Strichartz estimates for the Schrödinger equation on products of odd-dimensional spheres

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Abstract. We generalize the framework developed in [25] to prove certain Strichartz estimates which are scale-invariant up to an ε-loss on any product of odd-dimensional spheres. Some partial results toward such Strichartz estimates on a general compact globally symmetric space are also given, including a kernel estimate sharp up to an ε-loss near rational times and near corners of a maximal torus.

1. Introduction

Let M be a compact Riemannian manifold and let Δ denote the Laplace-Beltrami operator. Let $e_1, e_2, \ldots$ denote an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of $-\Delta$ with respect to the eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \ldots$, and define the Sobolev spaces

$$H^s(M) := \left\{ f \in L^2(M): \ f = \sum_i f_i e_i, \ \|f\|_{H^s(M)} := \left(\sum_i |f_i|^2 (\lambda_i^s + 1) \right)^{1/2} < \infty \right\}.$$

We may define the one-parameter unitary group $e^{it\Delta} : H^s(M) \to H^s(M)$ $(t \in \mathbb{R})$, such that

$$e^{it\Delta} f = \sum_i e^{-it\lambda_i} f_i e_i, \text{ for } f = \sum_i f_i e_i.$$

Then $u(t,x) = e^{it\Delta} f(x)$ provides the solution to the linear Schrödinger equation

$$\begin{cases} i\partial_t u(t,x) + \Delta u(t,x) = 0, \\ u(0,x) = f(x). \end{cases}$$

By Sobolev embedding, $H^s(M) \subset L^{\frac{2d}{d-2s}}(M)$ for $s < \frac{d}{2}$. An interesting behavior of the Schrödinger flow $e^{it\Delta}$ is that although for each fixed $t \in \mathbb{R}$ and $f \in H^s(M)$, $e^{it\Delta} f$ may not lie in $L^q(M)$ for any $q > \frac{2d}{d-2s}$, but when averaging over t in any (finite) interval I, we expect estimates of the form

$$\|e^{it\Delta} f\|_{L^p(I, L^q(M))} \leq C \|f\|_{H^s(M)} \tag{1.1}$$

for some $q > \frac{2d}{d-2s}$, thus the solution $e^{it\Delta} f$ actually gains integrability almost surely in time. We refer to such estimates as Strichartz estimates, which characterize the dispersive nature of solutions and have important applications in well-posedness and scattering theory for the nonlinear Schrödinger equations. If the underlying manifold M is noncompact and satisfies some geometric restraints such as nontrapping for geodesics, these estimates are usually proved by first establishing $L^\infty(M)$ decay of solutions with $L^1(M)$ initial data as $t \to \infty$, making use of the assumptions on M so that the solutions would genuinely "disperse to infinity". For reference, see the original work [12, 19] on the Euclidean space, and [3, 21, 1, 18, 2, 11] on other noncompact manifolds such as hyperbolic spaces, Damek-Ricci spaces, and some locally symmetric spaces. However, in the compact picture, naively, there is no "infinity" for the solutions to disperse into, so the global geometry of M should become very relevant, and techniques in proving the Strichartz estimates should somehow combine global geometry with the local dispersion phenomenon of solutions in a nice way.

We should distinguish two classes of Strichartz estimates. By a scale consideration, a necessary condition for the above Strichartz estimates to hold is

$$s \geq \frac{d}{2} - \frac{2}{p} - \frac{d}{q}, \tag{1.2}$$

$p, q \geq 2$. Here d denotes the dimension of M. If the above equality holds, we say the Strichartz estimates are scale-invariant, and non-scale-invariant otherwise. A fundamental result from [8] establishes that on a general compact Riemannian manifold (1.1) holds for all admissible pairs (p, q)

$$\frac{d}{2} = \frac{2}{p} + \frac{d}{q},\ p, q \geq 2,\ (p, q, d) \neq (2, \infty, 2),$$

with $s = \frac{1}{p}$. These estimates are non-scale-invariant, and are sharp for $(p, q, s) = (2, \frac{2d}{d-2}, \frac{1}{2})$ on spheres of dimension $d \geq 3$. The proof is in its essence a local argument, which establishes the same kind of L^∞ decay as in the noncompact setting for solutions with frequency localized initial data, but only for a short period of time due to the compact nature of M, which results in the non-scale-invariance of the estimates. See also [10] for more sharp non-scale-invariant estimates on spheres. On the other hand, known results of scale-invariant estimates (and their ε-loss versions) are all established by truly global arguments. Consider the special cases when $p = q$, then the scale-invariant estimates read

$$\|e^{it\Delta} f\|_{L^p(I\times M)} \leq C\|f\|_{H^{\frac{d}{2}-\frac{d+2}{p}}(M)}. \tag{1.3}$$

We summarize known results of the above type:

- In [9, 17], (1.3) is established for $p > 4$ for any sphere $\mathbb{S}^d$ with standard metric (and generally for any Zoll manifolds) with dimension $d \geq 3$, and $p \geq 6$ for the two sphere (or any Zoll surface); $p > 4$ is also the optimal range for $\mathbb{S}^d$ $(d \geq 3)$ ([8]). The global geometry of the spheres and the Zoll manifolds is explored in the proof in terms of the explicit spectral distribution of the Laplacian.
- In [5, 6, 7, 20], (1.3) is established for the optimal range $p > \frac{2(d+2)}{d}$ for rectangular tori, with the same estimates but with an ε-loss established for nonrectangular tori. The proofs use up-to-date tools of harmonic analysis on tori. Note also in [25], it is observed that the ε-loss can be eliminated for rational nonrectangular tori.
- In [25], (1.3) is established for $p \geq \frac{2(r+4)}{r}$ for any compact Lie group with a rational metric. Here r is the dimension of a maximally embedded torus of the group. The proof also explicitly applies harmonic/global analysis on groups. This range of p is not expected to be optimal.

We now continue our study after [25] and ask the question of whether the application of harmonic analysis on compact Lie groups as in [25] can be generalized to the setting of compact globally symmetric spaces. As in [25], we relate the Strichartz estimates to some Lie theoretic Weyl-type quadratic exponential sums as follows:

$$K_N(t, H) = \sum_{\lambda\in\Lambda^+} \varphi\Big(\frac{-|\lambda+\rho|^2 + |\rho|^2}{N^2}\Big) e^{it(-|\lambda+\rho|^2+|\rho|^2)} d_\lambda \Phi_\lambda(H),\ H \in \mathfrak{a},$$

where Λ^+ is the part of the weight lattice $\Lambda \subset \mathfrak{a}^*$ inside of a Weyl chamber associated to a root system, φ is any smooth cutoff function, d_λ is a polynomial function in λ (the dimensions of spherical representations), and most importantly

$$\Phi_\lambda(H) = \sum_{i=1}^{q} c_i e^{i\lambda_i(H)},\ \lambda_i \in \Lambda,\ c_i \geq 0,\ H \in \mathfrak{a},$$

are some special linear exponential sums (the spherical functions). Such $K_N(t,H)$ will appear as the mollified kernel function in

$$e^{it\Delta}\varphi(N^{-2}\Delta)f = f * K_N.$$

When the underlying manifold is actually a compact Lie group, the spherical functions Φ_λ are given explicitly by the Weyl character formula, which allows us to establish in [25] the following "major-arc" (i.e., near rational times) estimates for the above quadratic exponential sums, using the classical technique of Weyl differencing. Let d be the dimension of M and r be its rank (i.e., dimension of a maximal totally geodesic submanifold).

Theorem 1 ([25])**.** *Let $\mathbb{S}^1$ stand for the standard circle of unit length, and let $\|\cdot\|$ stand for the distance from 0 on $\mathbb{S}^1$. Define the major arcs*

$$\mathcal{M}_{a,q} = \{t \in \mathbb{S}^1 : \ \left\|t - \frac{a}{q}\right\| < \frac{1}{qN}\}$$

where

$$a \in \mathbb{Z}_{\geq 0},\ q \in \mathbb{N},\ a < q,\ (a,q) = 1,\ q < N.$$

Given any compact Lie group (with a rational metric), consider the associated exponential sum $K_N(t,H)$. Then there is some $T > 0$ such that $K_N(t,H) = K_N(t+T,H)$ for any $t,H,$ and

$$|K_N(t,H)| \leq C\frac{N^d}{[\sqrt{q}(1 + N\|\frac{t}{T} - \frac{a}{q}\|^{1/2})]^r} \tag{1.4}$$

for $\frac{t}{T} \in \mathcal{M}_{a,q}$, uniformly in $H \in \mathfrak{a}$.

In this paper, we generalize this result partially to the setting of compact symmetric spaces.

Theorem 2. *Given any compact globally symmetric space (with a rational metric), consider the associated exponential sum $K_N(t,H)$. Then there is some $T > 0$ such that $K_N(t,H) = K_N(t+T,H)$ for any $t,H,$ and estimates as* (1.4) *hold with an N^ε loss for $\frac{t}{T} \in \mathcal{M}_{a,q}$, uniformly for H within a constant times N^{-1} of distance from some corner in $\mathfrak{a}$.*

In addition to the techniques in [25] of proving Theorem 1, the proof of this theorem involves two expressions of the spherical functions Φ_λ. One is in terms of the Jacobi polynomials associated to root systems, the other an integral formula near the identity coset of the symmetric space. We conjecture that the above estimates hold uniformly for H in $\mathfrak{a}$, and the missing ingredients for a proof should include other useful expressions for the spherical functions.

Conjecture 3. *The estimates* (1.4) *in the above theorem hold (with a possible ε-loss) uniformly for H in $\mathfrak{a}$.*

We will prove in this paper this conjecture implies certain scale-invariant Strichartz estimates (with a possible ε-loss) on any compact globally symmetric space. The proof will be a generalization of the group theoretic framework in [25] to a symmetric-space one, and as in [25] a combination of the Stein-Tomas Fourier restriction argument and the major-minor-arc Hardy-Littlewood circle method, first employed by Jean Bourgain [4, 5] on problems on tori.

Theorem 4. *The above conjecture implies scale-invariant Strichartz estimates* (1.3) *(with a possible ε-loss) for any $p \geq \frac{2(r+4)}{r}$, r the rank, on any compact globally symmetric space with a rational metric.*

In particular, using explicit formulas of ultraspherical polynomials, we are able to stretch Theorem 1 further and establish the following.

