On Fourier restriction type problems on compact Lie groups

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Abstract. In this article, we obtain new results for Fourier restriction type problems on compact Lie groups. We first provide a sharp form of L^p estimates of irreducible characters in terms of their Laplace-Beltrami eigenvalue and as a consequence provide some sharp L^p estimates of joint eigenfunctions for the ring of invariant differential operators. Then we improve upon the previous range of exponent for scale-invariant Strichartz estimates for the Schrödinger equation, and prove L^p bounds of Laplace-Beltrami eigenfunctions in terms of their eigenvalue matching the known bounds on tori. The main novelties in our approach consist of a barycentric-semiclassical subdivision of the Weyl alcove and sharp L^p estimates on each component of this subdivision of some weight functions coming out of the Weyl denominator.

1. Introduction

1.1. **Three problems.** The goal of this article is to obtain new results for problems of Fourier restriction type on the setting of compact Lie groups. On Euclidean spaces, Fourier restriction estimates were first explicitly posed and studied by Stein and Tomas [29], as special types of oscillatory integrals. Let $d\mu$ be a measure on $\mathbb{R}^n$ for example the surface measure on a hypersurface such as a sphere or paraboloid. Fourier restriction estimates ask about decay properties of the (inverse) Fourier transform of $d\mu$, which may be quantified in terms of Lebesgue spaces via an inequality of the form

$$\|\widehat{f\cdot d\mu}\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^q(d\mu)}.$$

These estimates have broad applications in analysis and partial differential equations, and are currently under intensive study with an abundance of hard open problems; we refer to [10] for a recent survey. In particular, when $d\mu$ is the surface measure on the standard sphere, the above inequality provides L^p-bounded eigenfunctions of the Laplacian; or if $d\mu$ is the surface measure on the paraboloid $\xi_n = \xi_1^2+\xi_2^2+\cdots+\xi_{n-1}^2$, the above inequality becomes the Strichartz estimate for the Schrödinger equation

$$\|e^{it\Delta}f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^q(\mathbb{R}^{n-1})}.$$

On the other hand, on the setting of a compact manifold M, Sogge [25] initiated a study of the so-called discrete Fourier restriction problems, which is to consider the norm of spectral projectors on either a band of eigenvalues or an individual eigenvalue of the Laplace-Beltrami operator. For individual eigenvalues, it amounts to L^p bounds of Laplace-Beltrami eigenfunctions. Analogous to the Euclidean setting and because of the discreteness of the spectrum, we may call this problem as discrete Fourier restriction for the sphere. We may also pose Strichartz estimates for the Schrödinger equation

$$\|e^{it\Delta}f\|_{L^p(I\times M)} \leq C\|f\|_{L^q(M)},$$

where I could be $\mathbb{R}$, a finite interval, or a circle $\mathbb{T}$ if Δ has rational eigenvalues which is the case for rational tori or more generally compact Lie groups and symmetric spaces equipped with canonical metrics; for the latter, analogous to the Euclidean setting, the above estimates may be viewed as a discrete Fourier restriction problem for the parabola on the product manifold $\mathbb{T}\times M$.

Due to the specific structure or symmetry of the manifold, we may pose another closely related problem. Many manifolds are equipped with additional operators that commute with the Laplace-Beltrami operator. For symmetric spaces, there is a full commutative ring of differential operators invariant under the symmetry group, which include the Laplace-Beltrami operator as a special element; to go further, for arithmetic locally symmetric spaces, there are additionally Hecke operators that commute with the ring of invariant differential operators. Thus instead of restricting on individual eigenvalues of the Laplace-Beltrami operator as considered in Sogge's work, we may restrict on individual spectral parameters of the full ring of invariant operators, and this amounts to bounds of joint eigenfunctions of this ring. Compared with the previous two Fourier restriction problems, this one is more of a purely representation theoretic flavor.

We have now introduced three problems of Fourier restriction type on the setting of compact manifolds, and we summarize them as follows.

Problem 1. Joint eigenfunction bounds for a commutative ring of operators that commute with the Laplace-Beltrami operator.

Problem 2. Strichartz estimates for the Schrödinger equation.

Problem 3. Laplace-Beltrami eigenfunction bounds.

These problems are closely related to each other and together they have close interactions with many other problems in the field of partial differential equations as well as analytic number theory. Problem 1 has direct applications to Problem 2 and 3, provided there is a good knowledge of how to express the Laplace-Beltrami eigenvalue as a function of the spectral parameter for the operator ring, as is the case for compact Lie groups and globally symmetric spaces. Problem 3 is directly applicable to Problem 2, as long as we have a good knowledge of distribution of Laplace-Beltrami eigenvalues; in fact, this is how the optimal Strichartz estimate may be obtained from Laplace-Beltrami eigenfunction bounds on Zoll manifolds as in [16]. The reverse application of Problem 2 to Problem 3 is also available, an example of which is the Hardy-Littlewood circle method on the case of tori as applied in [3], and we will see in this paper that a generalization to compact Lie groups is also available. Now we provide a more detailed review of known results for the above three problems on compact manifolds in the literature.

1.2. **Literature review.** Let M be a compact manifold of dimension d throughout this article.

1.2.1. *Problem 1.* For general compact symmetric spaces of either noncompact or compact type, the seminal contribution by Sarnak [23] states that, for a joint eigenfunction ψ of the full ring of invariant differential operators of Laplace-Beltrami eigenvalue of size $N^2 \geq 1$, the sharp pointwise bound as follows holds

$$\|\psi\|_{L^\infty(M)} \leq CN^{\frac{d-r}{2}}\|\psi\|_{L^2(M)}. \tag{1.1}$$

For L^p bounds of joint eigenfunctions ψ, we have the conditional results of Marshall [21] which state that under the regularity assumption that the spectral parameter of ψ stays within a fixed cone away from the walls of the Weyl chamber, then it holds true that

$$\|\psi\|_{L^p(M)} \leq CN^{\gamma(d,r,p)}\|\psi\|_{L^2(M)}, \tag{1.2}$$

for the sharp exponent (on irreducible spaces M) as follows

$$\gamma(d,r,p) = \begin{cases} \frac{d-r}{2} - \frac{d}{p}, & \text{if } p > \frac{2(d+r)}{d-r}, \\ \frac{d-r}{2}\left(\frac{1}{2} - \frac{1}{p}\right), & \text{if } 2 \leq p < \frac{2(d+r)}{d-r}. \end{cases}$$

In particular, if the rank of M is one and thus the ring of invariant differential operators is solely generated by the Laplace-Beltrami operator, the above bound matches Sogge's Laplace-Beltrami eigenfunction bound

(1.5) on a general compact manifold. However, it remains a challenge how to remove the above regularity assumption for higher-rank spaces.

For the special case when ψ is a spherical function on compact symmetric spaces or in particular a character on groups, we may expect better estimates. As established in [27] (see also [11]), any irreducible character χ on a compact simple Lie group of rank r satisfies the bound

$$\|\chi\|_{L^p(M)} \leq C, \text{ for } p < \frac{2d}{d-r}.$$

Let d_χ be the associated dimension of representation, then we have the bound

$$|\chi| \leq |d_\chi| \leq CN^{\frac{d-r}{2}}$$

by an application of the Weyl dimension formula. By interpolation, we then have the bound

$$\|\chi\|_{L^p(M)} \leq \begin{cases} C_\varepsilon N^{\frac{d-r}{2}-\frac{d}{p}+\varepsilon}, & \text{for } p \geq \frac{2d}{d-r}, \\ C, & \text{for } p < \frac{2d}{d-r}. \end{cases} \tag{1.3}$$

The above exponent of N without the ε can be checked to be sharp, by testing a character on a small neighborhood of the origin and choosing the spectral parameter regular enough. Similar results for spherical functions on arbitrary symmetric spaces of compact type are naturally conjectured to be true, but they seem still missing in the literature.

Lastly, for arithmetic locally symmetric spaces, there is a richer theory; this is beyond our ability to make a thorough survey, and we only mention some results that look fundamental to our own eyes. A large part of literature has been focused on pointwise estimates of Hecke-Masse forms which are defined as joint eigenfunctions of the ring of invariant differential operators and the Hecke operators, with the purpose of improving the exponent in (1.1). We have the seminal contribution of Iwaniec and Sarnak [19] for hyperbolic surfaces which are of rank one, and the work of Blomer and Pohl [1] in rank two, and of Marshall [22] in arbitrary rank.

1.2.2. *Problem 2.* There are two approaches to this problem, one via semiclassical analysis and one via exponential sums. The semiclassical references are mainly the work [9] of Burq, Gérard, and Tzvetkov and [26] of Staffilani and Tataru, where it was established for a finite interval I

$$\|e^{it\Delta}f\|_{L^p(I,L^q(M))} \leq C\|f\|_{H^{1/p}(M)}$$

for all admissible pairs (p,q) such that

$$\frac{2}{p}+\frac{d}{q}=\frac{d}{2},\ p,q\geq 2,\ (p,q,d)\neq(2,\infty,2).$$

Such estimates are non-scale-invariant: if M is replaced by the Euclidean space $\mathbb{R}^d$, then the Sobolev space $H^{1/p}(M)$ in the above estimate may be replaced by $L^2(\mathbb{R}^d)$, as established by Ginibre and Velo [12] and Keel and Tao [20], and the resulting estimate is then invariant under the scaling symmetry $u(t,x)\mapsto u(\lambda^2 t,\lambda x)$, which results in the above condition of admissibility for the pair (p,q). These estimates have applications to local well-posedness theory for nonlinear Schrödinger equations with initial data of low yet scaling-subcritical regularity; see Proposition 3.1 in [9]. However, due to the non-scale-invariant nature of these estimates, they can never be used to solve the more interesting and difficult problem of local-wellposedness of scaling-critical regularity. There are scale-invariant Strichartz estimates on compact manifolds in the literature, but such results are rare. We first have the seminal contribution of Bourgain [4] on tori $M = \mathbb{T}^d$

$$\|e^{it\Delta} f\|_{L^p(I\times M)} \leq C\|f\|_{H^{d/2-(d+2)/p}(M)} \tag{1.4}$$

for a limited range of p using the Hardy-Littlewood circle method, which is then enlarged to the optimal range $p > 2 + \frac{4}{d}$ by Bourgain and Demeter [7] by the new powerful method of decoupling theory. For spheres and Zoll manifolds of dimension at least three, the same estimate as (1.4) holds true for any $p > 4$, as can be established by the argument in Herr [16]; this range is also optimal as observed in [9]. In [31], we established (1.4) on an arbitrary compact Lie group equipped with the canonical Killing metric, for any $p \geq 2 + \frac{8}{r}$; we conjecture that the optimal range should be $p > 2 + \frac{4}{d}$ as the case of tori for spaces of rank at least two, and we will provide some solid evidence in this article.

