \lesssim N^2 \), and \( |\mu_J^\perp| \lesssim N \), we have
\[ \# \{ \mu \in J \Lambda : -|\mu_J + \mu_J^\perp|^2 = m \} \leq N^{|J| - 2 + \varepsilon}. \]

**Proof.** Write \( \mu_J = n_1 \cdot J w_1 + \cdots + n_{|J|} \cdot J w_{|J|} \in J \Lambda \) as in Lemma 14. Suppose \( M|\mu_J + \mu_J^\perp|^2 = -Mm \). Since \( |\mu_J + \mu_J^\perp| \geq |\mu_J| \geq |n_j| \) for each \( j = 1, \ldots, |J| \), we have \( |n_j| \lesssim |m|^2 \lesssim N \). For \( j = 3, \ldots, |J| \), fix \( n_j \) such that \( |n_j| \lesssim N \). Then we can rewrite \( M|\mu_J + \mu_J^\perp|^2 = -Mm \) into \( P(n_1, n_2) = an_1^2 + bn_1 n_2 + cn_2^2 + dn_1 + en_2 + f = 0 \) where all the coefficients in \( P \) are integers bounded by \( \lesssim N^2 \), and \( \Delta = b^2 - 4ac < 0 \) (in particular \( a \neq 0 \)). By a standard result (see for example Lemma 8 in [4]), the number of integral solutions to \( P(n_1, n_2) = 0 \) is
\[ \sum_{j \mu, -|j \mu + j \mu^+|^2 = m} a_{j \mu} \chi_{j \mu}(H_J) \] 
\[ \lesssim_N N^{|J| - 2 + 2|\Sigma_j^+| + \varepsilon} \sum_{j \mu, -|j \mu + j \mu^+|^2 = m} |a_{j \mu}|^2. \]

Now (3.10) and (3.7) give
\[ \left\| \sum_{|j \mu + j \mu^+|^2 = m} e^{-it|j \mu|^2} a_{j \mu} \chi_{j \mu} \delta \right\|_{L^p(\mathbb{R}^n)} \]
\[ \lesssim e^{-|\Sigma_j^+|/2}(1 - |\Sigma_j^+|)(1 - \frac{1}{p}) + (2 + r - |J|)(1 - \frac{1}{p}) + \frac{|J| + 2 |\Sigma_j^+| + 1}{2} \varepsilon \|a_{j \mu}\|_2^2 \] 
\[ \lesssim e^{-|\Sigma_j^+|/2}(1 - |\Sigma_j^+|)(1 - \frac{1}{p}) + (2 + r - |J|)(1 - \frac{1}{p}) + \frac{|J| + 2 |\Sigma_j^+|}{2} \varepsilon \|a_{j \mu}\|_2^2 \] 
\[ \lesssim e^{-d_2 + \varepsilon} \|a_{j \mu}\|_2^2, \]
for any \( p \geq 2 \).

We have proved the following proposition.

**Proposition 16.**

(3.11)
\[ \left\| \sum_{|j \mu + j \mu^+|^2 = m} e^{-it|j \mu|^2} a_{j \mu} \chi_{j \mu} \delta \right\|_{L^p(\mathbb{R}^n)} \lesssim_N \frac{d_2 + \varepsilon}{p} \|a_{j \mu}\|_2. \]

hold (1) for \( p > 2 + \frac{4}{d_1} \) if \( J = \emptyset \), and (2) for \( p > 2 + \frac{4}{d_1} \) if \( |J| = 1 \), and (3) for \( p \geq 2 \) if \( |J| \geq 2 \).