Theorem 5 (Main). *Let M be a product of compact Lie groups and spheres of odd dimension, equipped with a rational metric, of dimension d and rank r. Then the kernel estimates* (1.4) *hold uniformly for $H \in \mathfrak{a}$ with an ε-loss, and as a consequence, Strichartz estimates* (1.3) *hold with an ε-loss*

$$\|e^{it\Delta} f\|_{L^p(I\times M)} \leq C_\varepsilon \|f\|_{H^{\frac{d}{2}-\frac{d+2}{p}+\varepsilon}(M)}$$

for any $p \geq \frac{2(r+4)}{r}$.

We now list some notations that are applied throughout the paper.

- $A \lesssim B$ means $A \leq CB$ for some constant C, and $A \lesssim_{a,b,\ldots} B$ means $A \leq CB$ for some constant C that depends on $a, b, \ldots$.
- Let $\mathbb{T} = \mathbb{R}/T\mathbb{Z}$. For $f \in L^1(\mathbb{T})$, let $\widehat{f}$ denote the Fourier transform of f such that $\widehat{f}(n) = \frac{1}{T}\int_0^T f(t)e^{-int}\, dt$, $n \in \frac{2\pi}{T}\mathbb{Z}$.
- p' denotes the number such that $\frac{1}{p} + \frac{1}{p'} = 1$.

2. Preliminaries

Let $M = U/K$ be a compact globally symmetric space. Then equivalently we may view M as finitely covered by $\tilde{M} = \mathbb{T}^n \times N$ where $\mathbb{T}^n$ is the n-dimensional torus and N is a simply connected symmetric space of the compact type ([23] p. 108-109). Since Strichartz estimates for the finitely Riemannian covering manifold implies the same estimates for the base manifold (Proposition 3.3 in [25]), we may assume for our purpose that $M = \tilde{M}$. As a simply connected Riemannian globally symmetric space of the compact type, N is a direct product $U_1/K_1 \times U_2/K_2 \times \cdots \times U_m/K_m$ of irreducible simply connected symmetric spaces of the compact type (see Proposition 5.5 in Ch. VIII in [15]).

Now let U/K be a simply connected symmetric space of the compact type. Let the Lie algebras of U, K be $\mathfrak{u}, \mathfrak{k}$ respectively, and we consider the dual symmetric pair $\mathfrak{g}, \mathfrak{k}$ of Lie algebras of the noncompact type such that both $\mathfrak{u}, \mathfrak{g}$ lie on the same complexification $\mathfrak{g}^{\mathbb{C}}$, and that we have the Cartan decompositions

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \tag{2.1}$$

$$\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}. \tag{2.2}$$

The negative of the *Killing form* $-\langle\ ,\ \rangle$ defined on $\mathfrak{u}$ induces a Riemannian metric on U/K invariant under the action of U.

We equip each irreducible factor U_i/K_i with such a metric g_i defined above. We assume the torus factor to be $\mathbb{T}^n \cong \mathbb{R}^n/2\pi\Gamma$, such that there exists some $D \in \mathbb{N}$ such that

$$\langle \lambda, \mu\rangle \in D^{-1}\mathbb{Z},\ \forall \lambda, \mu \in \Gamma.$$

Here $\langle \cdot, \cdot \rangle$ denote the inner product on $\mathbb{R}^n$ which induces a flat metric g_0 on $\mathbb{T}^n$. Then we equip $M = \mathbb{T}^n \times U_1/K_1 \times \cdots \times U_m/K_m$ the metric

$$g = \otimes_{j=0}^m \beta_j g_j, \tag{2.3}$$

$\beta_j > 0$, $j = 0, \ldots, m$.

Definition 6. *We call the above metric g rational provided the numbers $\beta_0, \cdots, \beta_m$ are rational multiples of each other.*

Let U/K be a simply connected symmetric space of the compact type, equipped with the push forward measure of the normalized Haar measure du of U. Let (δ, V_δ) be an irreducible unitary representation of U and let V_δ^K be the space of vectors $v \in V_\delta$ fixed under $\delta(K)$. We say δ is *spherical* if $V_\delta^K \neq 0$. Let δ be such an irreducible spherical representation of U. Then V_δ^K is spanned by a single unit vector $\mathbf{e}$, and let

$$H_\delta(U/K) = \{\langle \delta(u)\mathbf{e}, v\rangle_{V_\delta} : v \in V_\delta\}. \tag{2.4}$$

Let $\widehat{U}_K$ be the set of equivalence classes of spherical representations of U with respect to K. The theory of Peter-Weyl gives the Hilbert space decomposition

$$L^2(U/K) = \bigoplus_{\delta \in \widehat{U}_K} H_\delta(U/K).$$

Define the *spherical functions*

$$\Phi_\delta(u) := \langle \delta(u)\mathbf{e}, \mathbf{e}\rangle_{V_\delta} \in H_\delta(U/K),$$

then the L^2 projections $P_\delta : L^2(U/K) \to H_\delta(U/K)$ can be realized by convolution with $d_\delta \Phi_\delta$, so we have the L^2 *spherical Fourier series*

$$f = \sum_{\delta \in \widehat{U}_K} d_\delta f * \Phi_\delta = \sum_{\delta \in \widehat{U}_K} d_\delta \Phi_\delta * f.$$

Here the convolution on U/K is defined by pulling back the functions to U and then applying the group convolution.

More generally, let $M = \mathbb{R}^n/2\pi\Gamma \times U_1/K_1 \times \cdots \times U_m/K_m$ and let Λ be the dual lattice of Γ. Define the *Fourier dual* $\widehat{M}$ of M

$$\widehat{M} = \Lambda \times \widehat{U_1}_{K_1} \times \cdots \times \widehat{U_m}_{K_m}.$$

Let $\delta = (\lambda_0, \delta_1, \cdots, \delta_m) \in \widehat{M}$, $(H_0, x_1, \cdots, x_m) \in M$, and let

$$\Phi_\delta(H_0, x_1, \cdots, x_m) = e^{i\langle \lambda_0, H_0\rangle} \Phi_{\delta_1} \cdots \Phi_{\delta_m},$$
$$d_\delta = d_{\delta_1} \cdots d_{\delta_m}.$$

Then the spherical Fourier series reads

$$f = \sum_{\delta \in \widehat{M}} d_\delta \Phi_\delta * f = \sum_{\delta \in \widehat{M}} d_\delta f * \Phi_\delta,$$

where the convolution is defined component-wise. This gives the *Plancherel identity*

$$\|f\|_{L^2(M)}^2 = \sum_{\delta \in \widehat{M}} d_\delta^2 \|\Phi_\delta * f\|_{L^2(M)}.$$

The Young's convolution inequalities hold on compact symmetric spaces

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q},\ \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1,\ 1 \leq r, p, q \leq \infty.$$

This implies the Hausdorff-Young type inequality

$$\|f * \Phi_\delta\|_{L^2} \leq \|f\|_{L^1} \|\Phi_\delta\|_{L^2} = d_\delta^{-\frac{1}{2}} \|f\|_{L^1},\ \text{for any } \delta \in \widehat{M}. \tag{2.5}$$

Let $g = \sum_{\delta\in\widehat{M}} c_\delta d_\delta \Phi_\delta$, then $f * g = \sum_{\delta\in\widehat{M}} c_\delta d_\delta f * \Phi_\delta$, which implies

$$\|f * g\|_{L^2}^2 = \sum_{\delta\in\widehat{M}} |c_\delta|^2 d_\delta^2 \|f * \Phi_\delta\|^2, \tag{2.6}$$

$$\|f * g\|_{L^2} \leq (\sup_{\delta\in\widehat{M}} |c_\delta|) \cdot \|f\|_{L^2}. \tag{2.7}$$

Next we explicitly characterizes the Fourier dual. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition and $\mathfrak{a}$ be the maximal abelian subspace of $\mathfrak{p}$. Let $\Sigma \subset \mathfrak{a}^*$ denote the restricted root system and let $m_\lambda \in \mathbb{N}$ $(\lambda \in \Sigma)$ denote *multiplicity function.* Let $\mathfrak{b}$ be a maximal abelian subspace of the centralizer $\mathfrak{m}$ of $\mathfrak{a}$ in $\mathfrak{k}$ and let $\mathfrak{h}_\mathbb{R} = \mathfrak{a} + i\mathfrak{b}$. Let Σ^+ denote a set of positive restricted roots in Σ with respect to an order in $\mathfrak{a}^*$ compatible with one in $\mathfrak{h}_\mathbb{R}^*$. Then we have the Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k} \tag{2.8}$$

where $\mathfrak{n} = \sum_{\lambda\in\Sigma^+} \mathfrak{g}_\lambda$ is the direct sum of positive restricted root spaces $\mathfrak{g}_\lambda$. Let r and d be the rank and dimension of U/K respectively. The dimension of $\mathfrak{g}_\lambda$ being m_λ, the Iwasawa decomposition implies

$$\sum_{\lambda\in\Sigma^+} m_\lambda = d - r. \tag{2.9}$$

Let $\Sigma_* := \{\alpha \in \Sigma : 2\alpha \notin \Sigma\}$ be the root system consisting of inmultiplicable roots. Let the *weight lattice* Λ be

$$\Lambda := \{\lambda \in \mathfrak{a}^* : \frac{\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z}, \text{ for all } \alpha \in \Sigma_*\}. \tag{2.10}$$

Let Γ be the *restricted root lattice* generated by the root system $2 \cdot \Sigma$. Then $\Gamma \subset \Lambda$. Let $\Sigma_*^+ = \Sigma^+ \cap \Sigma_*$ be the set of positive roots in Σ_*. Let

$$\Lambda^+ := \{\lambda \in \mathfrak{a}^* : \frac{\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z}_{\geq 0}, \text{ for all } \alpha \in \Sigma_*^+\}$$

be the set of *dominant weights*. Given any irreducible spherical representation of $\delta \in \widehat{U}_K$, the highest weight of δ vanishes on $\mathfrak{b}$ and restricts on $\mathfrak{a}$ as an element in Λ^+. This gives the isomorphism

$$\Lambda^+ \cong \widehat{U}_K. \tag{2.11}$$

We can also express Λ, Λ^+ in terms of a basis. Let $\{\alpha_1, \cdots, \alpha_r\}$ be the set of simple roots in Σ_*^+. Let $\{w_1, \cdots, w_r\}$ be the fundamental weights, the dual basis to the (half) *coroot* basis $\{\frac{\alpha_1}{\langle\alpha_1,\alpha_1\rangle}, \cdots, \frac{\alpha_r}{\langle\alpha_r,\alpha_r\rangle}\}$. Then

$$\Lambda = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_r,$$
$$\Lambda^+ = \mathbb{Z}_{\geq 0}w_1 + \cdots + \mathbb{Z}_{\geq 0}w_r.$$