1.2.3. *Problem 3.* Let f be an eigenfunction for the Laplace-Beltrami operator of eigenvalue $-N^2$. The fundamental result of Sogge [25] states

$$\|f\|_{L^p(M)} \leq CN^{\gamma(d,p)}\|f\|_{L^2(M)} \tag{1.5}$$

for

$$\gamma(d,p) = \begin{cases} \frac{d-1}{2} - \frac{d}{p}, & \text{if } p \geq \frac{2(d+1)}{d-1}, \\ \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right), & \text{if } 2 \leq p \leq \frac{2(d+1)}{d-1}. \end{cases}$$

These exponents were shown to be optimal by Sogge [25] on the standard spheres. The major open question is then to find refinement of the above exponents for various kinds of geometry. For example, with the presence of negative curvature, Hassell and Tacy [15] established an ε-improvement (of $(\log N)^{-1/2}$ to be precise), which may be seen as the effect of chaotic properties of the geodesic flow. At the other extreme of the fully integrable system the square tori $M = \mathbb{T}^d$, we first have the result of Zygmund [32] where it was shown that (1.5) holds with $\gamma(2,4) = 0$. Then Bourgain [3] conjectured (1.5) should hold with $\gamma(2,p) = 0$ for all $p < \infty$, and with

$$\gamma(d,p) = \frac{d-2}{2} - \frac{d}{p} \tag{1.6}$$

for $p > 2d/(d-2)$ when $d \geq 3$, with an N^ε-loss for $d = 3, 4$. These conjectures for $p = \infty$ are indeed true, which are consequences of counting representations of integers as sums of squares, as observed in [3]. A similar result holds true for arbitrary symmetric spaces of compact type. By counting representations of an integer by a positive definite integral quadratic form, we may use Sarnak's bound (1.1) to establish the pointwise eigenfunction estimate

$$\|f\|_{L^\infty(M)} \leq CN^{\frac{d-2}{2}}\|f\|_{L^2(M)},$$

provided the rank r is at least 2, and with an N^ε-loss if $r = 2, 3, 4$. It is also worth noticing that by combining this counting argument with Marshall's bound (1.5), some L^p Laplace-Beltrami eigenfunction bounds with the same exponent as (1.6) may be established for arbitrary products of rank-one spaces, though such an argument provides a rather poor range of exponent. Then in a series of papers, Bourgain [3, 5], Bourgain and Demeter [6, 7, 8] established the conjectured estimates on tori with an ε-loss for $p \geq 2(d-1)/(d-3)$ when $d \geq 4$.

1.3. **Main results.** In this article, we prove new results for all the above three Fourier restriction type problems on the setting of compact Lie groups, via a rather uniform approach.

1.3.1. *On Problem 1.* We establish the sharp form of the character bound (1.3) by removing the ε factor, and provide as a corollary some sharp joint eigenfunction bounds.

Theorem 1.1. *Let M be a compact simple Lie group of dimension d and rank r.*
(i) Suppose χ_μ is any irreducible character with Laplace-Beltrami eigenvalue $-N^2$. Then

$$\|\chi_\mu\|_{L^p(M)} \leq \begin{cases} CN^{\frac{d-r}{2}-\frac{d}{p}}, & \text{for } p > \frac{2d}{d-r}, \\ C, & \text{for } p < \frac{2d}{d-r}. \end{cases}$$

(ii) Suppose ψ is any joint eigenfunction of the ring of invariant differential operators with Laplace-Beltrami eigenvalue $-N^2$. Then

$$\|\psi\|_{L^p(M)} \leq CN^{\frac{d-r}{2}-\frac{d}{p}}\|\psi\|_{L^2(M)}, \text{ for } p > \frac{4d}{d-r}.$$

Note that the exponent in the estimates in (ii) is sharp and matches that of (1.9) albeit in a limited range of p. However, they are established without any regularity assumptions on the spectral parameter and serve as the first such results on higher-rank spaces beyond products of rank-ones for $p < \infty$.

We will establish the above results by a careful study of the behavior of characters over the Weyl alcove (or called the Weyl cell). Compared with the work of [27], the novelty of our approach is to take care of the behavior of characters both near the walls and near different vertices of the alcove. To achieve this, we develop a so-called barycentric-semiclassical subdivision of the alcove, and obtain key sharp L^p estimates on each component of this subdivision of some weight functions coming out of the Weyl denominator.

1.3.2. *On Problem 2.* As mentioned above, we proved in [31] the following scale-invariant Strichartz estimate for the Schrödinger equation on any compact Lie group M of rank r equipped with the canonical Killing metric

$$\|e^{it\Delta}f\|_{L^p(I\times M)} \leq C\|f\|_{H^{d/2-(d+2)/p}(M)}, \text{ for } p \geq 2+8/r. \tag{1.7}$$

The proof adapts the framework of Bourgain [4] on the setting of tori. We studied the spectrally localized Schrödinger kernel $\mathscr{K}_N$ defined as

$$\phi\left(\frac{-\Delta}{N^2}\right)e^{it\Delta}f = f * \mathscr{K}_N(t,\cdot)$$

where $\phi\left(\frac{-\Delta}{N^2}\right)$ is a standard spectral localization operator on M. We realized this kernel as a Weyl type exponential sum and derived its pointwise bound as follows

$$\|\mathscr{K}_N(t,\cdot)\|_{L^\infty(M)} \leq C\frac{N^d}{\left(\sqrt{q}\left(1+N\left\|\frac{t}{\mathcal{T}}-\frac{a}{q}\right\|^{\frac{1}{2}}\right)\right)^r} \tag{1.8}$$

on major arcs of the time variable t

$$\left\|\frac{t}{\mathcal{T}}-\frac{a}{q}\right\| \leq \frac{1}{qN}$$

centered at the fraction a/q for $(a,q) = 1$ and $q < N$. Here $\mathcal{T}$ is a period for the Schödinger propagator $e^{it\Delta}$. The proof also applies interpolation for the operator norm between $L^1 \to L^\infty$ and $L^2 \to L^2$ in order to exploit oscillation on both physical and frequency spaces, a classical harmonic analytic method as employed in the ancestor theorem of Stein and Tomas [29]. In this paper, we improve the range of p in (1.7) for compact semisimple Lie groups. A distinction between flat tori and compact semisimple Lie groups as well as the more general symmetric spaces of compact type is that joint eigenfunctions of invariant differential operators for the latter tend to be concentrated on conjugate points, as an example the zonal spherical harmonics on spheres blowing up at the north and south poles as the eigenvalue goes to infinity, while the characters on tori are uniform in size. This is behind the previously mentioned Marshall's conditional L^p-upgrades of

Sarnak's pointwise bound (1.1) into

$$\|\psi\|_{L^p(M)} \le CN^{\frac{d-r}{2}-\frac{d}{p}}\|\psi\|_{L^2(M)}, \quad \text{for } p > \frac{2(d+r)}{d-r}, \tag{1.9}$$

assuming that M as a compact symmetric space is irreducible. Note in particular the extra term $N^{-\frac{d}{p}}$ in the above inequality compared with (1.1) may be considered as the "scale-invariant" factor, as ψ of Laplace-Beltrami eigenvalue of size N^2 is heuristically localized in the physical space a region of size about N^{-1} by the uncertainty principle. In comparison, the only such scale-invariant estimates valid on tori is when $p=\infty$. In a similar vein, the following scale-invariant L^p-upgrades of (1.8) is expected

$$\|\mathscr{K}_N(t,\cdot)\|_{L^p(M)} \le C\frac{N^{d-\frac{d}{p}}}{\left(\sqrt{q}\left(1+N\left\|\frac{t}{T}-\frac{a}{q}\right\|^{\frac{1}{2}}\right)\right)^r} \tag{1.10}$$

to hold true for a range of $p<\infty$. We confirm it on arbitrary compact semisimple Lie groups for a sharp range of p as follows.

Theorem 1.2. *Suppose M is a compact simply connected simple Lie group. Then for any $p>\frac{2d}{d-r}$, inequality (1.10) holds uniformly for $\left\|\frac{t}{T}-\frac{a}{q}\right\| \le \frac{1}{qN}$. More generally, let M be a compact simply connected semisimple Lie group. For each irreducible factor M_0 of M, set*

$$s_0 = \frac{2d_0}{d_0-r_0}$$

where d_0, r_0 are respectively the dimension and rank of M_0, and then let s be the largest among the s_0's. Then (1.10) holds for any $p>s$.

This theorem will also be proved by an application of the above mentioned barycentric-semiclassical subdivision of the alcove and sharp L^p estimates of some weight functions on the alcove. Then we can incorporate these L^p estimates into Strichartz estimates. We replace the major-minor arc decomposition as in [4, 31] by the Farey dissection into major arcs only, observing that the contributions from the minor arcs would not fall in the right scale for L^p estimates. We are then able to obtain the following improved scale-invariant Strichartz estimates on compact semisimple Lie groups, and they seem to saturate the method of [4] on this setting.

Theorem 1.3. *Let M be a compact semisimple Lie group of rank $r\ge 2$. Then*

$$\|e^{it\Delta}f\|_{L^p(I\times M)} \le C\|f\|_{H^{d/2-(d+2)/p}(M)} \tag{1.11}$$

holds for any $p > 2+\frac{8(s-1)}{sr}$.

What would be the optimal range of p in the above estimate? By looking at class functions on compact Lie groups, we realize that the estimates can be reduced to the following well conjectured Strichartz estimates for mixed Lebesgue norms on tori. Such estimates are indeed true on Euclidean spaces, as established in [12, 20].

Conjecture 1.4. *Let $(\cdot,\cdot)$ denote a positive-definite quadratic form of integral coefficients. Then we have*

$$\left\|\sum_{\xi\in\mathbb{Z}^r,\ |\xi|\le N} a_\xi e^{it(\xi,\xi)+i(\xi,x)}\right\|_{L^p((I,dt),L^q(B,dx))} \le CN^{\frac{r}{2}-\frac{2}{p}-\frac{r}{q}}\|a_\xi\|_{l^2(\mathbb{Z}^r)}$$

for all pairs $p,q\ge 2$ with $\frac{r}{2}-\frac{2}{p}-\frac{r}{q}>0$.

With the work [7] in mind, the resolution of this conjecture would require a decoupling theory for mixed-Lebesgue norms, which seems still missing in the literature. We will show that the above conjecture implies the following conjecture to hold for class functions on compact Lie groups, using again the key subdivision of the alcove and sharp L^p estimates of weight functions.

Conjecture 1.5. *Estimate* (1.11) *holds on any compact Lie group for any* $p > 2 + \frac{4}{d}$.

In other words, for rank $r \geq 2$, the estimate (1.11) would hold with an arbitrary ε-loss of derivatives, which is in contrast with the rank-one case of spheres of dimension $d \geq 3$, the latter enjoying only the optimal range of $p > 4$ (see [9]) which is more than a $(2 - 4/d)$-loss of derivatives.

1.3.3. *On Problem 3.* We first present another application of Theorem 1.2. We add the following L^p Laplace-Beltrami eigenfunction bound on compact Lie groups to the existing literature, matching the exponent as in (1.6) for tori. We will establish it using Theorem 1.2 and the circle method.

Theorem 1.6. *Let the assumptions be as in Theorem 1.3. Then we have the eigenfunction estimate*

$$\|f\|_{L^p(M)} \leq CN^{\frac{d-2}{2} - \frac{d}{p}} \|f\|_{L^2(M)} \tag{1.12}$$

for any $p > \frac{2sr}{sr-4s+4}$ *when* $r \geq 5$.