In particular, (3.3) holds for \( p > 2 + \frac{4}{d_1} \) with an \( N^\varepsilon \) loss. This loss can be eliminated in a standard way outlined as follows. Let \( f \) be an \( L^2 \) class function on \( U \). From (2.4), we know that the level set measure \( m_\lambda = \{(t, x) \in I \times U : |P_N e^{it\Delta} f(x)| > \lambda \} \) satisfies
\[ m_\lambda \lesssim N^{d_2 - (d + 2)} \lambda^{-p} \|f\|_{L^p(U)}^p, \forall \lambda \gtrsim N^{d_2 - \varepsilon}, \forall p > 2 + \frac{4}{d_1}. \]

Since Theorem 9 has already been shown to hold with an \( \varepsilon \) loss, we have
\[ m_\lambda \lesssim \varepsilon N^{d_2 - (d + 2) + \varepsilon} \lambda^{-p} \|f\|_{L^p(U)}^p, \forall \lambda > 0, \forall p > 2 + \frac{4}{d_1}. \]

The result follows if we write \( \int_{I \times U} |P_N e^{it\Delta} f|^p \, dt \, du \approx \int_0^\infty \lambda^{p-1} m_\lambda \, d\lambda = \int_{\lambda \geq CN^{d_2 - \frac{2}{d_1}}} \lambda^{-\frac{4}{d_1} + \frac{4}{d_1}} \lambda^{p-1} m_\lambda \, d\lambda + \int_{\lambda \leq CN^{d_2 - \frac{2}{d_1}}} \lambda^{p-1} m_\lambda \, d\lambda, \) and apply the above two estimates to these two integrals respectively.

We now indicate the necessary changes in the above proof of Theorem 9 for a general group. Since Strichartz estimates are preserved under taking a finite Riemannian cover of the underlying manifold ([25, Proposition 3.3]), we may assume \( U \) is a product of simply connected simple Lie groups and a torus, and we
assume it is equipped with rational metrics. Then an alcove $A$ for $U$ is a product of the alcoves for each of the simple group factors and a torus, and all these factors are orthogonal to each other. We may decompose each alcove factor as before, which yields a decomposition of the alcove $A$. We still distinguish three cases according to how many faces the points are close to, and it is now apparent that we may slightly adapt the above argument for each case to make it work.

3.2. An Approach to Further Improvement. We propose an improvement towards Case 1 and 2 in the previous section. Looking back at $\delta_J$ as estimated in (3.8), we observe that $\delta_J$ enjoys larger lower bound more away from the faces of the alcove, a fact we could quantify by computing the $L^p(P_J)$ bounds ($p < \infty$) of $1/\delta_J$. This piece of information could be applied to derive $L^p(I \times P_J)$ estimates on (3.6) with the help of Hölder’s inequality, if we conjecture the following exponential sum estimates with respect to mixed Lebesgue norms, based on a scale invariance consideration.

**Conjecture 17.** Let $\mathbb{R}^d$ be equipped with an inner product $(\cdot, \cdot)$ and $\Gamma$ a rational rank-$d$ lattice in $\mathbb{R}^d$. Let $B$ be a bounded domain in $\mathbb{R}^d$. Then

$$\tag{3.12} \left\| \sum_{\mu \in \Gamma, |\mu| \leq N} e^{it(\mu, \xi)} a_{\mu} \right\|_{L^p(I, L^q(B))} \lesssim N^{\frac{d}{2} - \frac{2}{p} - \frac{d}{q}} \|a_{\mu}\|_{\ell^2(\Gamma)}$$

for all pairs $p, q \geq 2$ such that $\frac{d}{2} - \frac{2}{p} - \frac{d}{q} > 0$. Furthermore, let $B^{d-1}$ be a bounded domain on any hyperplane in $\mathbb{R}^d$. Then

$$\tag{3.13} \left\| \sum_{\mu \in \Gamma, |\mu| \leq N} e^{it(\mu, \xi)} a_{\mu} \right\|_{L^p(I, L^q(B^{d-1}_B))} \lesssim N^{\frac{d}{2} - \frac{2}{p} - \frac{d-1}{q}} \|a_{\mu}\|_{\ell^2(\Gamma)}$$

for all pairs $p, q \geq 2$ such that $\frac{d}{2} - \frac{2}{p} - \frac{d-1}{q} > 0$.