Let $C = \mathbb{R}_{>0}w_1 + \cdots + \mathbb{R}_{>0}w_r$ be *fundamental Weyl chamber.*

Consider the map $i\mathfrak{a} \to U/K,\ iH \mapsto \exp(iH)K$. Let A denote the image of the map, then

$$A \cong i\mathfrak{a}/\Gamma^\vee$$

where $\Gamma^\vee = \{iH \in i\mathfrak{a} : \exp(iH) \in K\}$ is a lattice of $i\mathfrak{a}$. Then

$$\Gamma^\vee = 2\pi i\mathbb{Z}\frac{H_{\alpha_1}}{\langle\alpha_1,\alpha_1\rangle} + \cdots + 2\pi i\mathbb{Z}\frac{H_{\alpha_r}}{\langle\alpha_r,\alpha_r\rangle}.$$

Here $H_{\alpha_i} \in \mathfrak{a}$ corresponds to $\alpha_i \in \mathfrak{a}^*$ via the Killing form on $\mathfrak{a}$. We have the isomorphism between Λ and the character group $\widehat{A}$ of A

$$\Lambda \xrightarrow{\sim} \widehat{A},\ \lambda \mapsto e^{\lambda}.$$

Define the *cells* in A to be the connected components of $A \setminus \cup_{\alpha\in\Sigma}\{[iH] \in A : \langle \alpha, H\rangle \in \pi\mathbb{Z}\}$, and the hyperplanes in $\{[iH] \in A : \langle \alpha, H\rangle \in \pi\mathbb{Z}\}$ are called *cell walls*. Let

$$Q = \bigcap_{\alpha\in\Sigma^+} \{[iH] \in A : \langle \alpha, H\rangle \in (0, \pi)\},$$

be the *fundamental cell*. The Weyl group W acts simply transitively on the set of cells, and $W\bar{Q}$ covers A. Moreover, the K-orbits of A cover the whole space U/K, hence the values of any K-biinvariant function on U are determined by its restriction on $\bar{Q}$.

Lastly, for $H \in \mathfrak{a}$, we say $[iH] \in A$ is a *corner* if $\alpha(H) \in \pi\mathbb{Z}$ for all $\alpha \in \Sigma$. Every corner is fixed under the action of K and the set of corners in A is isomorphic to the finite set $\Lambda^\vee/\Gamma^\vee$.

Example 7. *Let $M = U/K$ be a simply connected compact symmetric space of rank 1. Then the restricted root system Σ is either $\{\pm\alpha\}$ or $\{\pm\frac{\alpha}{2}, \pm\alpha\}$. In both cases, the weight lattice $\Lambda = \mathbb{Z}\alpha$. Let $A = \mathbb{R}/2\pi\mathbb{Z}$ be the maximal torus, then $e^{n\alpha} = e^{in\theta}$, $\theta \in A$. The two cells of A are $(0, \pi)$ and $(\pi, 2\pi)$, with $0, \pi$ being the two corners. Let m_α and $m_{\frac{\alpha}{2}}$ be respectively the multiplicity of α and $\frac{\alpha}{2}$ (if the restricted root system is $\{\pm\alpha\}$, then let $m_{\frac{\alpha}{2}} = 0$). Then for $n \in \mathbb{Z}_{\geq 0} \cong \mathbb{Z}_{\geq 0}\alpha \cong \Lambda^+$, the spherical function Φ_n restricted on A is (see Theorem 4.5 of Chapter V in* [14]*)*

$$\Phi_n = \binom{n+a}{n}^{-1} P_n^{(a,b)}(\cos\theta),$$

where $\{P_n^{(a,b)} : n \in \mathbb{Z}_{\geq 0}\}$ is the set of Jacobi polynomials (see [22]*) with parameters*

$$a = \frac{1}{2}(m_{\frac{\alpha}{2}} + m_\alpha - 1),\ b = \frac{1}{2}(m_\alpha - 1).$$

The cases when $m_{\frac{\alpha}{2}} = 0$ correspond to spheres of dimension $d = m_\alpha + 1$, and the Jacobi polynomials in these cases are usually called ultraspherical polynomials. If d is odd, we have explicit formulas for these polynomials. Let $\{\Phi_n^{(\lambda)}, n \in \mathbb{Z}_{\geq 0}\}$ denote the spherical functions on the $(2\lambda + 1)$-dimensional sphere, $\lambda \in \mathbb{N}$, then (see Equation (4.7.3) and (8.4.13) in [22]*)*

$$\Phi_n^{(\lambda)}(\theta) = 2\binom{n+2\lambda-1}{n}^{-1} \alpha_n \sum_{\nu=0}^{\lambda-1} \alpha_\nu \frac{(1-\lambda)\cdots(\nu-\lambda)}{(n+\lambda-1)\cdots(n+\lambda-\nu)} \cdot \frac{\cos((n-\nu+\lambda)\theta - (\nu+\lambda)\pi/2)}{(2\sin\theta)^{\nu+\lambda}} \tag{2.12}$$

where $\alpha_n := \binom{n+\lambda-1}{n}$.

The following lemma is a direct consequence of the Weyl dimension formula.

Lemma 8. *Let $\Phi \subset \mathfrak{h}_{\mathbb{R}}^*$ denotes the root system associated to $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h})$ which restrict on $\mathfrak{a}$ gives $\Sigma \cup \{0\}$. Let Φ^+ be the set of positive roots with respect to the ordering on $\mathfrak{h}_{\mathbb{R}}^*$ compatible with $\mathfrak{a}^*$. Then the dimension d_λ ($\lambda \in \Lambda^+ \cong \widehat{U}_K$) equals*

$$d_\lambda = \frac{\prod_{\alpha\in\Phi^+, \alpha|_{\mathfrak{a}}\neq 0} \langle \lambda + \rho', \alpha\rangle}{\prod_{\alpha\in\Phi^+, \alpha|_{\mathfrak{a}}\neq 0} \langle \rho', \alpha\rangle}, \quad \textit{for } \rho' = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha. \tag{2.13}$$

We now review the functional calculus of the Laplace-Beltrami operator. Let $\lambda \in \Lambda^+ \cong \widehat{U}_K$ and $H_\lambda(U/K)$ be the space of matrix coefficients associated to λ as in (2.4). For any $f \in H_\lambda(U/K)$, we have

$$\Delta f = (-\langle \lambda+\rho, \lambda+\rho\rangle + \langle \rho, \rho\rangle) \cdot f, \tag{2.14}$$

where

$$\rho = \frac{1}{2} \sum_{\alpha\in\Sigma^+} m_\alpha \alpha. \tag{2.15}$$

Let $f \in L^2(U/K)$ and consider the spherical Fourier series $f = \sum_{\lambda\in\Lambda^+} d_\lambda f * \Phi_\lambda$. Then for any bounded Borel function $F : \mathbb{R} \to \mathbb{C}$, we have

$$F(\Delta)f = \sum_{\lambda\in\Lambda^+} F(-|\lambda+\rho|^2 + |\rho|^2) d_\lambda f * \Phi_\lambda.$$

In particular, we have

$$e^{it\Delta} f = \sum_{\lambda\in\Lambda^+} e^{it(-|\lambda+\rho|^2+|\rho|^2)} d_\lambda f * \Phi_\lambda. \tag{2.16}$$

Define $P_N f = \varphi(N^{-2}\Delta)f$, then

$$P_N e^{it\Delta} f = \sum_{\lambda\in\Lambda^+} \varphi\left(\frac{-|\lambda+\rho|^2 + |\rho|^2}{N^2}\right) e^{it(-|\lambda+\rho|^2+|\rho|^2)} d_\lambda f * \Phi_\lambda. \tag{2.17}$$

In particular, let

$$K_N(t, x) = \sum_{\lambda\in\Lambda^+} \varphi\left(\frac{-|\lambda+\rho|^2 + |\rho|^2}{N^2}\right) e^{it(-|\lambda+\rho|^2+|\rho|^2)} d_\lambda \Phi_\lambda, \tag{2.18}$$

then we have

$$P_N e^{it\Delta} f = f * K_N(t, \cdot) = K_N(t, \cdot) * f. \tag{2.19}$$

We call $K_N(t, x)$ as the (mollified) *Schrödinger kernel* on U/K. More generally, let $M = \mathbb{T}^n \times U_1/K_1 \times \cdots \times U_m/K_m$ equipped with a rational metric g. Let Λ_j be the weight lattice for U_j/K_j and identify $\widehat{U_j}_{K_j} \cong \Lambda_j^+$, $1 \le j \le m$. Let $\mathbf{P}_N = \otimes_{j=0}^m \varphi_j(N^{-2}\Delta_j)$ be a Littlewood-Paley projection of the product type (see Section 3C of [25]). Define the (mollified) *Schrödinger kernel* $\mathbf{K}_N$ on M by

$$\mathbf{P}_N e^{it\Delta} f = f * \mathbf{K}_N(t, \cdot) = \mathbf{K}_N(t, \cdot) * f. \tag{2.20}$$

Then

$$\mathbf{K}_N = \prod_{j=0}^m K_{N,j}, \tag{2.21}$$

where the $K_{N,j}$'s are respectively the Schrödinger kernel on each component

$$K_{N,0} = \sum_{\lambda_0\in\Lambda} \varphi_0\left(\frac{-|\lambda_0|^2}{\beta_0 N^2}\right) e^{-it\beta_0^{-1}|\lambda_0|^2} e^{i\langle\lambda_0, H_0\rangle},$$

$$K_{N,j} = \sum_{\lambda_j\in\Lambda_j^+} \varphi_j\left(\frac{-|\lambda_j+\rho_j|^2 + |\rho_j|^2}{\beta_j N^2}\right) e^{it\beta_j^{-1}(-|\lambda_j+\rho_j|^2+|\rho_j|^2)} d_{\lambda_j} \Phi_{\lambda_j},$$

$j = 1, \cdots, m$. Here the ρ_j's are defined in terms of (2.15). We also write

$$\mathbf{K}_N = \sum_{\lambda \in \widehat{M}} \varphi(\lambda, N) e^{-it\|\lambda\|^2} d_\lambda \Phi_\lambda,$$

where

$$\lambda = (\lambda_0, \ldots, \lambda_m) \in \widehat{M} = \Lambda \times \Lambda_1^+ \times \cdots \times \Lambda_m^+,$$

$$-\|\lambda\|^2 = -\beta_0^{-1}|\lambda_0|^2 + \sum_{j=1}^{m} \beta_j^{-1}(-|\lambda_j + \rho_j|^2 + |\rho_j|^2), \tag{2.22}$$

$$\varphi(\lambda, N) = \varphi_0\left(\frac{-|\lambda_0|^2}{\beta_0 N^2}\right) \cdot \prod_{j=1}^{n} \varphi_j\left(\frac{-|\lambda_j + \rho_j|^2 + |\rho_j|^2}{\beta_j N^2}\right), \tag{2.23}$$

$$d_\lambda = \prod_{j=1}^{m} d_{\lambda_j}, \ \Phi_\lambda = e^{i\langle\lambda_0, H_0\rangle} \prod_{j=1}^{m} \Phi_{\lambda_j}.$$

Lemma 9. *Let d, r be respectively the dimension and rank of M.*
(i) $|\{\lambda \in \widehat{M} : \ \|\lambda\|^2 \lesssim N^2\}| \lesssim N^r$.
(ii) $d_\lambda \lesssim N^{d-r}$, uniformly for all $\|\lambda\|^2 \lesssim N^2$.