These results seem to be the first such unconditional L^p-bounds for $p < \infty$ for genuine higher-rank spaces beyond the case of products of rank-ones such as tori, and together they form the only known examples that improve upon Sogge's bound (1.5) by a polynomial factor of N. These bounds may first look surprising, since they are established on a manifold of nonnegative curvature, and they are better than those ε-improvement results on manifolds of negative curvature as in the previously mentioned work of Hassell and Tacy [15]. We observe that it is not because of the chaotic dynamical behavior of the geodesic flow but instead because of the high integrability of the Laplace-Beltrami operator as provided by a high rank.

If an ε-loss is allowed in the above estimate, we are then able to prove it for a larger range of p as follows, by fusing to full extent the tools developed in this paper and the argument as in [3].

Theorem 1.7. *Let the assumptions be as in Theorem 1.3. Then we have the eigenfunction estimate*

$$\|f\|_{L^p(M)} \leq C_\varepsilon N^{\frac{d-2}{2} - \frac{d}{p} + \varepsilon} \|f\|_{L^2(M)} \tag{1.13}$$

for any $p > \frac{2s(r+1)}{sr-3s+4}$ *when* $r \geq 4$.

Similar to the above discussion on Strichartz estimates, we will provide evidence for the following conjecture on the optimal range for spaces of rank $r \geq 2$, by establishing it for class functions assuming the optimal eigenfunction bounds on tori as conjectured by Bourgain.

Conjecture 1.8. *Let M be a compact Lie group of rank $r \geq 2$. Then* (1.13) *holds for any* $p > 2 + \frac{4}{d-2}$, *with an ε-loss if* $2 \leq r \leq 4$.

It seems reasonable to conjecture that all the above theorems obtained for compact Lie groups should extend to compact globally symmetric spaces. A similar analysis as for the characters of the behavior of spherical functions near walls and near different vertices of the alcove would be needed for these extensions, but it is harder as spherical functions are in general less explicit.

1.4. **Overview of paper.** We provide an overview of the remaining of the paper as follows. In Section 2, we review the fundamental structures of compact Lie groups and in particular the affine Weyl groups and the geometry of the Weyl alcove. In Section 3, we develop the key geometric tool of a so-called barycentric-semiclassical subdivision of the alcove, in order to distinguish points of different distance from the walls and near different vertices of the alcove, as eigenfunctions such as the characters behave differently on these different points. Associated to each component of this subdivision are some weight functions coming out of the Weyl denominator, and we obtain some preliminary estimates on them. In Section 4, we rework some of arguments in [31] to decompose the characters according to this barycentric-semiclassical subdivision. Section 5 forms the technical heart of this paper, where we obtain sharp L^p estimates of the weight functions on each component of the subdivision, and they will be used throughout later sections for various L^p estimates. In Section 6, we discuss orthogonal projections of the weight lattice with respect to parabolic root subsystems to further analyze the characters, and to prepare for the analysis of the Schrödinger kernel. In Section 7, we prove Theorem 1.1. In Section 8, to prepare the treatment of Strichartz estimates and Laplace-Beltrami eigenfunction bounds, we rework some of arguments in [31] to obtain formulas for the Schrödinger kernel on each component of the subdivision of alcove, in which the weight functions and Weyl type exponential sums appear. In Section 9, we prove Theorem 1.2. As an application, we prove Theorem 1.3 in Section 11, and Theorem 1.6 in Section 12, with the help of Farey dissection which we review in Section 10. In Section 13, we prove Theorem 1.7. Lastly, in Section 14, we conjecture the optimal range for both Strichartz estimates and Laplace-Beltrami eigenfunction bounds on compact Lie groups, and prove them specialized to class functions conditional on the conjectured optimal Strichartz estimates (for the mixed Lebesgue norm) and Laplace eigenfunction bounds on tori.

We list a few notations that are used throughout this paper. We use $a \lesssim b$ to mean $a \leq Cb$ for some positive constant C, $a \lesssim_\varepsilon b$ to mean $a \leq C(\varepsilon)b$ for some function $C(\varepsilon)$ of ε, and $a \asymp b$ to mean $|a| \lesssim |b| \lesssim |a|$. We use $:=$ to mean equality by definition and $=$ to mean equality not by definition. We also use $\sqcup$ to mean disjoint union.

2. Geometry of the Weyl alcove

We refer to [2, 30, 24, 17] for information on analysis on compact Lie groups and in particular affine Weyl groups and Weyl alcoves that we review in this section without proof. Let U be a compact simply connected simple Lie group with Lie algebra $\mathfrak{u}$. Let $\mathfrak{t}$ be a Cartan subalgebra, i.e. a maximal abelian subalgebra of $\mathfrak{u}$ and let T be the corresponding analytic subgroup which is a maximal torus of U. Let $\mathfrak{t}^*$ denote the real dual space of $\mathfrak{t}$ and let i denote the imaginary unit so that $i\mathfrak{t}^*$ is the space of linear forms on $\mathfrak{t}$ that take imaginary values. Let $\Sigma \subset i\mathfrak{t}^*$ be the root system of $(\mathfrak{u}, \mathfrak{t})$. Pick a positive system $\Sigma^+ \subset \Sigma$, and let $\{\alpha_1, \ldots, \alpha_r\} \subset \Sigma^+$ be the corresponding simple system. Let $-\alpha_0 \in \Sigma^+$ be the corresponding highest root and we call α_0 the lowest root. For $\alpha \in \Sigma^+$ and $n \in \mathbb{Z}$, define the root hyperplanes

$$\mathfrak{p}_{\alpha,n} := \{H \in \mathfrak{t} : \ \alpha(H)/2\pi i + n = 0\}.$$

These hyperplanes cut the ambient space $\mathfrak{t}$ into alcoves. Let

$$A := \{H \in \mathfrak{t} : \ \alpha_j(H)/2\pi i + \delta_{0j} > 0 \ \forall j = 0, \ldots, r\}$$

be the open fundamental alcove and

$$\bar{A} := \{H \in \mathfrak{t} : \ \alpha_j(H)/2\pi i + \delta_{0j} \geq 0 \ \forall j = 0, \ldots, r\}$$

be the closed fundamental alcove. Here δ_{0j} is the Kronecker delta. Let W denote the finite Weyl group that acts on $\mathfrak{t}$ as well as $\mathfrak{t}^*$. The Weyl group translates sA ($s \in W$) of A are disjointly embedded in T and form the so-called regular elements of T, such that $T \setminus \bigsqcup_{s\in W} sA$ contains the non-regular elements in T and is of zero measure in T. These non-regular elements in T are also the conjugate points in T of the origin on U as a Riemannian manifold. We recall Weyl's integration formula, which is the basic tool to be used to evaluate the L^p norm of class functions.

Lemma 2.1. *For class functions f on U, Weyl's integration formula can be written as*

$$\int_U f(u)\ du = \int_A f(\exp H)|\delta(H)|^2\ dH \tag{2.1}$$

where $\delta(H)$ is the so-called Weyl denominator as follows

$$\delta(H) := \prod_{\alpha\in\Sigma^+} \left(e^{\frac{\alpha(H)}{2}} - e^{-\frac{\alpha(H)}{2}}\right), \text{ for } H \in \mathfrak{t}. \tag{2.2}$$

Let

$$\rho := \frac{1}{2}\sum_{\alpha\in\Sigma^+} \alpha$$

be the Weyl vector, then we may also express the above $\delta(H)$ as

$$\delta(H) = \sum_{s\in W} \det s\ e^{(s\rho)(H)}, \text{ for } H \in \mathfrak{t}.$$

The fundamental alcove (as well as any alcove) is a simplex whose geometry may be described using the extended Dynkin diagram for Σ. Each α_j ($j = 0, \dots, r$) corresponds to a node in the extended Dynkin diagram (Figure 1), and for each proper subset J of $\{0, \dots, r\}$, $\{\alpha_j,\ j \in J\}$ is a simple system for a root subsystem Σ_J whose Dynkin diagram can be obtained from the extended Dynkin diagram of Σ by removing all the nodes not belonging to J. These Σ_J's are usually called the parabolic subsystems of Σ. For $j = 0, \dots, r$, let $\tilde{s}_j : \mathfrak{t} \to \mathfrak{t}$ denote the reflection across the hyperplane

$$\mathfrak{p}_j := \mathfrak{p}_{\alpha_j,\delta_{0j}} = \{H \in \mathfrak{t} :\ \alpha_j(H)/2\pi i + \delta_{0j} = 0\}.$$

These hyperplanes form the walls of the alcove $\bar{A}$ as its non-regular elements. For each $J \subset \{0, \dots, r\}$, let $\tilde{W}_J$ be the finite group generated by the reflections $\{\tilde{s}_j,\ j \in J\}$. $\tilde{W} := \tilde{W}_{\{0,\dots,r\}}$ is called the affine Weyl group associated to Σ and the $\tilde{W}_J$'s may be called the parabolic subgroups of $\tilde{W}$. The facets of $\bar{A}$ correspond to proper subsets of $\{0, \dots, r\}$: for $J \subsetneqq \{0, \dots, r\}$,

$$A_J : = \{H \in \bar{A} :\ \alpha_j(H)/2\pi i + \delta_{0j} = 0\ \forall j \in J,\ \alpha_j(H)/2\pi i + \delta_{0j} > 0\ \forall j \notin J\}$$

is the corresponding $(r - |J|)$-dimensional facet. In particular, the $r + 1$ vertices of $\bar{A}$ are of the form A_I where I ranges through cardinality-r subsets of $\{0, \dots, r\}$, and $A_\varnothing = A$. We have

$$\bar{A} = \bigsqcup_{J\subset\{0,\dots,r\}} A_J.$$

The stabilizer in $\tilde{W}$ of any point of A_J coincides with $\tilde{W}_J$. Let W_J denote the Weyl group associated to the parabolic subsystem Σ_J. Then $\tilde{W}_J$ is isomorphic to W_J under the translation map $\tilde{s} \mapsto \tilde{s} - \tilde{s}(0)$. For $J \subsetneqq \{0, \dots, r\}$, the root hyperplanes that cross the facet A_J are

$$\left\{\mathfrak{p}_{\alpha,0} :\ \alpha \in \Sigma^+_{J\setminus\{0\}}\right\} \bigcup \left\{\mathfrak{p}_{\alpha,1} :\ \alpha \in \Sigma^+_J \setminus \Sigma^+_{J\setminus\{0\}}\right\}.$$