For $p = q$, (3.12) is the established Strichartz estimates on tori (except for a possible $N^\varepsilon$ loss for irrational nonrectangular tori). Note that (3.12) with the exponents $(p, q, d - 1)$ (so that $\frac{d-1}{2} - \frac{2}{p} - \frac{d-1}{q} > 0$) implies (3.13) (but not necessarily in the larger range $\frac{d}{2} - \frac{2}{p} - \frac{d-1}{q} > 0$), by an argument similar to one in the demonstration for Case 2 of last section.

The Euclidean analogues of this conjecture are indeed true. The Euclidean version of (3.12) above is the established mixed norm Strichartz estimates. That of (3.13) can be obtained by a change of variables (letting $m = (\mu, \mu)$) and an application of Littlewood-Paley theory and Bernstein’s inequalities for mixed Lebesgue norms, similar to the argument in establishing Case 3 of last section.

We now show how this conjecture would imply better Strichartz estimates for class functions on compact Lie groups. Since the estimates are conditional, we will avoid thorough discussion over all types of compact Lie groups to minimize technicality. Also to put things in a more concrete setting, instead, we will pick the special unitary group SU$(r + 1)$ (which in a certain sense is the simplest case for our computation, due to the simplicity of its extended Dynkin diagram, cf. the proof of Lemma 19), and demonstrate the following proposition.

**Proposition 18.** Conjecture 17 implies (3.3) on SU$(r + 1)$ for all $p > 2 + \frac{4}{d}$ with $r \geq 2$, which is the largest possible range (except for the endpoint).

This would imply that the three sphere (the SU(2) case), which would be the only case among the special unitary groups for which we have the optimal range $p > 4$ different from $p > 2 + \frac{4}{d}$, is merely a peculiarity. This may be explained by looking at the case when $J = \{1, \ldots, r\}$ so that the polytope $P_J$ is an $N^{-1}$
neighborhood of the origin \( H = 0 \). For \( r \geq 2 \), this belongs to Case 3 of last section, where a standard estimate of number of integral solutions to a positive quadratic form of \( r \) variables is applicable to yield the desired estimates on the \( L^p(I \times \mathcal{P}) \) norm for any \( p \geq 2 \). But for \( r = 1 \), the quadratic form becomes a square of one variable, so counting its integral solutions is meaningless, while the following exponential sum estimate ([16, Lemma 3.1]) becomes crucial for the rank one case

\[
\left\| \sum_{|n| \leq N} e^{t \pi n^2} a_n \right\|_{L^p(I)} \lesssim N^{\frac{p}{2} - \frac{p}{2}} |a_n|, \forall p > 4.
\]

We now prove Proposition 18 by demonstrating (3.11) for all \( p > 2 + \frac{4}{3} \), for \( J = \emptyset \) and \(|J| = 1\) as a consequence of (3.12) and (3.13) in the full expected range of \( p, q \), respectively.

**Case 1.** \( J = \emptyset \). For (3.2), we compute the \( L^p \) norms of \( 1/\delta \) over the simplex \( \mathcal{P}_{\emptyset} \). The shape of \( \mathcal{P}_{\emptyset} \) looks exactly like the alcove \( A \), but slightly shrunk. Since the size of \( \delta \) is sensitive to which of the faces \( \{ H \in \Omega : \alpha_k(H)/i + 2\pi \delta_{ok} = 0 \} \) \( (k = 0, \ldots, r) \) the points are close to, we decompose \( \mathcal{P}_{\emptyset} \) into \( r \) regions, each close to a vertex of \( \mathcal{P}_{\emptyset} \). We perform a version of barycentric subdivision of the alcove \( A \). For \( l \in \{0, \ldots, r\} \) and the vertex \( A_{\{0, \ldots, r\}\setminus\{l\}} \) of \( A \), let \( C_l \) be the convex hull of the barycenters of all the facets of \( A \) that has \( A_{\{0, \ldots, r\}\setminus\{l\}} \) as a boundary point, and set \( \mathcal{P}_{\emptyset, l} = \mathcal{P}_{\emptyset} \cap C_l \). Then \( \mathcal{P}_{\emptyset} = \bigcup_{l=0}^r \mathcal{P}_{\emptyset, l} \) (see Figure 1). Let \( \Sigma^l \) denote the root system generated by \( \{\alpha_k, k \neq l\} \). It is known for the case of \( SU(r + 1) \), \( \Sigma^l \cong \Sigma^0 = \Sigma \) for any \( l = 0, \ldots, r \).