Proof. Note that $\lambda \in \widehat{M}$ lies in a lattice of dimension r, then (i) is a direct consequence of the definition of $\|\lambda\|^2$. For (ii), let d_j, r_j, Σ_j be respectively the dimension, rank, and the set of restricted roots of U_j/K_j, $j = 1, \cdots, m$. For $\lambda_j \in \Lambda_j^+$, (2.13) implies that d_{λ_j} is a polynomial in λ_j of degree equal to the number of positive restricted roots counting multiplicities, which is equal to $d_j - r_j$ by (2.9). Thus $d_\lambda = d_{\lambda_1} \cdots d_{\lambda_m}$ is a polynomial in λ of degree $\sum_{j=1}^m (d_j - r_j) = d - r$. In view of the definition of $\|\lambda\|^2$ again, we get (ii). □

We have the following lemma which explains why we take Definition 6 as rationality of a metric.

Lemma 10. *Let Σ be the restricted root system equipped with the Killing form $\langle \ , \ \rangle$ and Λ the associated weight lattice. Then there exists some $D \in \mathbb{N}$, such that $\langle \alpha, \beta \rangle \in D^{-1}\mathbb{Z}$ for all $\alpha, \beta \in \Lambda$.*

Proof. Let $\{\alpha_1, \cdots, \alpha_r\}$ be a set of simple roots for Σ_*. Let $\{w_1, \cdots, w_r\}$ be the dual basis of the coroot basis $\{\frac{\alpha_1}{\langle\alpha_1,\alpha_1\rangle}, \cdots, \frac{\alpha_r}{\langle\alpha_r,\alpha_r\rangle}\}$ so that $\Lambda = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_r$. Then it suffices to prove that $\langle w_i, w_j \rangle \in D^{-1}\mathbb{Z}$ for all $1 \le i, j \le r$, for some $D \in \mathbb{N}$, which then reduces to proving the rationality of $\langle w_i, w_j \rangle$, which further reduces to proving the rationality of $\langle \alpha, \beta \rangle$ for all $\alpha, \beta \in \Sigma$. Since Σ is a root system, $2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$ for all $\alpha, \beta \in \Sigma$, thus it suffices to prove the rationality of $\langle \alpha, \alpha \rangle$ for all $\alpha \in \Sigma$. Let α be a restricted root in Σ, and let $\alpha' \in \Delta$ be a root such that $\alpha'|_{\mathfrak{a}} = \alpha$. By Lemma 4.3.5 in [24], $\langle \alpha', \alpha' \rangle$ is rational. Then by Lemma 8.4 of Ch. VII in [15], $\langle \alpha, \alpha \rangle$ is also rational. This finishes the proof. □

Finally, we observe that the Strichartz estimates can be reduced to those about the mollified operator $\mathbf{P}_N e^{it\Delta}$.

Proposition 11 (Section 3C in [25])**.** *(i) Strichartz estimate* (1.3) *is equivalent to the following mollified one:*

$$\|\mathbf{P}_N e^{it\Delta} f\|_{L^p(I\times M)} \lesssim N^{\frac{d}{2} - \frac{d+2}{p}} \|f\|_{L^2(M)}, \text{ for any } N \gtrsim 1. \tag{2.24}$$

(ii) We have the following Bernstein type inequality for $\mathbf{P}_N$. For $2 \le r \le \infty$,

$$\|\mathbf{P}_N f\|_{L^r(M)} \lesssim N^{d(\frac{1}{2} - \frac{1}{r})} \|f\|_{L^2(M)}. \tag{2.25}$$

3. Strichartz and dispersive estimates

Let $M = \mathbb{R}^n/2\pi\Gamma \times U_1/K_1 \times \cdots \times U_m/K_m$ be equipped with a rational metric g. By Lemma 10, there exists for each $j = 1, \cdots, m$ some $D_j \in \mathbb{N}$ such that $\langle \lambda, \mu\rangle \in 2D_j^{-1}\mathbb{Z}$ for all $\lambda, \mu \in \Lambda_j^+ \cong \widehat{U_j}_{K_j}$, which implies by (2.15) that $-|\lambda_j + \rho_j|^2 + |\rho_j|^2 = -|\lambda_j|^2 - \langle \lambda_j, 2\rho_j\rangle \in D_j^{-1}\mathbb{Z}$ for all $\lambda_j \in \Lambda_j$. Also recall that we require that there exists some $D \in \mathbb{N}$ such that $\langle u, v\rangle \in D^{-1}\mathbb{Z}$ for all $u, v \in \Gamma$. This implies that there also exists some $D_0 \in \mathbb{N}$ such that $\langle \lambda, \mu\rangle \in D_0^{-1}\mathbb{Z}$ for all λ, μ in the dual lattice Λ of Γ. By the definition of a rational metric, there exists some $D_* > 0$ such that

$$\beta_0^{-1}, \cdots, \beta_m^{-1} \in D_*^{-1}\mathbb{N}.$$

Define

$$T = 2\pi D_* \cdot \prod_{j=0}^{m} D_j. \tag{3.1}$$

Then (2.22) implies that $T\|\lambda\|^2 \in 2\pi\mathbb{Z}$, and hence the Schrödinger kernel (2.21) is periodic in t with a period of T. Thus we may view the time variable t as living on the circle $\mathbb{T} = \mathbb{R}/T\mathbb{Z}$. Now the formal dual to the operator

$$\mathbf{T} : L^2(M) \to L^p(\mathbb{T} \times M), \quad f \mapsto \mathbf{P}_N e^{it\Delta} \tag{3.2}$$

is computed to be

$$\mathbf{T}^* : L^{p'}(\mathbb{T} \times M) \to L^2(M), \quad F \mapsto \int_{\mathbb{T}} \mathbf{P}_N e^{-is\Delta} F(s, \cdot)\, \frac{ds}{T}, \tag{3.3}$$

and thus

$$\mathbf{T}\mathbf{T}^* : L^{p'}(\mathbb{T} \times M) \to L^p(\mathbb{T} \times M), \quad F \mapsto \int_{\mathbb{T}} \mathbf{P}_N^2 e^{i(t-s)\Delta} F(s, \cdot)\, \frac{ds}{T} = \tilde{\mathbf{K}}_N \times F, \tag{3.4}$$

where

$$\tilde{\mathbf{K}}_N = \sum_{\lambda \in \widehat{M}} \varphi^2(\lambda, N) e^{-it\|\lambda\|^2} d_\lambda \Phi_\lambda = \mathbf{K}_N \times \mathbf{K}_N, \tag{3.5}$$

and the symbol $\times$ is understood as convolution on the “space-time” $\mathbb{T} \times M$.

The cutoff function $\varphi^2(\lambda, N)$ (see (2.23)) still defines a Littlewood-Paley projection $\mathbf{P}_N$ of the product type, and $\tilde{\mathbf{K}}_N$ is the Schrödinger kernel associated to $\mathbf{P}_N$. By the $\mathbf{T}\mathbf{T}^*$ argument, boundedness of the operators (3.2), (3.3) and (3.4) are all equivalent, thus the Strichartz estimate in (2.24) is equivalent to the following *space-time Strichartz estimate*

$$\|\mathbf{K}_N \times F\|_{L^p(\mathbb{T}\times M)} \lesssim N^{d - \frac{2(d+2)}{p}} \|F\|_{L^{p'}(\mathbb{T}\times M)}, \tag{3.6}$$

which can be interpreted as Fourier restriction estimates on the product symmetric space $\mathbb{T} \times M$.

We have the *space-time spherical Fourier series* as follows. For $F \in L^2(\mathbb{T} \times M)$, we have

$$F = \sum_{\substack{n \in \frac{2\pi}{T}\mathbb{Z}, \\ \lambda \in \widehat{M}}} d_\lambda F \times [e^{itn}\Phi_\lambda].$$

Let $m = \sum_{n \in \frac{2\pi}{T}\mathbb{Z}} \widehat{m}(n) e^{itn}$, then

$$m \cdot \mathbf{K}_N = \sum_{\substack{n \in \frac{2\pi}{T}\mathbb{Z}, \\ \lambda \in \widehat{M}}} \varphi(\lambda, N) \widehat{m}(n + \|\lambda\|^2) d_\lambda e^{itn} \Phi_\lambda. \tag{3.7}$$

Our strategy to prove (3.6) is to first explore L^∞ estimates of K_N. Let $\mathbb{S}^1$ stand for the standard circle of unit length, and $\|\cdot\|$ the distance from 0 on $\mathbb{S}^1$. Define the major arcs

$$\mathcal{M}_{a,q} := \{t \in \mathbb{S}^1 : \left\| t - \frac{a}{q} \right\| < \frac{1}{qN}\}$$

where

$$a \in \mathbb{Z}_{\geq 0}, \quad q \in \mathbb{N}, \quad a < q, \quad (a,q) = 1, \quad q < N.$$

In [5], Jean Bourgain shows that for the Schrödinger kernel on the standard torus $\mathbb{T}^n$

$$\mathbf{K}_N(t,\mathbf{t}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} \varphi(\mathbf{k},N) e^{-it|\mathbf{k}|^2 + i\mathbf{k}\cdot\mathbf{t}},$$

it holds that for any $D \in \mathbb{N}$,

$$|\mathbf{K}_N(t,\mathbf{t})| \lesssim \frac{N^r}{[\sqrt{q}(1 + N\|\frac{t}{2\pi D} - \frac{a}{q}\|^{1/2})]^r} \tag{3.8}$$

for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$, uniformly in $\mathbf{t} \in \mathbb{T}^n$. Inspired by this, we conjecture a general dispersive estimate as follows.