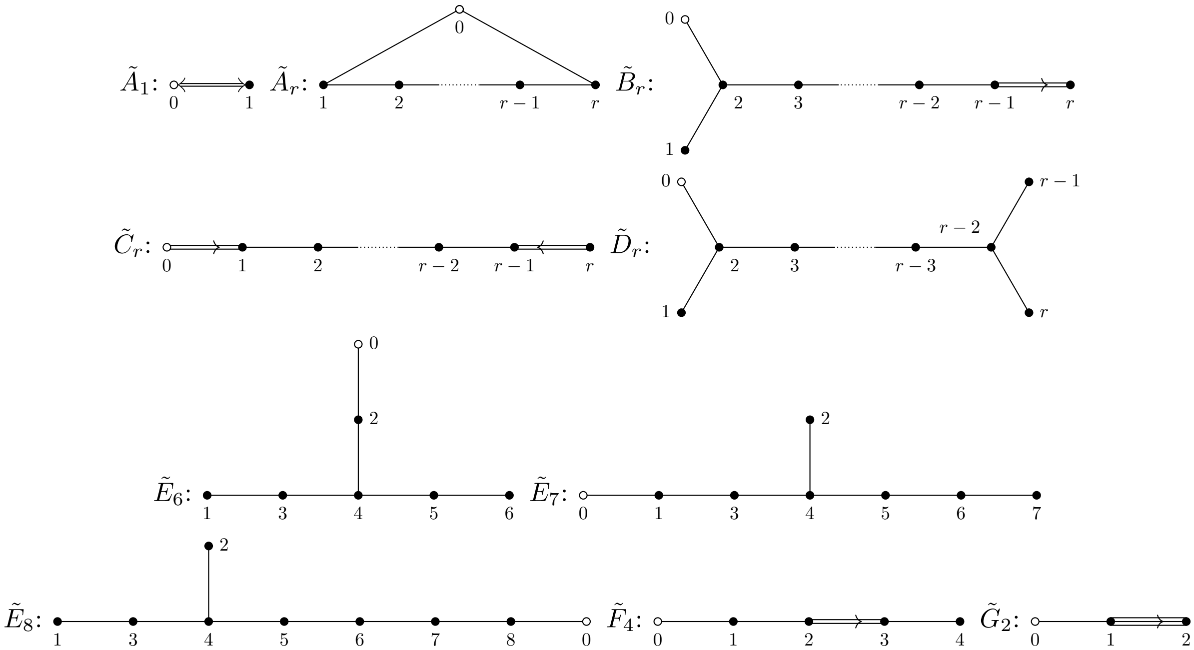


Figure 1. Extended Dynkin diagrams

3. Barycentric-semiclassical subdivision

From the semiclassical perspective, Laplace-Beltrami eigenfunctions such as characters tend to be concentrated near conjugate points of the origin. In the alcove, these conjugate points are the walls of $\bar{A}$, thus in order to get L^p estimates of these eigenfunctions especially the characters, their behavior near each facet of $\bar{A}$ needs to be clarified. We achieve this by making a so-called semiclassical subdivision of the alcove according to how close the points are from each facet. Let $N \gtrsim 1$ be a fixed large parameter. Let $J \subsetneq \{0, \ldots, r\}$ and let A_J be the corresponding facet. We define a subset P_J of A that consists of points close to A_J but away from all the other facets. Let

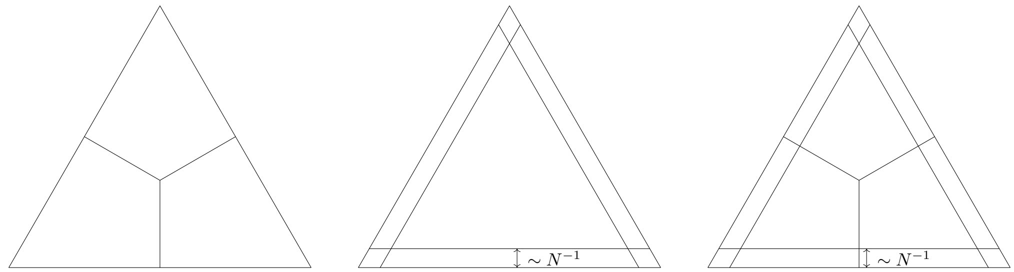
$$P_J := \{H \in A: \ \alpha_j(H)/2\pi i + \delta_{0j} \le N^{-1} \ \forall j \in J, \ \alpha_j(H)/2\pi i + \delta_{0j} > N^{-1} \ \forall j \notin J\}.$$

In other words, P_J consists of points in the alcove that are $\le N^{-1}$ close to the walls $\mathfrak{p}_j$ for $j \in J$ and are $> N^{-1}$ far from the other walls $\mathfrak{p}_j$ for $j \notin J$. We record the following self-evident fact as a lemma.

Lemma 3.1 (Semiclassical subdivision)**.** *We have*

$$A = \bigsqcup_{J \subsetneq \{0,\ldots,r\}} P_J.$$

We need yet another subdivision of the alcove in order to evaluate L^p norms of eigenfunctions. This one is technical in nature and concerns the classification of root systems. Its necessity will be transparent in its applications in later sections; the idea is that eigenfunctions such as the characters also behave differently near different vertices of the alcove, which motivates the following version of barycentric subdivision of the alcove. For each vertex A_I ($|I| = r$) of $\bar{A}$, consider the convex hull C_I of the barycenters of the facets A_J such that $J \subset I$, in other words, facets that contain A_I in their boundary.



(a) (b) (c)

FIGURE 2. (a) Barycentric subdivision (b) Semiclassical subdivision (c) Barycentric-semiclassical subdivision

Lemma 3.2 (Barycentric subdivision). *We have*

$$A = \bigsqcup_{|I|=r} C_I.$$

The above disjoint union is understood modulo a lower-dimensional set. For $J \subsetneq \{0,\ldots,r\}$ and $I \subset \{0,\ldots,r\}$ such that $|I| = r$, set

$$\begin{aligned} P_{I,J} :&= P_J \cap C_I \\ &= \{H \in C_I :\ \alpha_j(H)/2\pi i + \delta_{0j} \le N^{-1}\ \forall j \in J,\ \alpha_j(H)/2\pi i + \delta_{0j} > N^{-1}\ \forall j \in I \setminus J\}. \end{aligned}$$

In other words, $P_{I,J}$ consists of points in C_I that are $\le N^{-1}$ close to the hyperplanes $\mathfrak{p}_j$ for $j \in J$ and $> N^{-1}$ far from the hyperplanes $\mathfrak{p}_j$ for $j \in I \setminus J$. Note that $P_{I,J} = \varnothing$ if J is not a subset of I, if we pick N^{-1} small enough. For $J \subset I$, we now have

$$C_I = \bigsqcup_{J \subset I} P_{I,J},\ \ P_J = \bigsqcup_{I \supset J} P_{I,J},$$

and:

Lemma 3.3 (Barycentric-semiclassical subdivision). *We have*

$$A = \bigsqcup_{J,I \subset \{0,\ldots,r\},\ |I|=r,\ J \subset I} P_{I,J}.$$

For $j \in \{0,\ldots,r\}$, let

$$t_j(H) := \alpha_j(H)/2\pi i + \delta_{0j}.$$

Then for each $I \subset \{0,\ldots,r\}$ such that $|I| = r$, $\{t_j,\ j \in I\}$ provide a natural coordinate system for C_I, and there exists uniform positive constants

$$c_1, c_2 < 1,$$

such that

$$\{H \in \mathfrak{t} :\ 0 \le t_j(H) \le c_1\ \forall j \in I\} \subset C_I \subset \{H \in \mathfrak{t} :\ 0 \le t_j(H) \le c_2\ \forall j \in I\}.$$

Also for each $J \subset I$, we have

$$\begin{aligned} &\{H \in \mathfrak{t} :\ 0 \le t_j(H) \le N^{-1}\ \forall j \in J,\ N^{-1} < t_j(H) \le c_1\ \forall j \in I \setminus J\} \\ &\subset P_{I,J} \subset \{H \in \mathfrak{t} :\ 0 \le t_j(H) \le N^{-1}\ \forall j \in J,\ N^{-1} < t_j(H) \le c_2\ \forall j \in I \setminus J\}. \end{aligned} \tag{3.1}$$

Associated to the above barycentric-semiclassical subdivision are some naturally defined weight functions which appear as factors of the function $\delta(H)$ as in (2.2).

Definition 3.4 (Weight functions)**.** *Set*

$$\delta(H) = \delta_I(H) \cdot \delta_{I,J}(H) \cdot \delta^J(H),$$

where

$$\delta_I(H) := \prod_{\alpha \in \Sigma^+ \setminus \Sigma_I^+} \left(e^{\frac{\alpha(H)}{2}} - e^{-\frac{\alpha(H)}{2}} \right),$$
$$\delta^J(H) := \prod_{\alpha \in \Sigma_J^+} \left(e^{\frac{\alpha(H)}{2}} - e^{-\frac{\alpha(H)}{2}} \right),$$
$$\delta_{I,J}(H) := \prod_{\alpha \in \Sigma_I^+ \setminus \Sigma_J^+} \left(e^{\frac{\alpha(H)}{2}} - e^{-\frac{\alpha(H)}{2}} \right).$$

We derive some preliminary estimates for these weight functions.

Lemma 3.5. *We have*

$$|\delta_I(H)| \gtrsim 1, \ \text{ for } H \in C_I, \tag{3.2}$$

$$|\delta^J(H)| \lesssim N^{-|\Sigma_J^+|}, \ \text{ for } H \in P_J, \tag{3.3}$$

$$|\delta_{I,J}(H)| \gtrsim N^{|\Sigma_J^+| - |\Sigma^+|}, \ \text{ for } H \in P_{I,J}. \tag{3.4}$$

Proof. First, for $\alpha \in \Sigma^+$, $\left| e^{\frac{\alpha(H)}{2}} - e^{-\frac{\alpha(H)}{2}} \right| = \|\alpha(H)/2\pi i\|$ is comparable to the shortest distance from the root hyperplanes $\mathfrak{p}_{\alpha,n}$ among $n \in \mathbb{Z}$. Here $\| \cdot \|$ denotes the distance from the nearest integer. By definition, $C_I = \bigcup_{J \subset I} P_{I,J} \subset \bigcup_{J \subset I} A_J$. As reviewed in Section 2, the only hyperplanes that cross the facets $\bigcup_{J \subset I} A_J$ are of the form $\mathfrak{p}_{\alpha,n}$ for $\alpha \in \Sigma_I^+$. Thus C_I as a compact set stays away from all root hyperplanes $\mathfrak{p}_{\alpha,n}$ for $\alpha \in \Sigma^+ \setminus \Sigma_I^+$ by a fixed distance, which yields the first estimate. The second estimate follows since each $\alpha \in \Sigma_J^+$ is a linear combination of simple roots α_j, $j \in J$, and then the definition of P_J assures that $\|\alpha(H)/2\pi i\| \lesssim N^{-1}$ for $H \in P_J$. For the last estimate, since $|\Sigma_I^+| \leq |\Sigma^+|$, it suffices to show

$$|\delta_{I,J}(H)| \gtrsim N^{|\Sigma_J^+| - |\Sigma_I^+|}, \ \text{ for } H \in P_{I,J}.$$

This follows if $P_{I,J}$ stays away from the root hyperplanes $\mathfrak{p}_{\alpha,n}$ for $\alpha \in \Sigma_I^+ \setminus \Sigma_J^+$ by a distance of at least $\sim N^{-1}$. It suffices to check this in a neighborhood $\mathcal{N}_I$ of the vertex A_I, since away from $\mathcal{N}_I$, $P_{I,J}$ is only close to hyperplanes of the form $\mathfrak{p}_j$ for $j \in J$. But in such a neighborhood $\mathcal{N}_I$ of A_I, any root α in $\Sigma_I^+ \setminus \Sigma_J^+$ is a linear combination of α_j, $j \in I$ with nonnegative integral coefficients where at least one of α_j, $j \in I \setminus J$ has positive coefficient. By the definition of $P_{I,J}$, this implies that for any $H \in \mathcal{N}_I \cap P_{I,J}$, $\|\alpha(H)/2\pi i\| \gtrsim N^{-1}$ for any $\alpha \in \Sigma_I^+ \setminus \Sigma_J^+$, which concludes the proof.