**Lemma 19.** For \( SU(r + 1) \), we have

\[
\left\| \frac{1}{\delta} \right\|_{L^p(\mathcal{P}_{\emptyset, l})} \asymp \begin{cases} N^{\frac{p}{2} - \frac{p}{2}}, & \text{if } \frac{p}{2} - \frac{p}{2} > 0, \\ \log^{\frac{p}{2}} N, & \text{if } \frac{p}{2} - \frac{p}{2} = 0, \\ 1, & \text{if } \frac{p}{2} - \frac{p}{2} < 0. \end{cases}
\]

**Proof.** Let \( t_k(H) = \alpha_k(H)/i + 2\pi \delta_{ok} \) for each \( k = 0, \ldots, r \). The functions \( \{t_k, k \neq l\} \) provide a coordinate system for \( \mathcal{P}_{\emptyset, l} \), and \( \mathcal{P}_{\emptyset, l} \) is contained in a parallelootope \( \{N^{-1} \leq t_k(H) \leq C \forall k \neq l\} \), as well as contains such a parallelootope \( \{N^{-1} \leq t_k(H) \leq c \forall k \neq l\} \), for some universal constants \( C, c \in (0, 2\pi) \) that depend only on the ambient root system \( \Sigma \). For any \( H \in \mathcal{P}_{\emptyset, l} \), there exists some permutation \( (k_1, \ldots, k_r) \) of \( \{k = 0, \ldots, r, k \neq l\} \) such that \( t_{k_1}(H) \leq t_{k_2}(H) \leq \cdots \leq t_{k_r}(H) \). We claim that

\[
|\delta(H)| \gtrsim (t_{k_1}(H))^{r-1}(H) \cdots t_{k_1}(H), \text{ uniformly in } H.
\]

We need the specific structure of the root system for \( SU(r+1) \). In terms of an orthonormal basis \( \{e_1, \ldots, e_{r+1}\} \) of \( \mathbb{R}^{l+1} \), we may write the root system \( \Sigma \) as \( \{e_i - e_j, i \neq j\} \) with \( e_j = e_j - e_{j+1} \) \( (j = 0, \ldots, r) \). Here \( e_0 = e_{r+1} \), and for convenience, for \( k = (r+1)m + j \) \( (j = 1, \ldots, r+1, m \in \mathbb{Z}) \), let \( e_k = e_j \). The roots \( \beta_j = e_{i+j} - e_{i+j+1} \) \( (j = 1, \ldots, r) \) form a simple system for \( \Sigma^l \), and the corresponding positive system \( (\Sigma^l)^+ \) consists of the roots \( \beta_i + \beta_{i+1} + \cdots + \beta_j, 1 \leq i \leq j \leq r \). Let \( s_j(H) = t_{i+j}(H), j = 1, \ldots, r \). Now for \( H \in \mathcal{P}_{\emptyset, l} \),

\[
|\delta(H)| \asymp \prod_{\alpha \in \Sigma^l} \left\| \frac{\alpha(H)}{2\pi i} \right\|_2 \asymp \prod_{1 \leq i \leq j \leq r} (s_i(H) + s_{i+1}(H) + \cdots + s_j(H)).
\]

Now assume \( s_{j_1}(H) \leq s_{j_2}(H) \leq \cdots \leq s_{j_r}(H) \) for some permutation \( (j_1, \ldots, j_r) \) of \( \{1, \ldots, r\} \). We induct on \( r \) to prove that