Conjecture 12. *Let $\mathbf{K}_N$ be the Schrödinger kernel* (2.21) *and T be the period* (3.1)*, associated to any compact symmetric space of the form $M = \mathbb{T}^n \times U_1/K_1 \times \cdots U_m/K_m$. Then*

$$|\mathbf{K}_N(t,x)| \lesssim \frac{N^d}{[\sqrt{q}(1 + N\|\frac{t}{T} - \frac{a}{q}\|^{1/2})]^r} \tag{3.9}$$

for $\frac{t}{T} \in \mathcal{M}_{a,q}$, uniformly in $x \in M$.

Noting the product structure (2.21) of $\mathbf{K}_N$, the definition of the rank of the product space M, the definition (3.1) of T, the above conjecture reduces to the irreducible components of M. The cases when the irreducible component is either a semisimple compact Lie group or a torus are proved in [25]. Thus the missing parts are conjectured as follows.

Conjecture 13. *Let M be an irreducible simply connected symmetric space of compact type of rank r and dimension d (not of the compact group type), equipped with a rational metric. Let Λ be the weight lattice and Λ^+ the set of positive weights. Let D be a positive number such that $\langle \lambda, \mu \rangle \in D^{-1}\mathbb{Z}$ for all $\lambda, \mu \in \Lambda$. Let K_N be the Schrödinger kernel* (2.18)*. Then*

$$|K_N(t,x)| \lesssim \frac{N^d}{[\sqrt{q}(1 + N\|\frac{t}{2\pi D} - \frac{a}{q}\|^{1/2})]^r} \tag{3.10}$$

for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$, uniformly in $x \in M$.

We will prove the following special cases of this conjecture in the next section.

Theorem 14. *(1) In Conjecture 13,* (3.10) *holds with an ε-loss (i.e., to add an N^ε multiplicative factor to the right side of* (3.10)*), uniformly for $d(x,x_0) \lesssim N^{-1}$, $x \in M$. Here $d(\cdot,\cdot)$ denote the Riemannian distance on M and x_0 is any corner in the maximal torus A.*
(2) Conjecture 13 holds with an ε-loss when M is a sphere of odd dimension.

Note that the three sphere is the compact group $SU(2)$, thus Conjecture 13 holds without loss for the three sphere. Now we show how Conjecture 12 implies Strichartz estimates (2.24) for $p \geq 2 + \frac{8}{r}$.

Theorem 15. *Continue the assumptions in Conjecture 12. Let $f \in L^2(M)$, $\lambda > 0$, and define*

$$m_\lambda = \mu\{(t,x) \in \mathbb{T} \times M : |\mathbf{P}_N e^{it\Delta} f(x)| > \lambda|\}$$

where $\mu = dt \cdot d\mu_M$, $dt, d\mu_M$ being the canonical measures (normalized to be of total measure 1) on $\mathbb{T} = \mathbb{R}/T\mathbb{Z}$ and M respectively. Let

$$p_0 = \frac{2(r+2)}{r}.$$

Assuming truthfulness of Conjecture 12, the following statements hold true.
(I)

$$m_\lambda \lesssim_\varepsilon N^{\frac{dp_0}{2}-(d+2)+\varepsilon} \lambda^{-p_0} \|f\|_{L^2(M)}^{p_0}, \quad \textit{for all } \lambda \gtrsim N^{\frac{d}{2}-\frac{r}{4}}, \quad \varepsilon > 0.$$

(II)

$$m_\lambda \lesssim N^{\frac{dp}{2}-(d+2)} \lambda^{-p} \|f\|_{L^2(M)}^{p}, \quad \textit{for all } \lambda \gtrsim N^{\frac{d}{2}-\frac{r}{4}}, \quad p > p_0.$$

(III)

$$\|\mathbf{P}_N e^{it\Delta} f\|_{L^p(\mathbb{T}\times M)} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}} \|f\|_{L^2(M)} \tag{3.11}$$

holds for all $p \geq 2 + \frac{8}{r}$.
(IV) Assume it holds that

$$\|\mathbf{P}_N e^{it\Delta} f\|_{L^p(\mathbb{T}\times M)} \lesssim_\varepsilon N^{\frac{d}{2}-\frac{d+2}{p}+\varepsilon} \|f\|_{L^2(M)} \tag{3.12}$$

for some $p > p_0$, then (3.11) *holds for all $q > p$.*
Assuming only truthfulness of Conjecture 12 with ε-loss, then (I) holds, and (II), (III) hold with an ε-loss.

Combining Theorem 14, Theorem 15, and Theorem 6.2 in [25], we have the following main theorem of this paper.

Theorem 16 (Main)**.** *Let G be a simply connected compact Lie group and $\mathbb{S}^{d_j}$ $(1 \leq j \leq m)$ be the standard sphere of odd dimension d_j. Let M be the product manifold $\mathbb{T}^n \times G \times \mathbb{S}^{d_1} \times \cdots \times \mathbb{S}^{d_m}$ of dimension d and rank r, equipped with a rational metric. Then the Strichartz estimates* (1.3) *hold with an ε-loss for any $p \geq 2 + \frac{8}{r}$.*

The proof of Theorem 15 is a generalization of the compact-group theoretic framework as developed in [25] to a compact-symmetric-space one. In the remainder of this section, we point out the modifications needed in this generalization. We now inherit all the notations of and follow closely the proof of Theorem 6.3 in [25]. Recall the definition of the coefficients $\alpha_{Q,M}$

$$\left[\left(\sum_{\substack{(a,q)=1,\\ Q\leq q<2Q}} \delta_{a/q}\right) * \omega_{\frac{1}{NM}}\left(\frac{\cdot}{T}\right)\right]^{\wedge}(0) = \alpha_{Q,M}\widehat{\rho}(0), \tag{3.13}$$

and the major-arc components of $\mathbf{K}_N(t,x)$

$$\Lambda_{Q,M}(t,x) := \mathbf{K}_N(t,x)\left[\left(\left(\sum_{(a,q)=1, Q\leq q<2Q} \delta_{a/q}\right) * \omega_{\frac{1}{NM}}\left(\frac{\cdot}{T}\right)\right) - \alpha_{Q,M}\rho\right](t). \tag{3.14}$$

From (3.7), we have

$$\Lambda_{Q,M} = \sum_{\substack{n\in\frac{2\pi}{T}\mathbb{Z},\\ \lambda\in\widehat{M}}} \lambda_{Q,M}(n,\lambda) d_\lambda e^{itn}\Phi_\lambda. \tag{3.15}$$

where

$$\lambda_{Q,M}(n,\lambda) = \varphi(\lambda,N)\left[\left(\sum_{(a,q)=1, Q\le q<2Q} \delta_{a/q}\right)^{\wedge} \cdot \widehat{\omega}_{\frac{1}{NM}}(T\cdot) - \alpha_{Q,M}\widehat{\rho}\right](n+\|\lambda\|^2). \tag{3.16}$$

Note that (3.13) immediately implies

$$\lambda_{Q,M}(n,\lambda) = 0, \quad \text{for } n+\|\lambda\|^2 = 0. \tag{3.17}$$

We have

$$|\lambda_{Q,M}(n,\lambda)| \lesssim_\varepsilon \varphi(\lambda,N)\frac{Q^{1+\varepsilon}}{NM} d\left(\frac{T(n+\|\lambda\|^2)}{2\pi}, Q\right) + \frac{Q^2}{NM}|\widehat{\rho}(n+\|\lambda\|^2)|. \tag{3.18}$$

and then

$$\begin{aligned} |\lambda_{Q,M}(n,\lambda)| &\lesssim_\varepsilon \varphi(\lambda,N)\frac{Q}{NM}\left[Q^\varepsilon d(\frac{T(n+\|\lambda\|^2)}{2\pi}, Q) + \frac{Q}{N^{1-\varepsilon}}\right] \\ &\lesssim_\varepsilon \varphi(\lambda,N)\frac{QN^\varepsilon}{NM}, \quad \text{for } |n|\lesssim N^2. \end{aligned} \tag{3.19}$$

Proposition 17. *(i) Assume that $f\in L^1(\mathbb{T}\times M)$. Then*

$$\|f\times\Lambda_{Q,M}\|_{L^\infty(\mathbb{T}\times M)} \lesssim N^{d-\frac{r}{2}}\left(\frac{M}{Q}\right)^{r/2}\|f\|_{L^1(\mathbb{T}\times M)}. \tag{3.20}$$

(ii) Assume that $f\in L^2(\mathbb{T}\times M)$. Assume also

$$f\times[e^{itn}\Phi_\lambda] = 0, \quad \textit{for all } n\in\frac{2\pi}{T}\mathbb{Z} \textit{ such that } |n|\gtrsim N^2. \tag{3.21}$$

Then

$$\|f\times\Lambda_{Q,M}\|_{L^2(\mathbb{T}\times M)} \lesssim_\varepsilon \frac{QN^\varepsilon}{NM}\|f\|_{L^2(\mathbb{T}\times M)}, \tag{3.22}$$

and

$$\|f\times\Lambda_{Q,M}\|_{L^2(\mathbb{T}\times M)} \lesssim_{\tau,B} \frac{Q^{1+2\tau}L}{NM}\|f\|_{L^2(\mathbb{T}\times M)} + M^{-1}L^{-B/2}N^{d/2}\|f\|_{L^1(\mathbb{T}\times M)}. \tag{3.23}$$

for all

$$L>1, \quad 0<\tau<1, \quad B>\frac{6}{\tau}, \quad N>(LQ)^B. \tag{3.24}$$

Proof. (i) is proved identically as in [25]. (3.22) is a consequence of (2.7), (3.15), and (3.19). To prove (3.23), we use (2.6) and (3.15) to get

$$\|f\times\Lambda_{Q,M}\|_{L^2(\mathbb{T}\times M)} = \left(\sum_{n,\lambda} d_\lambda^2\|f\times[e^{itn}\Phi_\lambda]\|^2_{L^2(\mathbb{T}\times M)}\cdot|\lambda_{Q,M}(n,\lambda)|^2\right)^{1/2},$$

which combined with (3.17), (3.18), and the estimate $|\hat{\rho}(n)| \lesssim \frac{N^\varepsilon}{N}$ yields

$$\|f \times \Lambda_{Q,M}\|_{L^2(\mathbb{T}\times M)} \lesssim_\varepsilon \frac{Q^{1+\varepsilon}}{NM} \left(\sum_{n,\lambda} \varphi(\lambda, N)^2 d_\lambda^2 \|f \times [e^{itn}\Phi_\lambda]\|^2_{L^2(\mathbb{T}\times M)} d\left(\frac{T(n+\|\lambda\|^2)}{2\pi}, Q\right)^2 \right)^{1/2} + \frac{Q^2}{MN^{2-\varepsilon}} \|f\|_{L^2(\mathbb{T}\times M)}.$$