□

We have the following immediate corollary.

Lemma 3.6. *We have*

$$|\delta(H)| \gtrsim N^{-|\Sigma^+|}, \ \text{ for } H \in P_\varnothing. \tag{3.5}$$

Proof. Since $P_\varnothing = \bigcup_{I \subset \{0,\ldots,r\},\ |I|=r} P_{I,\varnothing}$, it suffices to prove the estimate for each $P_{I,\varnothing}$. Write $\delta = \delta_I \cdot \delta_{I,\varnothing}$, then the result follows by (3.2) and (3.4). □

4. Characters

In this section, adapted to the above barycentric-semiclassical subdivision, we give a formula of the character that illuminate its behavior on each component of this subdivision. Let $(\cdot,\cdot)$ denote the Killing form on $\mathfrak{t}$ as well as on $\mathfrak{t}^*$ (and also on $i\mathfrak{t}^*$ by linear extension) and $|\cdot|$ be the corresponding norm, for which the Weyl group W acts on $\mathfrak{t}$ as well as on $\mathfrak{t}^*$ by isometry. The weight lattice reads

$$\Lambda := \left\{\mu \in i\mathfrak{t}^* : \ \frac{2(\mu,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}\ \forall \alpha \in \Sigma\right\},$$

and let

$$\Lambda^+ := \left\{\mu \in i\mathfrak{t}^* : \ \frac{2(\mu,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}_{\geq 1}\ \forall \alpha \in \Sigma^+\right\}$$

be the subset of strictly dominant weights. We have chosen here the strictly dominant weights instead of the more standard larger set of dominant weights to slightly improve simplicity of the following presentation. Concerning the relation between Λ and Λ^+, we have the following standard lemma of root system theory. We say $\mu \in i\mathfrak{t}^*$ is regular provided $(\mu,\alpha) \neq 0$ for all $\alpha \in \Sigma$, and non-regular otherwise.

Lemma 4.1. *The regular elements of Λ form exactly the subset $\bigsqcup_{s\in W} s\Lambda^+$ so that we have*

$$\Lambda = \left(\bigsqcup_{s\in W} s\Lambda^+\right) \bigsqcup \{\mu \in \Lambda : \ (\mu,\alpha) = 0 \text{ for some } \alpha \in \Sigma\}.$$

Each $\mu \in \Lambda^+$ is associated with an irreducible representation of U of highest weight $\mu - \rho$, and the associated character χ_μ can be expressed by the following Weyl's formula

$$\chi_\mu(H) := \frac{\sum_{s\in W} \det s \ e^{(s\mu)(H)}}{\sum_{s\in W} \det s \ e^{(s\rho)(H)}} = \frac{\sum_{s\in W} \det s \ e^{(s\mu)(H)}}{\delta(H)}, \text{ for } H \in \mathfrak{t}.$$

Note that this formula make sense for any $\mu \in i\mathfrak{t}^*$ and in particular for any $\mu \in \Lambda$, though the characters are initially defined only for $\mu \in \Lambda^+$.

We now study the behavior of χ_μ near each facet of $\bar{A}$. For $J \subsetneqq \{0,\dots,r\}$, recall that A_J denotes the corresponding $(r-|J|)$-dimensional facet of the fundamental alcove A. Consider the subspace

$$\mathfrak{t}_J := \bigoplus_{j\in J} \mathbb{R} H_{\alpha_j} \tag{4.1}$$

of $\mathfrak{t}$, where $H_{\alpha_j} \in \mathfrak{t}$ is defined such that $(H_{\alpha_j}, H) := \alpha_j(H)/2\pi i$ for all $H \in \mathfrak{t}$. Let $H_J := \mathrm{Proj}_{\mathfrak{t}_J}(H)$ denote the orthogonal projection of $H \in \mathfrak{t}$ on $\mathfrak{t}_J$ with respect to the Killing form. Let

$$H_J^\perp := H - H_J,$$

which lies in the orthogonal complement

$$\mathfrak{t}_J^\perp := \mathfrak{t} \ominus \mathfrak{t}_J$$

of $\mathfrak{t}_J$ in $\mathfrak{t}$. Dual to $\mathfrak{t}_J$, we also consider the root subspace

$$V_J := \mathrm{span}_{\mathbb{R}} \Sigma_J = \bigoplus_{j\in J} \mathbb{R}\alpha_j$$

of $i\mathfrak{t}^*$ spanned by the parabolic subsystem Σ_J. Let $\mu_J := \mathrm{Proj}_{V_J}(\mu)$ denote the orthogonal projection of $\mu \in \Lambda$ on V_J. Let $\Sigma_J^+ := \Sigma^+ \cap \Sigma_J$ be the positive system for Σ_J and let Λ_J be the weight lattice for Σ_J. For $\gamma \in \Lambda_J$, let

$$\chi_\gamma^J := \frac{\sum_{s_J\in W_J} \det s_J \ e^{s_J\gamma}}{\sum_{s_J\in W_J} \det s_J \ e^{s_J\rho_J}} = \frac{\sum_{s_J\in W_J} \det s_J \ e^{s_J\gamma}}{\delta^J(H)}$$

be the associated character where $\rho_J := \frac{1}{2}\sum_{\alpha\in\Sigma_J^+}\alpha$ is the Weyl vector associated to Σ_J. Note that the above expression makes sense for any $\gamma\in V_J$. For each $j=0,\dots,r$, let

$$\mathfrak{t}_j := \mathbb{R}H_{\alpha_j},\ \mathfrak{t}_j^\perp := \mathfrak{t}\ominus\mathfrak{t}_j = \mathfrak{p}_{\alpha_j,0}.$$

Then the Weyl group W_J is generated by reflections across the hyperplanes $\mathfrak{t}_j^\perp$, $j\in J$. In particular, as $\mathfrak{t}_J^\perp = \bigcap_{j\in J}\mathfrak{t}_j^\perp$, any $s_J\in W_J$ fixes every point on $\mathfrak{t}_J^\perp$. We now derive the following key formula of characters.

Lemma 4.2. *For any $H\in\mathfrak{t}$ and $\mu\in i\mathfrak{t}^*$, we have*

$$\chi_\mu(H) = \frac{1}{|W_J|\delta_I(H)\delta_{I,J}(H)}\sum_{s\in W}\det s\ e^{(s\mu)(H_J^\perp)}\chi^J_{(s\mu)_J}(H_J). \tag{4.2}$$

Proof. We have for $H\in\mathfrak{t}$

$$\begin{aligned}\chi_\mu(H) &= \frac{\sum_{s\in W}\det s\ e^{(s\mu)(H)}}{\delta(H)}\\ &= \frac{\sum_{s\in W}\sum_{s_J\in W_J}\det(s_Js)\ e^{(s_Js\mu)(H)}}{|W_J|\prod_{\alpha\in\Sigma^+\setminus\Sigma_J^+}\left(e^{\frac{\alpha(H)}{2}}-e^{-\frac{\alpha(H)}{2}}\right)\prod_{\alpha\in\Sigma_J^+}\left(e^{\frac{\alpha(H)}{2}}-e^{-\frac{\alpha(H)}{2}}\right)}.\end{aligned}$$

Now write $H = H_J + H_J^\perp$, we have for $s\in W$ and $s_J\in W_J$ that

$$(s_Js\mu)(H) = (s_Js\mu)(H_J) + (s_Js\mu)(H_J^\perp).$$

But

$$(s_Js\mu)(H_J) = (s_J(s\mu)_J)(H_J)$$

since $(s\mu-(s\mu)_J)(H_J)=0$ by definition, while

$$(s_Js\mu)(H_J^\perp) = (s\mu)(s_J^{-1}H_J^\perp) = (s\mu)(H_J^\perp)$$

since s_J^{-1} as an element of W_J fixes any point on $\mathfrak{t}_J^\perp$. Also for $\alpha\in\Sigma_J$, $\alpha(H_J^\perp)=0$ thus $\alpha(H)=\alpha(H_J)+\alpha(H_J^\perp)=\alpha(H_J)$. Now we derive

$$\chi_\lambda = \frac{\sum_{s\in W}\det s\ e^{(s\mu)(H_J^\perp)}\sum_{s_J\in W_J}\det s_J e^{(s_J(s\mu)_J)(H_J)}}{|W_J|\prod_{\alpha\in\Sigma^+\setminus\Sigma_J^+}\left(e^{\frac{\alpha(H)}{2}}-e^{-\frac{\alpha(H)}{2}}\right)\prod_{\alpha\in\Sigma_J^+}\left(e^{\frac{\alpha(H_J)}{2}}-e^{-\frac{\alpha(H_J)}{2}}\right)}$$

which is (4.2), recalling Definition 3.4 of the weight functions. ☐

Note in particular that the above formula (4.2) for characters holds for any $\mu\in i\mathfrak{t}^*$ which is not necessarily regular. Thus it takes care of both non-regular physical parameters (i.e. near walls) and non-regular spectral parameters in $i\mathfrak{t}^*$, which is an important point in analysis on high-rank spaces.

Before we leave this section, we list a few basic properties about Fourier analysis on a compact Lie group. In our previous discussion, U is assumed a compact simple Lie group, but now we assume U can also be a product of a compact semisimple Lie group and a torus $\mathbb{T}^n$. In the latter case, the space of spectral parameters is of the form $\Lambda^+\times\mathbb{Z}^n$ where Λ^+ is the set of (strictly) dominant weights associated to the semisimple component. Let χ_μ and d_μ denote the character and dimension of representation for the spectral parameter μ respectively. Then we have the following version of Fourier series. For any $f\in L^2(U)$,

$$f = \sum_\mu f*(d_\mu\chi_\mu),$$

and we have the Parseval's identity

$$\|f\|_{L^2(U)}^2 = \sum_{\mu} \|f * (d_\mu \chi_\mu)\|_{L^2(U)}^2 .$$

The map $f \mapsto f * (d_\mu \chi_\mu)$ is also the projection onto the space of invariant differential operators of spectral parameter μ. For any L^2 class function κ, we may define its Fourier transform $\hat{\kappa}(\mu)$ such that

$$\kappa = \sum_{\mu} \hat{\kappa}(\mu) d_\mu \chi_\mu .$$

As a consequence, we have

$$f * \kappa = \sum_{\mu} \hat{\kappa}(\mu) f * (d_\mu \chi_\mu),$$

and that

$$\|f * \kappa\|_{L^2(U)} \leq \sup_{\mu} |\hat{\kappa}(\mu)| \cdot \|f\|_{L^2(U)} .$$

5. L^p norm of $1/\delta_{I,J}$

This section forms the technical heart of the paper. Let I be a subset of $\{0, \ldots, r\}$ with $|I| = r$ and J be a subset of I. Recall definition of the weight function

$$\delta_{I,J}(H) := \prod_{\alpha \in \Sigma_I^+ \setminus \Sigma_J^+} \left(e^{\frac{\alpha(H)}{2}} - e^{-\frac{\alpha(H)}{2}} \right).$$

As is clear from Lemma 4.2, it is crucial to estimate $\delta_{I,J}$ in order to estimate characters. We obtain sharp L^p-estimates for $1/\delta_{I,J}$ over the polytope $P_{I,J}$ in this section. We start from the following key lemma in root system theory.