\[
\prod_{1 \leq i \leq j \leq r} (s_i(H) + s_{i+1}(H) + \cdots + s_j(H)) \gtrsim s_{j_1}(H)s_{j_2}^{r-1}(H) \cdots s_{j_r}(H).
\]
The case \( r = 1 \) is clear. Suppose \( 1 = j_m \) for some \( m = 1, \ldots, r \). Then by the induction hypothesis,
\[
\prod_{2 \leq i < j \leq r} (s_i + \cdots + s_j) \gtrsim s_{j_r}^{-1} \cdots s_{j_{m+1}}^{-1} \cdot s_{j_m}^{m-1} \cdots s_{j_1}.
\]
Now we prove by another induction on \( r \) that
\[
\prod_{1 \leq i \leq r} (s_1 + \cdots + s_j) \gtrsim s_{j_{r-1}} \cdots s_{j_{m+1}} \cdot s_{j_m}^m.
\]
By induction, \( \prod_{1 \leq j \leq j_r-1} (s_1 + \cdots + s_j) \gtrsim s_{j_{r-1}}^{-1} s_{j_{r-2}} \cdots s_{i_n+1} \cdot s_{i_n}^n \), here \( s_{i_n} \geq s_{i_n-1} \geq \cdots \geq s_{i_t} \) for some permutation \((i_1, \ldots, i_{j_r - 1})\) of \( \{1, \ldots, j_r - 1\} \), with \( 1 = i_n \) for some \( n \in \{1, \ldots, j_r - 1\} \). Thus
\[
\prod_{1 \leq i \leq r} (s_1 + \cdots + s_j) = \left( \prod_{1 \leq j \leq j_r-1} \right) \cdot \left( \prod_{j \geq j_r} \right) \gtrsim s_{j_{r-1}}^{-1} \cdots s_{i_n+1} \cdot s_{i_n}^n \cdot s_{j_r}^{-r-j_r+1}
\]
which is \( \gtrsim s_{j_r} \cdots s_{j_{m+1}} \cdot s_{j_m}^m \). Now (3.15) and (3.16) imply
\[
\prod_{1 \leq i \leq j \leq r} (s_1 + \cdots + s_j) \gtrsim s_{j_{r-1}}^{-1} \cdots s_{j_{m+1}} \cdot s_{j_m}^m \cdot s_{j_{r-1}}^{-1} \cdots s_{j_1}.
\]
This implies (3.14). On the other hand, if \( H \in \mathbb{P}_{a,l} \) satisfies \( s_1(H) \leq s_2(H) \leq \cdots \leq s_r(H) \), then clearly
\[
\prod_{1 \leq i \leq j \leq r} (s_i(H) + \cdots + s_j(H)) \approx s_r^r(H) s_{r-1}^{-1}(H) \cdots s_1(H).
\]
We can now estimate
\[
\int \left| \delta(H) \right|^{-p} \, dH \lesssim \int_{N^{-1} \leq s_{j}(H) \leq C} \left| \delta(H) \right|^{-p} \, dH
\]
\[
\approx \sum_{(j_1, \ldots, j_r) \text{ a permutation of } (1, \ldots, r)} \int_{N^{-1} \leq s_{j_1} \leq \cdots \leq s_{j_r} \leq C} \left| \prod_{1 \leq i \leq j \leq r} (s_i(H) + \cdots + s_j(H)) \right|^{-p} \, dH.
\]
By (3.17) and (3.18), the above is
\[
\approx \int_{N^{-1} \leq s_1 \leq \cdots \leq s_r \leq C} s_r^{-rp} s_{r-1}^{-(r-1)p} \cdots s_1^{-p} \, ds_r \, ds_{r-1} \cdots ds_1.
\]
This then gives the desired result, noting that \( d - r = r(r + 1) \).