Using Lemma 3.47 in [5], we have

$$\begin{aligned} &\left| \left\{ (n,\lambda) : |n|, \|\lambda\|^2 \lesssim N^2, d\left(\frac{T(n+\|\lambda\|^2)}{2\pi}, Q\right) > D \right\} \right| \\ &\lesssim_{\tau,B} (D^{-B}Q^\tau N^2 + Q^B) \cdot \max_{|m|\lesssim N^2} |\{(n,\lambda) : n + \|\lambda\|^2 = m\}| \\ &\lesssim_{\tau,B} (D^{-B}Q^\tau N^2 + Q^B) \cdot |\{\lambda \in \widehat{M} : \|\lambda\|^2 \lesssim N^2\}| \\ &\lesssim_{\tau,B} (D^{-B}Q^\tau N^2 + Q^B) \cdot N^r. \end{aligned} \tag{3.25}$$

Here we used (i) of Lemma 9.

Now (2.5) gives

$$\|f \times [e^{itn}\Phi_\lambda]\|_{L^2(\mathbb{T}\times M)} \leq d_\lambda^{-\frac{1}{2}} \|f\|_{L^1(\mathbb{T}\times M)},$$

which together with (3.25), $d(\cdot, Q) \leq Q$, and (ii) of Lemma 9 implies

$$\|f \times \Lambda_{Q,M}\|_{L^2(\mathbb{T}\times M)} \lesssim_{\tau,B} \left(\frac{Q^{1+\varepsilon}D}{NM} + \frac{Q^2}{MN^{2-\varepsilon}} \right) \|f\|_{L^2(\mathbb{T}\times M)} + \frac{Q^{1+\varepsilon}}{NM} \cdot Q \cdot (D^{-B/2}Q^\tau N + Q^{B/2}) N^{d/2} \|f\|_{L^1(\mathbb{T}\times M)}.$$

This implies (3.23) assuming the conditions in (3.24). □

Using Proposition 17, the rest of the proof of Theorem 15 follows exactly as in [25].

4. Dispersive estimates near the corners

In this section, we prove Theorem 14 (1). First note that the Schrödinger kernel $K_N(t, \cdot)$ given by (2.18) as a linear combination of spherical functions is K-invariant, thus the values of $K_N(t, \cdot)$ are determined by its restriction on any maximal torus (and moreover on the closure of any cell). Thus it suffices to prove (3.10) uniformly on a fixed maximal torus. By Proposition 9.4 of Ch. III in [16], the spherical function Φ_λ for $\lambda \in \Lambda^+$ on a maximal torus equals

$$\Phi_\lambda = \sum_{i=1}^{q} c_i e^{\lambda_i}, \ \ \lambda_i \in \Lambda, c_i \geq 0.$$

This puts the Schrödinger kernel (2.18) in the form of an exponential sum. We now review parts of Section 7 in [25] necessary for the estimate of this exponential sum.

Definition 18. *Let $L = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_r$ be a lattice on an inner product space. We say L is a rational lattice provided that there exists some $D > 0$ such that $\langle w_i, w_j \rangle \in D^{-1}\mathbb{Z}$. We call the number D a period of L.*

By Lemma 10, the weight lattice Λ of U/K is a rational lattice with respect to the Killing form. As a sublattice of Λ, the restricted root lattice Γ is also rational.

Let f be a function on $\mathbb{Z}^r$ and define the *difference operator* D_i's by

$$D_i f(n_1, \cdots, n_r) := f(n_1, \cdots, n_{i-1}, n_i + 1, n_{i+1}, \cdots, n_r) - f(n_1, \cdots, n_r) \tag{4.1}$$

for $i = 1, \cdots, r$. The Leibniz rule for D_i reads

$$D_i(\prod_{j=1}^{n} f_j) = \sum_{l=1}^{n} \sum_{1 \le k_1 < \cdots < k_l \le n} D_i f_{k_1} \cdots D_i f_{k_l} \cdot \prod_{\substack{j \neq k_1, \cdots, k_l \\ 1 \le j \le n}} f_j. \tag{4.2}$$

Note that there are $2^n - 1$ terms in the right side of the above formula.

We have the following estimate on Weyl type sums. This is Lemma 7.4 in [25].

Lemma 19. *Let $L = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_r$ be a rational lattice in the inner product space $(V, \langle\ ,\ \rangle)$ with a period D. Let φ be a bump function on $\mathbb{R}$ and $N \ge 1$, $A \in \mathbb{R}$. Suppose f is a complex valued function on $L \cong \mathbb{Z}^r$ such that there exists a constant $A \in \mathbb{R}$ for which*

$$|D_1^{\varepsilon_1} \cdots D_r^{\varepsilon_r} f(n_1, \dots, n_r)| \lesssim N^{A - \varepsilon_1 - \cdots - \varepsilon_r}$$

for all $(\varepsilon_1, \dots, \varepsilon_r) \in \{0, 1, 2\}^r$, uniformly for $|n_i| \lesssim N$, $i = 1, \dots, r$. Let

$$F(t, H) = \sum_{\lambda \in L} e^{-it|\lambda|^2 + i\langle \lambda, H\rangle} \varphi(\frac{|\lambda|^2}{N^2}) \cdot f \tag{4.3}$$

for $t \in \mathbb{R}$ and $H \in V$. Then for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$, we have

$$|F(t, H)| \lesssim \frac{N^{A+r}}{[\sqrt{q}(1 + N\|\frac{t}{2\pi D} - \frac{a}{q}\|^{1/2})]^r} \tag{4.4}$$

uniformly in $H \in V$.

On the setting of compact Lie groups, we can rewrite the kernel function (2.18), originally a sum over the dominant weights, into a sum over the whole weight lattice, so that the above lemma can be applied to an estimate of this kernel. But for compact symmetric spaces, such rewriting seems unavailable, thus we need the following variant of the above lemma.

Lemma 20. *Let $L = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_r$ be a rational lattice in the inner product space $(V, \langle\ ,\ \rangle)$ with a period D. Let $L^+ = \mathbb{Z}_{\ge 0} w_1 + \cdots + \mathbb{Z}_{\ge 0} w_r$. Let φ be a bump function on $\mathbb{R}$ and $N \ge 1$. Let f be a complex valued function on $L^+ \cong (\mathbb{Z}_{\ge 0})^r$ such that there exists a constant $A \in \mathbb{R}$ for which*

$$|D_1^{\varepsilon_1} \cdots D_r^{\varepsilon_r} f(n_1, \dots, n_r)| \lesssim N^{A - \varepsilon_1 - \cdots - \varepsilon_r} \tag{4.5}$$

for all $(\varepsilon_1, \dots, \varepsilon_r) \in \{0, 1\}^r$, uniformly for $0 \le n_i \lesssim N$, $i = 1, \dots, r$. Let

$$F(t, H) = \sum_{\lambda \in L^+} e^{-it|\lambda|^2 + i\langle \lambda, H\rangle} \varphi\left(\frac{|\lambda|^2}{N^2}\right) \cdot f \tag{4.6}$$

for $t \in \mathbb{R}$ and $H \in V$. Then for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$, we have

$$|F(t, H)| \lesssim_{\varepsilon > 0} \frac{N^{A+r+\varepsilon}}{[\sqrt{q}(1 + N\|\frac{t}{2\pi D} - \frac{a}{q}\|^{1/2})]^r} \tag{4.7}$$

uniformly in $H \in V$.

Proof. By Weyl's differencing technique,

$$\begin{aligned}|F(t,H)|^2 &= \sum_{\lambda_1\in L^+,\lambda_2\in L^+} e^{-it(|\lambda_1|^2-|\lambda_2|^2)+i\langle\lambda_1-\lambda_2,H\rangle}\varphi\left(\frac{|\lambda_1|^2}{N^2}\right)\varphi\left(\frac{|\lambda_2|^2}{N^2}\right)f(\lambda_1)\overline{f(\lambda_2)}\\ &= \sum_{\substack{\mu\in L^+\\(\mu=\lambda_1+\lambda_2)}} e^{it|\mu|^2-i\langle\mu,H\rangle}\sum_{\substack{\lambda\in L^+\cap(\mu-L^+)\\(\lambda=\lambda_1)}} e^{2i[\langle\lambda,H\rangle-t\langle\lambda,\mu\rangle]}\varphi\left(\frac{|\mu|^2}{N^2}\right)\varphi\left(\frac{|\mu-\lambda|^2}{N^2}\right)f(\mu)\overline{f(\mu-\lambda)}\end{aligned}$$

$$\lesssim \sum_{\substack{\mu\in L^+\\(\mu=\lambda_1+\lambda_2)}}\left|\sum_{\substack{\lambda\in L^+\cap(\mu-L^+)\\(\lambda=\lambda_1)}} e^{2i[\langle\lambda,H\rangle-t\langle\lambda,\mu\rangle]}\varphi\left(\frac{|\mu|^2}{N^2}\right)\varphi\left(\frac{|\mu-\lambda|^2}{N^2}\right)f(\mu)\overline{f(\mu-\lambda)}\right|. \tag{4.8}$$

For $\mu\in L^+$, write

$$\mu = n_1^\mu w_1+\cdots+n_r^\mu w_r.$$

Then

$$\lambda\in\mu^+\cap(\mu-L^+) \text{ if and only if } \lambda=n_1w_1+\cdots+n_rw_r,\ 0\le n_j\le n_j^\mu,\ j=1,\dots,r.$$