Lemma 5.1. *Let Σ be an irreducible root system and let $\{\alpha_j,\ j = 1, \ldots, r\}$ be a simple system. Then there exists positive integers $p_r > p_{r-1} > \cdots > p_1 = 1$ with $p_1 + \cdots + p_r = |\Sigma^+|$ such that the following is true. Assume $t_j(H) := \alpha_j(H)/2\pi i > 0$ for each $j = 1, \ldots, r$ (which defines the Weyl chamber for Σ). Suppose $\{s_j(H),\ j = 1, \ldots, r\}$ is the permutation of $\{t_j(H),\ j = 1, \ldots, r\}$ such that $s_1(H) \leq s_2(H) \leq \cdots \leq s_r(H)$. Then we have*

$$\prod_{\alpha \in \Sigma^+} \alpha(H)/2\pi i \gtrsim s_r^{p_r}(H) s_{r-1}^{p_{r-1}}(H) \cdots s_1^{p_1}(H). \tag{5.1}$$

Proof. $\prod_{\alpha \in \Sigma^+} \alpha(H)/2\pi i$ may be expressed as a homogeneous polynomial in $\{t_j(H),\ j = 1, \ldots, r\}$ of degree $|\Sigma^+|$ with nonnegative integral coefficients. The result follows if for each permutation $\{s_j(H),\ j = 1, \ldots, r\}$ of $\{t_j(H),\ j = 1, \ldots, r\}$, $s_r^{p_r}(H) s_{r-1}^{p_{r-1}}(H) \cdots s_1^{p_1}(H)$ occurs with positive coefficient in the polynomial $\prod_{\alpha \in \Sigma^+} \alpha(H)/2\pi i$. This fact can be checked case by case using classification of irreducible root systems to provide explicit r-tuples $(p_1, p_2, \ldots, p_r)$. We are happy to have found for it the appendix of [27] as a precise reference. $\square$

We provide a few corollaries.

Corollary 5.2. *Inherit the assumptions in the above lemma. Consider the subsystem Σ_J of Σ for some $J \subset \{1, \ldots, r\}$. We assume furthermore*

$$0 < t_{j_1}(H) < t_{j_2}(H) < \cdots < t_{j_{|J|}}(H) \leq N^{-1}$$

for some fixed permutation $(j_1, \ldots, j_{|J|})$ of J, while

$$t_{j_r}(H) > t_{j_{r-1}}(H) > \cdots > t_{j_{|J|+1}}(H) > N^{-1}$$

for some fixed permutation $(j_{|J|+1}, \dots, j_r)$ *of* $\{1, \dots, r\} \setminus J$*. Then we have*

$$\prod_{\alpha \in \Sigma^+ \setminus \Sigma_J^+} \alpha(H)/2\pi i \asymp s_r^{q_r}(H) s_{r-1}^{q_{r-1}}(H) \cdots s_{|J|+1}^{q_{|J|+1}}(H) \tag{5.2}$$

for some nonnegative integral integers $q_r, q_{r-1}, \dots, q_{|J|+1}$ *with*

$$q_r + q_{r-1} + \cdots + q_{|J|+1} = |\Sigma^+| - |\Sigma_J^+|,$$

such that

$$q_r + q_{r-1} + \cdots + q_{j+1} \geq \frac{|\Sigma^+| \cdot (r-j)}{r}, \text{ for all } j = r-1, r-2, \dots, |J|, \tag{5.3}$$

where equality holds if and only if $j = 0 = |J|$*.*

Proof. For each $\alpha \in \Sigma^+$, $\alpha(H)/2\pi i$ is a linear function of the $t_j(H)$'s $(j = 1, \dots, r)$ with nonnegative integral coefficients. As α goes through Σ^+, let n_j be the number of such linear functions in which at least one of the variables $s_r(H), s_{r-1}(H), \dots, s_j(H)$ $(j = 1, \dots, r)$ occurs with positive coefficient. Then (5.1) may be restated as $n_j \geq p_r + p_{r-1} + \cdots + p_j$ for any $j = 1, \dots, r$. Since $p_j > p_{j-1}$ $(j = 2, \dots, r)$ and $\sum_{j=1}^r p_j = |\Sigma^+|$, we have $n_j > \frac{|\Sigma^+| \cdot (r-j+1)}{r}$ for any $j = 2, \dots, r$. Let $q_r = n_r$ and $q_j = n_j - n_{j+1}$ for $j = |J|+1, \dots, r-1$, then they satisfy (5.3). The reason (5.2) holds is that the assumptions on the $t_j(H)$'s imply that $\{s_r(H), s_{r-1}(H), \dots, s_{|J|+1}(H)\}$ is the same set as $\{t_j(H),\ j \in \{1, \dots, r\} \setminus J\}$, and thus in each linear function $\alpha(H)/2\pi i$ for $\alpha \in \Sigma^+ \setminus \Sigma_J^+$ occurs at least one of the variables $s_r(H), s_{r-1}(H), \dots, s_{|J|+1}(H)$ with positive coefficient. □

As a consequence, we have:

Corollary 5.3. *For any nonempty subset* J *of* $\{0, \dots, r\}$ *such that* $|J| \leq r-1$*, we have*

$$\frac{|J|}{|\Sigma_J^+|} > \frac{r}{|\Sigma^+|}.$$

Proof. We divide the proof into two cases.

Case 1. $0 \notin J$. Let $(j_1, j_2, \dots, j_r)$ be any permutation of $\{1, \dots, r\}$ such that $(j_1, j_2, \dots, j_{|J|})$ is a permutation of $\{1, \dots, |J|\}$. As α goes through Σ^+ let n_k be the number of linear functions $\alpha(H)/2\pi i$ in which occurs at least one of the variables $t_{j_r}(H), t_{j_{r-1}}(H), \dots, t_{j_k}(H)$ $(k = 1, \dots, r)$. Then $|\Sigma_J^+| = |\Sigma^+| - n_{|J|+1}$. We may pretend that the assumptions in the above corollary holds. Then as argued in the above proof, we have $n_{|J|+1} > \frac{|\Sigma^+| \cdot (r-|J|)}{r}$, which gives the desired result.

Case 2. $0 \in J$. We first check the cases when $|J| = r-1$. This amounts to removing two nodes in the the extended Dynkin diagram (Figure 1). A key observation is that by removing any node in the diagram, each of the resulting connected subgraphs (the total number of which could be one, two, or three) is a subgraph of a connected subgraph of r vertices, and in the language of root systems, this means that each of the resulting irreducible root subsystems (denoted Σ_i, $i \in \{1\}$ or $\{1,2\}$ or $\{1,2,3\}$) is a subsystem of some irreducible Σ_I with $(|I| = r)$. Denote the rank of Σ_i by r_i. By the previous discussion, this yields

$$\frac{r_i}{|\Sigma_i^+|} > \frac{r}{|\Sigma_I^+|} \geq \frac{r}{|\Sigma^+|}.$$

To get J, we remove another node say in Σ_1 to get a subsystem $\Sigma_{1,1}$ of Σ_1. Then

$$\Sigma_J = \left(\bigsqcup_{i \neq 1} \Sigma_i \right) \bigsqcup \Sigma_{1,1}.$$

Denote the rank of $\Sigma_{1,1}$ by $r_{1,1}$. Since Σ_1 is irreducible, the previous discussion tells that

$$\frac{r_{1,1}}{|\Sigma_{1,1}^+|} > \frac{r_1}{|\Sigma_1^+|} > \frac{r}{|\Sigma^+|}.$$

Now

$$\frac{|J|}{|\Sigma_J^+|} = \frac{(\sum_{i\neq 1} r_i) + r_{1,1}}{\sum_{i\neq 1} |\Sigma_i^+| + |\Sigma_{1,1}^+|},$$

which as an averaging of the above fractions is also bigger than $\frac{r}{|\Sigma^+|}$. For a general J, we can see inductively that every Σ_J is a disjoint union of irreducible root subsystems each of which satisfies the desired inequality, and then by averaging of the fractions this shows that Σ_J satisfies the desired inequality also. □

We are ready to prove the following main result of this section.

Proposition 5.4. *For $I \subset \{0, \dots, r\}$, $|I| = r$, $J \subset I$, we have*

$$\left\| \frac{1}{\delta_{I,J}} \right\|_{L^p(P_{I,J})} \begin{cases} \lesssim N^{|\Sigma^+| - |\Sigma_J^+| - \frac{r}{p}}, & \text{for } p > \frac{r}{|\Sigma^+|}, \\ \lesssim_\varepsilon N^{-|\Sigma_J^+| + \varepsilon}, & \text{for } p = \frac{r}{|\Sigma^+|}, \\ \lesssim N^{-|\Sigma_J^+|}, & \text{for } p < \frac{r}{|\Sigma^+|}. \end{cases}$$

Proof. First we claim that it suffices to obtain the above bound replacing $P_{I,J}$ by $P_{I,J} \cap \mathcal{N}_I$ where $\mathcal{N}_I$ is a small neighborhood of the vertex A_I. In fact, the set $P_{I,J} \setminus \mathcal{N}_I$ stays away from all root hyperplanes $\mathfrak{p}_{\alpha,n}$ for $\alpha \in \Sigma_I^+ \setminus \Sigma_J^+$ and $n \in \mathbb{Z}$ by a uniform nonzero distance, and thus

$$|\delta_{I,J}(H)| \gtrsim 1, \text{ for } H \in P_{I,J} \setminus \mathcal{N}_I.$$

This gives

$$\left\| \frac{1}{\delta_{I,J}} \right\|_{L^p(P_{I,J} \setminus \mathcal{N}_I)} \lesssim \|1\|_{L^p(P_{I,J} \setminus \mathcal{N}_I)} \lesssim N^{-\frac{|J|}{p}}.$$

By Corollary 5.3, we have

$$N^{-\frac{|J|}{p}} < N^{-\frac{r|\Sigma_J^+|}{p|\Sigma^+|}},$$

which is bounded by the above desired bound by an inspection of the exponents of N as functions of $\frac{1}{p}$. We refer to Figure 3 for this and later comparisons of exponents. For $P_{I,J} \cap \mathcal{N}_I$, we now divide into two cases $0 \notin I$ and $0 \in I$.