Now using Weyl’s character and denominator formulas, we estimate
\[
\left\| \sum_{\mu \in A^+, |\mu| \leq N} e^{-i(t|\mu|^2 + \mu \cdot \chi_n)} \right\|_{L_p(1 \times \mathbb{P}_{a,l})} \lesssim \sum_{s \in W} \left\| \sum_{\mu \in A^+, |\mu| \leq N} e^{-i(t|\mu|^2 + \mu \cdot H)} a_{\mu} \right\|_{L_p(1 \times \mathbb{P}_{a,l})} \|rac{1}{\delta^{1/p}}\|_{L^{p}(\mathbb{P}_{a,l})}.
\]
Here \( \frac{1}{q} + \frac{1}{\mu} = \frac{1}{p} \). Using the conjectured (3.12) and Lemma 19, the above is bounded by
\[
\lesssim N^{\frac{\nu}{2} - \frac{\nu}{q} + \frac{\nu}{2} (1 - \frac{1}{ \mu}) - \frac{1}{2} \|a_{\mu}\|_{L^2} = N^{\frac{\nu}{2} - \frac{\nu}{q} - \frac{1}{p} \|a_{\mu}\|_{L^2}}
\]
provided
\[
2 \leq p \leq q, \frac{r}{p} - \frac{2}{q} \geq 0, \frac{d - r}{2} \left( 1 - \frac{2}{p} \right) - r \left( \frac{1}{p} - \frac{1}{q} \right) > 0.
\]
An inspection of the above inequalities in the \( \left( \frac{1}{p}, \frac{1}{q} \right) \) plane (see Figure 2) shows that any \( p > 2 + \frac{4}{d} \) is admissible. Since \( P_\emptyset = \bigcup_J P_{\emptyset, J} \), we have shown that the conjectured (3.12) in the expected range imply (3.11) for \( J = \emptyset \) for all \( p > 2 + \frac{4}{d} \).

Case 2. \( |J| = 1 \). We compute the \( L^p \) bounds of \( 1/\delta_J \) over the frustum \( P_J \). Similar to Case 1, we decompose \( P_J = \bigcup_{l \in J} P_{j, l} \) for \( P_{J, l} = P_J \cap C_l \) (see Figure 1).

To integrate over \( P_{J, l} \), we use the functions \( t_k \ (k \neq l) \) and the relabelled ones \( s_n \ (n = 1, \ldots, r) \) as defined in Case 1. For \( J = \{ j \} \), assume that \( s_m = t_j \) for some \( m = 1, \ldots, r \). The functions \( \{ s_n, n = 1, \ldots, r \} \) provide a coordinate system for \( P_{j, l} \), such that for \( H \in P_{j, l} \) we have \( 0 \leq s_m \leq N^{-1} \) and \( N^{-1} \leq s_n(H) \leq C \) for \( n \neq m \), for some positive constant \( C < 2\pi \) depending only on the ambient root system \( \Sigma \).

Following the proof of Lemma 19, we now estimate

\[
\int_{P_{j, l}} |\delta_J|^{-p} dH \lesssim \int_{0 \leq s_m(H) \leq N^{-1}} |\delta(H)|^{-p} s_m^p dH \\
\lesssim \sum_{(n_1, \ldots, n_{r-1}) \text{ a permutation of } \{n \neq m\}} \int_{0 \leq s_m \leq N^{-1}} \int_{N^{-1} \leq s_{n_1} \leq \cdots \leq s_{n_{r-1}} \leq C} \left| \prod_{1 \leq i \leq r} (s_i + \cdots + s_j) \right|^{-p} s_m^p dH \\
\lesssim N^{\frac{d+r-1}{2} - p},
\]

provided \( d-r-2 - \frac{r+1}{p} > 0 \). Here we used \( \frac{r(r+1)}{2} = \frac{d-r}{2} \). Since \( P_J = \bigcup_{l \in J} P_{J, l} \), we have

\[
\left\| \frac{1}{\delta_J} \right\|_{L^p(P_J)} \lesssim N^{\frac{d+r-1}{2} - \frac{r}{p}}, \text{ provided } \frac{d-r}{2} - 1 - \frac{r-1}{p} > 0.
\]
Now using (3.6), $|δ^J| \lesssim N^{-[\frac{r}{2}]} = N^{-1}$, and the decomposition $P_J = \{H_j^+ + H_j^- : H_j^+ \in E_j^+(H_j), H_j^- \in E_j^-(H_j)\}$, we estimate

$$\sum_{|μ| ≤ N} \sum_{j} e^{-it|μ|^2} |a_μχ_μ| δ_j^p \right\|_{L^p(1 \times P_j)} \lesssim \sum_{s \in W} N^{-\frac{2}{p}} \left\| \sum_{|μ| ≤ N} e^{-it|μ|^2 + μ(H_j^+)} |a_μχ_μ| (H_j^-) \right\|_{L^p(1, L^q(E_j^+(H_j)))} \cdot \frac{1}{δ_j^{\frac{1}{p} - \frac{1}{q}}} \right\|_{L^q(E_j^+(H_j))} \right\|_{L^p(E_j)}.$$  