For $\lambda = n_1w_1+\cdots+n_rw_r$, let

$$g(n_1,\dots,n_r)=g(\lambda)=g(\lambda,N,\mu):=\phi\left(\frac{|\lambda|^2}{N^2}\right)\phi\left(\frac{|\mu-\lambda|^2}{N^2}\right)f_\lambda f_{\mu-\lambda}.$$

By the assumption on f and the Leibiniz rule for difference operators, we have

$$|D_1^{\varepsilon_1}\cdots D_r^{\varepsilon_r}g(n_1,\cdots,n_r)|\lesssim N^{2A-\varepsilon_1-\cdots-\varepsilon_r} \tag{4.9}$$

for all $(\varepsilon_1,\dots,\varepsilon_r)\in\{0,1\}^r$, uniformly for $0\le n_i\lesssim N$, $i=1,\dots,r$. Now let $F^\mu=F^\mu(t,H)$ be the sum in (4.8) inside of the absolute value. Then

$$F^\mu=\sum_{0\le n_r\le n_r^\mu}e^{in_r\theta_r}\cdots\sum_{0\le n_1\le n_1^\mu}e^{in_1\theta_1}g(\lambda)$$

where

$$\theta_j=\theta_j(t,H,\mu):=2[\langle w_j,H\rangle-t\langle w_j,\mu\rangle],\ j=1,\dots,r.$$

We can perform summation by parts on F^μ with respect to the variable n_1

$$\begin{aligned}\sum_{0\le n_1\le n_1^\mu}e^{in_1\theta_1}g(\lambda)&=\frac{1}{1-e^{i\theta_1}}\sum_{0\le n_1\le n_1^\mu}e^{i(n_1+1)\theta_1}D_1g(\lambda)\\&\quad+\frac{1}{1-e^{i\theta_1}}g(0,n_2,\dots,n_r)-\frac{e^{i(n_1^\mu+1)\theta_1}}{1-e^{i\theta_1}}g(n_1^\mu+1,n_2,\dots,n_r).\end{aligned}$$

Then we can perform summation by parts with respect to other variables $n_2,\dots,n_r$. But we require that only when

$$|1-e^{i\theta_j}|\ge N^{-1},$$

do we carry out the procedure to the variable n_j. Using (4.9), what we we end up with is an estimate

$$\begin{aligned}|F^\mu|^2 &\lesssim N^{2A}\prod_{j=1}^{r}\frac{1}{\max\{\frac{1}{N},|1-e^{i\theta_j}|\}}\\ &\lesssim N^{2A}\prod_{j=1}^{r}\frac{1}{\max\{\frac{1}{N},\|\frac{\theta_j}{2\pi}\|\}}\\ &\lesssim N^{2A}\prod_{j=1}^{r}\frac{1}{\max\{\frac{1}{N},\|\frac{w_j(H)}{\pi}-\frac{t\langle w_j,\mu\rangle}{\pi}\|\}}.\end{aligned}$$

Since D is a period of the lattice L, $-2\langle w_j,\mu\rangle\in D^{-1}\mathbb{Z},\ \forall\mu\in L,\ j=1,\dots,r$. Let

$$m_j=-2\langle w_j,\mu\rangle\cdot D,\ j=1,\dots,r.$$

Since the map $\Lambda\ni\mu\mapsto(m_1,\dots,m_r)\in\mathbb{Z}^r$ is one-one, we can write (4.8) into

$$\begin{aligned}|F(t,H)|^2 &\lesssim N^{2A}\sum_{|m_1|\lesssim N,\dots,|m_j|\lesssim N}\prod_{j=1}^{r}\frac{1}{\max\{\frac{1}{N},\|m_j\frac{t}{2\pi D}+\frac{w_j(H)}{\pi}\|\}}\\ &\lesssim N^{2A}\prod_{j=1}^{r}\left(\sum_{|m_j|\lesssim N}\frac{1}{\max\{\frac{1}{N},\|m_j\frac{t}{2\pi D}+\frac{w_j(H)}{\pi}\|\}}\right).\end{aligned}$$

By a standard estimate as in deriving the classical Weyl inequality in one dimension, we get

$$\sum_{|m_j|\lesssim N}\frac{1}{\max\{\frac{1}{N},\|m_j\frac{t}{2\pi D}+\frac{w_j(H)}{\pi}\|\}}\lesssim\frac{N^2\log N}{[\sqrt{q}(1+N\|\frac{t}{2\pi D}-\frac{a}{q}\|^{1/2})]^2}$$

for $\frac{t}{2\pi D}$ lying on the major arc $\mathcal{M}_{a,q}$. Hence

$$|F(t,H)|^2\lesssim\frac{N^{2A+2r}\log^r N}{[\sqrt{q}(1+N\|\frac{t}{2\pi D}-\frac{a}{q}\|^{1/2})]^{2r}}.$$

□

Remark 21. *Let λ_0 be any constant vector in V and C any constant real number. If we slightly generalize the form of the function $F(t,H)$ in Lemma 20 into*

$$F(t,H)=\sum_{\lambda\in L^+}e^{-it|\lambda+\lambda_0|^2+i\langle\lambda,H\rangle}\varphi\left(\frac{|\lambda+\lambda_0|^2+C}{N^2}\right)\cdot f$$

then the proof still works.

We have our first application of Lemma 20. Let U/K be a simply connected irreducible symmetric space of compact type. Let o denote the identity coset K in U/K and specialize the Schrödinger kernel (2.18) to $x=o$. Noting that $\Phi_\lambda(o)=1$, we have

$$K_N(t,o)=\sum_{\lambda\in\Lambda^+}\varphi\left(\frac{-|\lambda+\rho|^2+|\rho|^2}{N^2}\right)e^{it(-|\lambda+\rho|^2+|\rho|^2)}d_\lambda. \tag{4.10}$$

Proposition 22. *For all irreducible spaces U/K of compact type, we have*

$$|K_N(t,o)|\lesssim_{\varepsilon>0}\frac{N^{d+\varepsilon}}{[\sqrt{q}(1+N\|\frac{t}{2\pi D}-\frac{a}{q}\|^{1/2})]^r}$$

for $\frac{t}{2\pi D}\in\mathcal{M}_{a,q}$.

Proof. Recall that d_λ is a polynomial in $\lambda \in \Lambda$ of degree $d-r$. Thus d_λ as a function on $\Lambda^+ \cong (\mathbb{Z}_{\geq 0})^r$ satisfies (4.5) with $A = d-r$. Then this is a direct consequence of Lemma 20. □

We now strengthen Proposition 22.

Theorem 23. *Still, let M be a simply connected symmetric space of the compact type. Let D be a period of the weight lattice and let $d(\cdot,\cdot)$ be the Riemannian distance function on M. Then we have*

$$|K_N(t,x)| \lesssim_{\varepsilon>0} \frac{N^{d+\varepsilon}}{(\sqrt{q}(1+N\|\frac{t}{2\pi D}-\frac{a}{q}\|^{1/2}))^r} \tag{4.11}$$

for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$, uniformly for $d(x,o) \lesssim N^{-1}$.

The proof hinges on an integral representation of spherical functions in a neighborhood of o. Continue the notations in the Preliminaries, let $\mathfrak{n}^{\mathbb{C}}, \mathfrak{a}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}$ be respectively the complexification of $\mathfrak{n}, \mathfrak{a}, \mathfrak{k}$ in $\mathfrak{g}^{\mathbb{C}}$. Let $G^{\mathbb{C}}$ be a simply connected group whose Lie algebra is $\mathfrak{g}^{\mathbb{C}}$ so that U becomes an analytic subgroup of $G^{\mathbb{C}}$. By Section 9.2 Ch. III in [16], the mapping

$$(X,H,T) \mapsto \exp X \exp H \exp T,\ \ X \in \mathfrak{n}^{\mathbb{C}}, H \in \mathfrak{a}^{\mathbb{C}}, T \in \mathfrak{k}^{\mathbb{C}}$$

is a holomorphic diffeomorphism of a neighborhood $\mathcal{U}^{\mathbb{C}}$ of $G^{\mathbb{C}}$ such that $\mathcal{U} = \mathcal{U}^{\mathbb{C}} \cap U$ is invariant under the maps $u \mapsto kuk^{-1}$, $k \in K$. This induces the map

$$A : \exp X \exp H \exp T \to H$$

that sends $\mathcal{U}^{\mathbb{C}}$ into $\mathfrak{a}^{\mathbb{C}}$. Let Φ_λ be the spherical function associated to $\lambda \in \Lambda^+$. By Lemma 9.2 of Ch. III in [16],

$$\Phi_\lambda(u) = \int_K e^{-\lambda(A(ku^{-1}k^{-1}))}\, dk,\ \ u \in \mathcal{U}. \tag{4.12}$$

Note that the map $u \mapsto kuk^{-1}$ preserves the distance $d_U(\cdot, e)$ to the identity e of U. Let $N \geq 1$ be large enough so that $\{u \in U : d_U(u,e) \lesssim N^{-1}\} \subset \mathcal{U}$. Then

$$|A(ku^{-1}k^{-1})| \lesssim N^{-1} \tag{4.13}$$

uniformly for $d_U(u,e) \lesssim N^{-1}$ and $k \in K$. Here the norm on $\mathfrak{a}^{\mathbb{C}}$ of course comes from the Killing form. Write $\lambda = n_1 w_1 + \cdots + n_r w_r$, $n_i \in \mathbb{Z}_{\geq 0}$, viewing $\Phi_\lambda(u) = \Phi(\lambda, u)$ as a function of $\lambda \in (\mathbb{Z}_{\geq 0})^r$, (4.12) and (4.13) imply that $\Phi(\lambda, u)$ satisfies an equality of the type (4.5) as follows.

Lemma 24.

$$|D_{i_1} \cdots D_{i_n} \Phi(n_1, \cdots, n_r, u)| \lesssim N^{-n}$$

holds uniformly for $0 \leq n_i \lesssim N$ and $d_U(u,e) \lesssim N^{-1}$, for all $i_j = 1, \cdots, r$ and $n \in \mathbb{Z}_{\geq 0}$.

Proof of Theorem 23. Using Lemma 24 and the fact that d_λ is a polynomial in λ of degree $d-r$ and applying the Leibniz rule (4.2), we have $f = d_\lambda \Phi_\lambda$ satisfies (4.5) with $A = d-r$. Then we can apply Lemma 20 to get the result. □

Finally, we upgrade Theorem 23 to cover the cases when $x \in M$ is within the distance of about N^{-1} from any corner.