Case 1: $I = \{1, \dots, r\}$. For this case, $\Sigma_I = \Sigma$ which is the irreducible root system we started with. Recall that $\{t_j, \ j = 1, \dots, r\}$ provide a coordinate system for $P_{I,J}$ on which $0 \le t_j \le N^{-1}$ for any $j \in J$ and $1 > c > t_j > N^{-1}$ for any $j \in \{1, \dots, r\} \setminus J$; see (3.1). For $H \in P_{I,J} \cap \mathcal{N}_I$, we have

$$|\delta_{I,J}(H)| \asymp \prod_{\alpha \in \Sigma^+ \setminus \Sigma_J^+} \alpha(H)/2\pi i \asymp s_r^{q_r}(H) s_{r-1}^{q_{r-1}}(H) \cdots s_{|J|+1}^{q_{|J|+1}}(H),$$

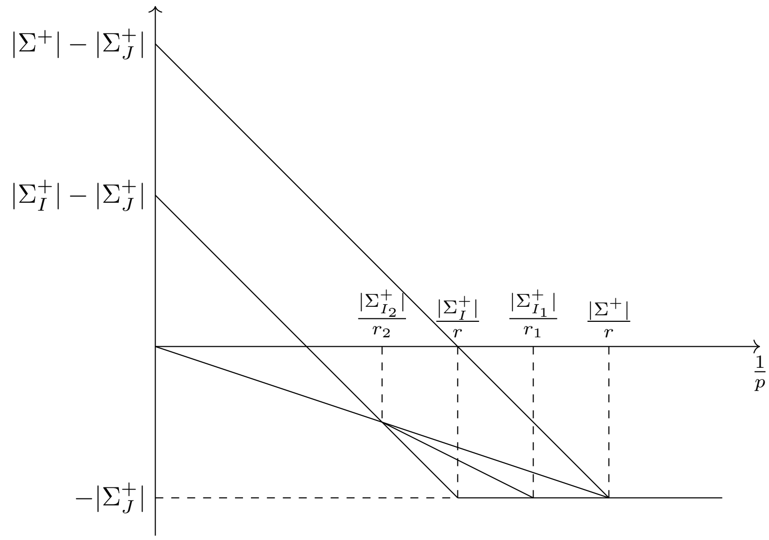


Figure 3. Graph of exponents

using (5.2). We estimate

$$\begin{aligned}
\int_{P_{I,J}\cap\mathcal{N}_I}\left|\frac{1}{\delta_{I,J}(H)}\right|^p dH &\lesssim \sum_{\substack{(n_{|J|+1},\dots,n_r)\text{ a}\\ \text{permutation of }\{1,\dots,r\}\setminus J}} \int_{\substack{0\le t_j\le N^{-1},\ j\in J\\ N^{-1}<t_{n_{|J|+1}}\le\cdots\le t_{n_r}\le c}} t_{n_r}^{-q_r p}\cdots t_{n_{|J|+1}}^{-q_{|J|+1}p}\, dt_1\cdots dt_r \\
&\lesssim \sum \int_{\substack{0\le t_j\le N^{-1},\ j\in J\\ N^{-1}<t_{n_{|J|+1}}\le\cdots\le t_{n_{r-1}}\le c}} t_{n_{r-1}}^{-(q_r+q_{r-1})p+1}\cdots t_{n_{|J|+1}}^{-q_{|J|+1}p}\, dt_1\cdots dt_r \\
&\lesssim \sum \cdots \\
&\lesssim \sum N^{(q_r+q_{r-1}+\cdots+q_{|J|+1})p-(r-|J|)} \int_{0\le t_j\le N^{-1},\ j\in J} dt_1\cdots dt_r \\
&\lesssim N^{(q_r+q_{r-1}+\cdots+q_{|J|+1})p-(r-|J|)-|J|} = N^{(|\Sigma^+|-|\Sigma_J^+|)p-r}.
\end{aligned}$$

We have evaluated the above integral in an iterated manner first with respect to t_{n_r}, then $t_{n_{r-1}}$, and so on all the way to $t_{n_{|J|+1}}$, and have used $-(q_r+q_{r-1}+\cdots+q_{j+1})p+r-j<0$ for each $j=|J|,\dots,r-1$, which is a consequence of (5.3) and the assumption $p>\frac{2r}{d-r}=\frac{r}{|\Sigma^+|}$. This proves the desired L^p-bound for this case. An inspection also reveals that in this case

$$\left\|\frac{1}{\delta_{I,J}}\right\|_{L^p(P_{I,J}\cap\mathcal{N}_I)} \begin{cases} \lesssim_\varepsilon N^{-|\Sigma_J^+|+\varepsilon}, & \text{for } p=\frac{r}{|\Sigma^+|},\\ \lesssim N^{-|\Sigma_J^+|}, & \text{for } p<\frac{r}{|\Sigma^+|}.\end{cases}$$

Case 2: $I\neq\{1,\dots,r\}$, or equivalently $0\in I$. The main new technicality for this case is that Σ_I is not necessarily irreducible. By removing the node in the extended Dynkin diagram not belonging to I, we obtain the Dynkin diagram for the root system Σ_I. Checking Figure 1, Σ_I may be irreducible, or a product of two or three irreducible root systems. If Σ_I is irreducible, following a similar argument as the case $I=\{1,\dots,r\}$

above, we obtain

$$\left\|\frac{1}{\delta_{I,J}}\right\|_{L^p(P_{I,J}\cap\mathcal{N}_I)}\begin{cases}\lesssim N^{|\Sigma_I^+|-|\Sigma_J^+|-\frac{r}{p}}, & \text{for } p>\frac{r}{|\Sigma_I^+|},\\ \lesssim_\varepsilon N^{-|\Sigma_J^+|+\varepsilon}, & \text{for } p=\frac{r}{|\Sigma_I^+|},\\ \lesssim N^{-|\Sigma_J^+|}, & \text{for } p<\frac{r}{|\Sigma_I^+|}.\end{cases}$$

Since $|\Sigma_I^+|\leq|\Sigma^+|$, the above is bounded by the desired bound. Next we demonstrate the necessary modifications for the argument when Σ_I is a product of two irreducibles, and the case of three irreducibles may be treated similarly. See Figure 4 on rank-two root systems, where it can be seen that for types B_2 and G_2 the reducible rank-two root system of product type $A_1\times A_1$ appears as one of these Σ_I's. Now suppose $\Sigma_I=\Sigma_{I_1}\bigsqcup\Sigma_{I_2}$ where Σ_{I_1} and Σ_{I_2} are nonempty, irreducible and orthogonal to each other, with $I=I_1\bigsqcup I_2$. Associated to this is also the direct product decomposition of the ambient linear space $\mathfrak{t}=\mathfrak{t}_1\times\mathfrak{t}_2$. Let $J_i=I_i\bigcap J$ $(i=1,2)$, then $J=J_1\bigsqcup J_2$. The polytope $P_{I,J}$ is now more of less the orthogonal product $P_{I_1,J_1}\times P_{I_2,J_2}$; more precisely, by (3.1), $P_{I,J}$ is contained in the product domain

$$\underbrace{\{H_1\in\mathfrak{t}_1:\ 0\leq t_j(H_1)\leq N^{-1}\ \forall j\in J,\ N^{-1}<t_j(H_1)\leq c\ \forall j\in I_1\setminus J_1\}}_{=:P'_{I_1,J_1}}$$
$$\times\underbrace{\{H_2\in\mathfrak{t}_2:\ 0\leq t_j(H_2)\leq N^{-1}\ \forall j\in J,\ N^{-1}<t_j(H_2)\leq c\ \forall j\in I_2\setminus J_2\}}_{=:P'_{I_2,J_2}}$$

(as well as contains such a domain with a smaller constant c). The vertex A_I may be expressed as $A_{I_1}\times A_{I_2}$ where A_{I_i} is a point in $\mathfrak{t}_i$ $(i=1,2)$, and the neighborhood $\mathcal{N}_I$ may be contained in a product $\mathcal{N}_{I_1}\times\mathcal{N}_{I_2}$ where $\mathcal{N}_{I_i}$ is a neighborhood of A_{I_i} in the space $\mathfrak{t}_i$ $(i=1,2)$. With the positive systems also decomposed as $\Sigma_I^+=\Sigma_{I_1}^+\bigsqcup\Sigma_{I_2}^+$, $\Sigma_J^+=\Sigma_{J_1}^+\bigsqcup\Sigma_{J_2}^+$, we have $\delta_{I,J}=\delta_{I_1,J_1}\cdot\delta_{I_2,J_2}$. Apply the proof in Case 1 for irreducible root systems, we obtain for $i=1,2$

$$\left\|\frac{1}{\delta_{I_i,J_i}}\right\|_{L^p(P'_{I_i,J_i}\cap\mathcal{N}_{I_i})}\begin{cases}\lesssim N^{|\Sigma_{I_i}^+|-|\Sigma_{J_i}^+|-\frac{r_i}{p}}, & \text{for } p>\frac{r_i}{|\Sigma_{I_i}^+|},\\ \lesssim_\varepsilon N^{-|\Sigma_{J_i}^+|+\varepsilon}, & \text{for } p=\frac{r_i}{|\Sigma_{I_i}^+|},\\ \lesssim N^{-|\Sigma_{J_i}^+|}, & \text{for } p<\frac{r_i}{|\Sigma_{I_i}^+|},\end{cases}$$

Here $r_i=|I_i|$ is the rank of Σ_{I_i} $(i=1,2)$. By Corollary 5.3, we may assume

$$\frac{r}{|\Sigma^+|}<\frac{r_1}{|\Sigma_{I_1}^+|}\leq\frac{r_2}{|\Sigma_{I_2}^+|}. \tag{5.4}$$

Then

$$\left\|\frac{1}{\delta_{I,J}}\right\|_{L^p(P_{I,J}\cap\mathcal{N}_I)}\leq\left\|\frac{1}{\delta_{I_1,J_1}}\right\|_{L^p(P'_{I_1,J_1}\cap\mathcal{N}_{I_1})}\left\|\frac{1}{\delta_{I_2,J_2}}\right\|_{L^p(P'_{I_2,J_2}\cap\mathcal{N}_{I_2})}\begin{cases}\lesssim N^{|\Sigma_I^+|-|\Sigma_J^+|-\frac{r}{p}}, & \text{for } p>\frac{r_2}{|\Sigma_{I_2}^+|},\\ \lesssim_\varepsilon N^{|\Sigma_I^+|-|\Sigma_J^+|-\frac{r}{p}+\varepsilon}, & \text{for } p=\frac{r_2}{|\Sigma_{I_2}^+|},\\ \lesssim N^{|\Sigma_{I_1}^+|-|\Sigma_J^+|-\frac{r_1}{p}}, & \text{for} \frac{r_1}{|\Sigma_{I_1}^+|}<p<\frac{r_2}{|\Sigma_{I_2}^+|},\\ \lesssim_\varepsilon N^{-|\Sigma_J^+|+\varepsilon}, & \text{for } p=\frac{r_1}{|\Sigma_{I_1}^+|},\\ \lesssim N^{-|\Sigma_J^+|}, & \text{for } p<\frac{r_1}{|\Sigma_{I_1}^+|}.\end{cases}$$