Here $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Since $E_J^+(H_j)$ is a uniformly bounded region in the hyperplane $\{H ∈ t : \alpha_j(H) = 0\}$ ($J = \{j\}$), assuming the conjectured estimates (3.13), we have the above is bounded by

$$\sum_{s \in W} N^{-\frac{2}{p} + \frac{2}{q} - \frac{2}{r}} \left\| |a_μχ_μ| (H_j^-) \right\|_{L^p} \cdot \left\| \frac{1}{δ_j^{\frac{1}{p} - \frac{1}{q}}} \right\|_{L^q(E_j^+(H_j))} \right\|_{L^p(E_j)}.$$  

By (3.8), the above is bounded by

$$\lesssim N^{1 - \frac{2}{p} + \frac{2}{q} - \frac{2}{r}} \|a_μ\|_{L^2} \cdot \left\| \frac{1}{δ_j^{\frac{1}{p} - \frac{1}{q}}} \right\|_{L^q(E_j^+(H_j))} \right\|_{L^p(E_j)}.$$  

which is bounded via Hölder’s inequality by

$$\lesssim N^{1 - \frac{2}{p} + \frac{2}{q} - \frac{2}{r} - \frac{r - 1}{q}} \|a_μ\|_{L^2} \cdot \left\| \frac{1}{δ_j^{\frac{1}{p} - \frac{1}{q}}} \right\|_{L^q(E_j^+(H_j))} \right\|_{L^p(E_j)}.$$  

$$\lesssim N^{1 - \frac{2}{p} + \frac{2}{q} - \frac{2}{r} - \frac{r - 1}{q} - \frac{1}{p} - \frac{r - 1}{q}} \|a_μ\|_{L^2} \cdot \left\| \frac{1}{δ_j^{\frac{1}{p} - \frac{1}{q}}} \right\|_{L^q(E_j^+(H_j))} \right\|_{L^p(P_j)}.$$  

which is then bounded via (3.19) by

$$\lesssim N^{1 - \frac{2}{p} + \frac{2}{q} - \frac{2}{r} - \frac{r - 1}{q} + \left(\frac{2 - r}{p} - 1\right) \left(1 - \frac{2}{p}\right) - (r - 1) \left(\frac{1}{p} - \frac{1}{q}\right)} \|a_μ\|_{L^2} = N^{\frac{2}{p} - \frac{4}{d} + \frac{2}{p}} \|a_μ\|_{L^2},$$  

provided

$$2 \leq p ≤ q, \frac{r}{2} - \frac{2}{p} - \frac{r - 1}{q} ≥ 0, \left(\frac{d - r}{2} - 1\right) \left(1 - \frac{2}{p}\right) - (r - 1) \left(\frac{1}{p} - \frac{1}{q}\right) > 0.$$  

An inspection of the above inequalities on the $\left(\frac{1}{p}, \frac{1}{q}\right)$ plane (see Figure 2) reveals that any $p > 2 + \frac{2}{d-2}$ is admissible. We have shown that the conjectured (3.13) in the expected range imply (3.11) for $|J| = 1$ for all $p > 2 + \frac{2}{d-2}$.

Note that $2 + \frac{2}{d-2} ≤ 2 + \frac{4}{d}$ for any $d ≥ 4$, which is the case when $r ≥ 2$. Combining Case 1 and 2, we conclude that Conjecture 17 implies (3.11) for any $J ⊆ \{0, \ldots, r\}$, and thus the Strichartz estimates (3.1) for class functions on SU($r+1$) ($r ≥ 2$), in the maximal possible range $p > 2 + \frac{4}{d}$ (except the endpoint).

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References


Department of Mathematics, University of Connecticut, Storrs, CT 06269

Email address: yunfeng.zhang@uconn.edu