Theorem 25. *Let $[iH_0] \in A$ be any corner. Then* (4.11) *holds uniformly for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$ and $d(x,[iH_0]) \lesssim N^{-1}$ $(x \in M)$.*

To prove this theorem, we describe another important characterization of spherical functions. For $\mu, \lambda \in \Lambda$, let $\mu \leq \lambda$ denote the statement that $\lambda - \mu \in 2\mathbb{Z}_{\geq 0}\alpha_1 + \cdots + 2\mathbb{Z}_{\geq 0}\alpha_r$. For $\mu \in \Lambda^+$, define

$$M(\mu) = \sum_{s \in W} e^{s\mu}.$$

Then define the *Heckman-Opdam polynomials* $P(\lambda)$, $\lambda \in \Lambda^+$, by

$$P(\lambda) = \sum_{\mu \in \Lambda^+, \mu \leq \lambda} c_{\lambda,\mu} M(\mu),\ c_{\lambda,\lambda} = 1$$

such that

$$\int_A P(\lambda) \cdot \overline{M(\mu)} \cdot \left| \prod_{\alpha \in \Sigma^+} (e^{\alpha} - e^{-\alpha})^{m_\alpha} \right| da = 0, \text{ for any } \mu \in \Lambda^+ \text{ such that } \mu < \lambda.$$

Here da is the normalized Haar measure on the A. Normalize $P(\lambda)$ by

$$\tilde{P}(\lambda) = \frac{P(\lambda)}{P(\lambda)(o)}.$$

Theorem 26 (Corollary 5.2.3 in Part I of [13])**.** *The spherical functions on U/K restricted on A are given by the normalized Heckman-Opdam polynomials:*

$$\Phi_\lambda = \tilde{P}(\lambda),\ \lambda \in \Lambda^+.$$

Corollary 27. *Let $[iH_0] \in A$ be a corner. Then*

$$\Phi_\lambda(iH + iH_0) = e^{i\lambda(H_0)} \Phi_\lambda(iH),\ H \in \mathfrak{a},\ \lambda \in \Lambda^+.$$

Proof. By the above theorem and the definition of Heckman-Opdam polynomials, it suffices to show that for any $\lambda, \mu \in \Lambda^+$ such that $\mu \leq \lambda$ and $s \in W$,

$$e^{i(s\mu)(H_0)} = e^{i\lambda(H_0)}.$$

This is reduced to showing $(s\mu - \lambda)(H_0) \in 2\pi\mathbb{Z}$, and by the definition of $[iH_0]$ as a corner, it is further reduced to $s\mu - \lambda \in 2\mathbb{Z}\alpha_1 + \cdots + 2\mathbb{Z}\alpha_r$. By the fact $\mu \leq \lambda$, it then suffices to show $s\mu - \mu \in 2\mathbb{Z}\alpha_1 + \cdots + 2\mathbb{Z}\alpha_r$ for any $\mu \in \Lambda$ and $s \in W$. But this is a standard fact of root system theory (see Corollary 4.13.3 in [24]). ☐

Let $\Gamma = 2\mathbb{Z}\alpha_1 + \cdots + 2\mathbb{Z}\alpha_r$. The above corollary implies that for $\lambda \in \Gamma$ and $\mu \in \Lambda^+$ such that $\lambda + \mu \in \Lambda^+$,

$$\Phi_{\lambda+\mu}(iH + iH_0) = e^{i\mu(H_0)} \Phi_{\lambda+\mu}(iH). \tag{4.14}$$

This inspires a decomposition of Λ^+ and thus of the Schrödinger kernel (2.18), so to make applicable the techniques in proving Theorem 23 for the proof of Theorem 25.

Proof of Theorem 25. First note that all the fundamental weights $w_1, \cdots, w_r$ are rational linear combinations of the roots. Thus there exists some $B \in \mathbb{N}$ such that $Bw_i \in \Gamma$ for all i. Define

$$\Gamma_1^+ = \mathbb{Z}_{\geq 0} Bw_1 + \cdots + \mathbb{Z}_{\geq 0} Bw_r.$$

Then $\Lambda^+/\Gamma_1^+ = \{n_1 w_1 + \cdots + n_r w_r : n_i = 0, \cdots, B-1, i = 1, \cdots, r\}$ and we decompose

$$\Lambda^+ = \bigsqcup_{\mu \in \Lambda^+/\Gamma_1^+} (\Gamma_1^+ + \mu).$$

This yields a decomposition of the Schrödinger kernel

$$K_N = \sum_{\mu\in\Lambda^+/\Gamma_1^+} K_N^\mu, \tag{4.15}$$

$$K_N^\mu = \sum_{\lambda\in\Gamma_1^+} \varphi\left(\frac{-|\lambda+\mu+\rho|^2+|\rho|^2}{N^2}\right) e^{it(-|\lambda+\mu+\rho|^2+|\rho|^2)} d_{\lambda+\mu}\Phi_{\lambda+\mu}.$$

By the finiteness of Λ^+/Γ_1^+, it suffices to prove (4.11) replacing K_N by K_N^μ. By (4.14),

$$K_N^\mu(t, iH+iH_0) = e^{i\mu(H_0)} \sum_{\lambda\in\Gamma_1^+} \varphi(\frac{-|\lambda+\mu+\rho|^2+|\rho|^2}{N^2}) e^{it(-|\lambda+\mu+\rho|^2+|\rho|^2)} d_{\lambda+\mu}\Phi_{\lambda+\mu}(iH).$$

Now for $x = [iH + iH_0]$, the assumption $d(x, [iH_0]) \lesssim N^{-1}$ gives $|H| \lesssim N^{-1}$. We apply Lemma 20 to $f(\lambda) = d_{\lambda+\mu}\Phi_{\lambda+\mu}(iH)$, and then the rest of the proof follows exactly as the proof of Theorem 23. ☐

5. Dispersive estimates away from the corners

We have not yet developed enough tools to prove (3.10) (with an ε-loss) uniformly for all $x \in M$ that stays away from the corners by a distance of about N^{-1} for a general symmetric space of the compact type. One main obstacle is the lack of explicit formulas for the spherical functions or the more general Heckman-Opdam polynomials. The group case solved in [25] relies on the Weyl character formula which is further analyzed by the tools of either the BBG-Demazure operators or the Harish-Chandra integral formula. In this section, we record another solved case which is the odd dimensional spheres, where explicit formulas of the ultraspherical polynomials could be used rather conveniently.

Proof of Theorem 14 (2). Let M be the sphere of dimension $d = 2\lambda + 1$, $\lambda \in \mathbb{N}$, and continue the notations in Example 7. The Schrödinger kernel reads

$$K_N(t,\theta) = \sum_{n\in\mathbb{Z}_{\geq 0}} \varphi\left(\frac{(n+\lambda)^2-\lambda^2}{N^2}\right) e^{-it[(n+\lambda)^2-\lambda^2]} d_n \Phi_n^{(\lambda)}(\theta).$$

To prove (3.10) with ε-loss, first realize that $K_N(t,\theta)$ is invariant under the Weyl group action $\theta \mapsto 2\pi - \theta$, thus it suffices to prove (3.10) uniformly for θ in the closed cell $[0, \pi]$. Since Theorem 25 implies (3.10) with ε-loss uniformly for $|\theta| \lesssim N^{-1}$ or $|\theta - \pi| \lesssim N^{-1}$, thus it suffices to prove (3.10) with ε-loss uniformly for θ away from $0, \pi$ by a distance $\gtrsim N^{-1}$. By (2.12), it then suffices to prove for all $\nu = 0, 1, \ldots, \lambda - 1$, we have

$$|K_N^{(\nu)}(t,\theta)| \lesssim_\varepsilon \frac{N^{2\lambda+1+\varepsilon}}{\sqrt{q}(1+N\|\frac{t}{2\pi D}-\frac{a}{q}\|^{1/2})}$$

for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$, uniformly in $CN^{-1} \leq \theta \leq \pi - CN^{-1}$, $C > 0$, where

$$K_N^{(\nu)}(t,\theta) = \frac{2}{(2\sin\theta)^{\nu+\lambda}} \sum_{n\in\mathbb{Z}_{\geq 0}} \varphi(\frac{(n+\lambda)^2-\lambda^2}{N^2}) e^{-it[(n+\lambda)^2-\lambda^2]} d_n C_{n,\nu} \cos((n-\nu+\lambda)\theta - (\nu+\lambda)\pi/2),$$

with

$$C_{n,\nu} = \binom{n+2\lambda-1}{n}^{-1} \binom{n+\lambda-1}{n} \binom{\nu+\lambda-1}{\nu} \frac{(1-\lambda)\cdots(\nu-\lambda)}{(n+\lambda-1)\cdots(n+\lambda-\nu)}.$$

As $CN^{-1} \leq \theta \leq \pi - CN^{-1}$, we have

$$\left|\frac{2}{(2\sin\theta)^{\nu+\lambda}}\right| \lesssim N^{\nu+\lambda},\ \nu = 0, \cdots, \lambda - 1.$$

Rewriting $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$, it then suffices to prove

$$\left| \sum_{n\in\mathbb{Z}_{\geq 0}} \varphi(\frac{(n+\lambda)^2-\lambda^2}{N^2}) e^{-it[(n+\lambda)^2-\lambda^2]\pm i(n-\nu+\lambda)\theta \mp i(\nu+\lambda)\pi/2} d_n C_{n,\nu} \right| \lesssim_\varepsilon \frac{N^{\lambda-\nu+1+\varepsilon}}{\sqrt{q}(1+N\|\frac{t}{2\pi D}-\frac{a}{q}\|^{1/2})} \tag{5.1}$$

uniformly in $\theta \in [CN^{-1}, \pi - CN^{-1}]$. Note that d_n is polynomial in n of degree $d-1 = 2\lambda$, then we can write $d_n C_{n,v} = \frac{f(n)}{g(n)}$ such that $f(n)$ and $g(n)$ are polynomials of degree $3\lambda - 1$ and $2\lambda - 1 + \nu$ respectively. Note that $2\lambda - 1 + \nu \leq 3\lambda - 2$. This implies that $d_n C_{n,v}$ satisfies an estimate of the form (4.5)

$$|D^\mu(d_n C_{n,\nu})| \lesssim N^{\lambda-\nu-\mu}$$

for $\mu = 0, 1$, uniformly for $0 \leq n \lesssim N$. Then we apply Lemma 20 to (5.1) and finish the proof. □

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