Let $e_1(\frac{1}{p})$ denote the above exponent of N as a piecewise linear function of $\frac{1}{p}$. e_1 is a continuous convex function modulo two ε-sized discontinuities. We compare it with the linear function $e_2(\frac{1}{p})=|\Sigma^+|-|\Sigma_J^+|-\frac{r}{p}$ of $\frac{1}{p}$ on the range $0<\frac{1}{p}<\frac{|\Sigma^+|}{r}$. First for p large enough, since $|\Sigma^+|>|\Sigma_I^+|$ ($\Sigma_I\subsetneq\Sigma$ since Σ_I is reducible),

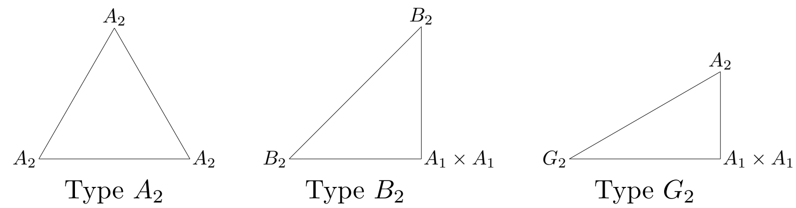


FIGURE 4. Alcoves and parabolic subsystems

it holds $e_1(\frac{1}{p}) < e_2(\frac{1}{p})$; on the other hand, for any p near but still larger than $\frac{r}{|\Sigma^+|}$, it is also clear that $e_1(\frac{1}{p}) < e_2(\frac{1}{p})$, noting (5.4). By convexity (modulo ε-sized discontinuities) of e_1 and linearity of e_2, we get $e_1(\frac{1}{p}) < e_2(\frac{1}{p})$ for all $p > \frac{r}{|\Sigma^+|}$, which yields the desired estimate on this range. We have in fact shown for this case the slightly improved estimate

$$\left\|\frac{1}{\delta_{I,J}}\right\|_{L^p(P_{I,J}\cap\mathcal{N}_I)} \lesssim \begin{cases} N^{|\Sigma^+|-|\Sigma_J^+|-\frac{r}{p}}, & \text{for } p \geq \frac{r}{|\Sigma^+|}, \\ N^{-|\Sigma_J^+|}, & \text{for } p < \frac{r}{|\Sigma^+|}. \end{cases}$$

$\square$

Remark 5.5. *Considering that the inequality* (5.3) *holds strictly when $J \neq \varnothing$, an inspection of the proof shows that the ε-factor in the above estimate is only necessary when $J = \varnothing$.*

Remark 5.6. *We have a natural extension of the above discussion to arbitrary root systems that are not necessary irreducible. Let Σ_j $(j = 1, \dots, k)$ be irreducible root systems of rank r_j and consider the product root system $\Sigma := \bigsqcup_j \Sigma_j$ of rank $\sum_j r_j$. Let Σ_j^+ be a positive system of Σ_j and then $\Sigma^+ = \bigsqcup_j \Sigma_j^+$ is a positive system of Σ. The associated Weyl alcove A to Σ may be defined as the product $A := A_1 \times \cdots \times A_k$ where A_j is the alcove for Σ_j. Let I_j denote a subset of $\{0, \dots, r_j\}$ such that $|I_j| = r_j$ and let J_j be a subset of I_j. Let $I := \bigsqcup_j I_j$ and $J := \bigsqcup_j J_j$. Set*

$$C_I := C_{I_1} \times \cdots \times C_{I_k},$$

$$P_J := P_{J_1} \times \cdots \times P_{J_k},$$

$$P_{I,J} := P_J \cap C_I = P_{I_1,J_1} \times \cdots \times P_{I_k,J_k}.$$

Set $\delta_I := \prod_j \delta_{I_j}$, $\delta^J := \prod_j \delta^{J_j}$, and $\delta_{I,J} := \prod_j \delta_{I_j,J_j}$. Then Lemma 3.5 may be generalized to this setting without change, and the above proposition implies that

$$\left\|\frac{1}{\delta_{I,J}}\right\|_{L^p(P_{I,J})} \lesssim N^{|\Sigma^+|-|\Sigma_J^+|-\frac{r}{p}}, \text{ for } p > \max\left\{\frac{r_j}{|\Sigma_j^+|},\ j = 1, \dots, k\right\}.$$

6. Projections of the weight lattice

For each proper subset J of $\{0, \dots, r\}$, recall that V_J denotes the the $\mathbb{R}$-subspace of $i\mathfrak{t}^*$ spanned by Σ_J, Λ_J denotes the weight lattice associated to Σ_J, and $\mathrm{Proj}_{V_J} : i\mathfrak{t}^* \to i\mathfrak{t}^*$ denotes orthogonal projection onto V_J. With Lemma 4.2 in mind, to examine the behavior of the characters, we still need to understand the image set $\mathrm{Proj}_{V_J}(\Lambda)$ in detail. We have the following characterization. Let $\Gamma := \mathrm{span}_{\mathbb{Z}}\Sigma$, $\Gamma_J := \mathrm{span}_{\mathbb{Z}}\Sigma_J$ be the root lattices generated by the root systems Σ, Σ_J respectively.

Lemma 6.1. *We have $\Lambda_J \supset \mathrm{Proj}_{V_J}(\Lambda) \supset \Gamma_J$.*

Proof. $\Lambda_J \supset \mathrm{Proj}_{V_J}(\Lambda)$ follows by definition of the weight lattices. Then $\mathrm{Proj}_{V_J}(\Lambda) \supset \mathrm{Proj}_{V_J}(\Gamma) \supset \mathrm{Proj}_{V_J}(\Gamma_J) = \Gamma_J$. $\square$

In particular, since Λ_J and Γ_J are both lattices of rank $|J|$, so is $\mathrm{Proj}_{V_J}(\Lambda)$. This has the following as an immediate consequence.

Lemma 6.2. *We have the $\mathbb{Z}$-linear direct sum*

$$\Lambda = {}^J\Lambda \bigoplus {}^J\Lambda^\perp,$$

such that $\mathrm{Proj}_{V_J}({}^J\Lambda^\perp) = 0$ *while* $\mathrm{Proj}_{V_J} : {}^J\Lambda \xrightarrow{\sim} \mathrm{Proj}_{V_J}(\Lambda)$ *is an isomorphism of rank-$|J|$ lattices.*

We now wish to evaluate the L^p norm of the characters χ_μ as in the formula (4.2). With Lemma 3.5, Proposition 5.4, and Lemma 6.1 in mind, we now analyze characters of the form $\chi^J_\lambda(H_J)$ for $\lambda \in \Lambda_J$ and $H \in P_J$. Let $H \in P_J$. By the definition of H_J and P_J we know that

$$0 < \alpha_j(H_J)/2\pi i + \delta_{0j} \le N^{-1} \text{ for } j \in J.$$

If $0 \notin J$, then the above already implies that $|H_J| \lesssim N^{-1}$. In general, we pick any $H_J^0 \in t_J$ such that $\alpha_j(H_J^0)/2\pi i + \delta_{0j} = 0$. Then

$$H_J = H_J^1 + H_J^0$$

such that $|\alpha_j(H_J^1)| \le N^{-1}$ for all $j \in J$, and thus

$$|H_J^1| \lesssim N^{-1}.$$

Lemma 6.3. *For $H \in P_J$ and $\lambda \in \Lambda_J$, $\chi^J_\lambda(H_J) = e^{(\lambda+\rho_J)(H_J^0)}\chi^J_\lambda(H_J^1)$.*

Proof. We have $\chi^J_\lambda = (\delta^J)^{-1}\sum_{s\in W_J} \det s\ e^{s\lambda}$. By a standard fact in root system theory, $s\lambda - \lambda \in \Gamma_J$ for each $s \in W_J$ and $\lambda \in \Lambda_J$. By definition $\alpha(H_J^0)/2\pi i \in \mathbb{Z}$ for all $\alpha \in \Gamma_J$, we thus have $e^{s\lambda(H_J^0)} = e^{\lambda(H_J^0)}$, which contributes the factor $e^{\lambda(H_J^0)}$ to the right side of the desired equation. Now we can write

$$\delta^J(H_J) = e^{-\rho_J(H_J)} \prod_{\alpha\in\Sigma_J^+} \left(e^{\alpha(H_J)} - 1\right) = e^{-\rho_J(H_J^0)} e^{-\rho_J(H_J^1)} \prod_{\alpha\in\Sigma_J^+} \left(e^{\alpha(H_J^1)} - 1\right) = e^{-\rho_J(H_J^0)}\delta^J(H_J^1).$$

This contributes the other factor $e^{\rho_J(H_J^0)}$ and thus concludes the proof. □

Lastly, to analyze $\chi^J_\lambda(H_J^1)$, we apply:

Lemma 6.4. [14, Harish-Chandra's integral formula] *Let U be a compact semisimple Lie group, and let $\mathfrak{u}$ be its Lie algebra and let $\mathfrak{t}$ be its Cartan subalgebra. Let $\mathfrak{u}^*_{\mathbb{C}}, \mathfrak{t}^*_{\mathbb{C}}$ be the spaces of complex linear forms on $\mathfrak{u}, \mathfrak{t}$ respectively. For $u \in U$, let Ad_u denotes the adjoint action of u on $\mathfrak{u}$ as well as on $\mathfrak{u}^*_{\mathbb{C}}$ such that $(\mathrm{Ad}_u\lambda)(X) = \lambda(\mathrm{Ad}_u^{-1}X)$ for any $\lambda \in \mathfrak{u}^*_{\mathbb{C}}$ and $X \in \mathfrak{u}$. Then for any $\lambda, \mu \in \mathfrak{t}^*_{\mathbb{C}}$, we have*

$$\sum_{s\in W} \det s\ e^{(s\lambda,\mu)} = \frac{\prod_{\alpha\in\Sigma^+}(\alpha,\lambda)\cdot\prod_{\alpha\in\Sigma^+}(\alpha,\mu)}{\prod_{\alpha\in\Sigma^+}(\alpha,\rho)} \int_U e^{(\mathrm{Ad}_u\lambda,\mu)}\ du.$$

As a consequence, we have:

Lemma 6.5. *Let U_J be a compact semisimple Lie group whose root system is Σ_J (which always exists thanks to Lie's third theorem). Then*

$$\chi^J_\lambda(H_J^1) = \frac{\prod_{\alpha\in\Sigma_J^+}\alpha(H_J^1)}{\delta^J(H_J^1)} \cdot \frac{\prod_{\alpha\in\Sigma_J^+}(\alpha,\lambda)}{\prod_{\alpha\in\Sigma_J^+}(\alpha,\rho_J)} \cdot \int_{U_J} e^{\lambda(\mathrm{Ad}_u H_J^1)}\ du.$$

In particular, we have the following character bound.

Lemma 6.6. *For $|\mu| \lesssim N$, $\mu \in \Lambda$, $J \subsetneq \{0,\dots,r\}$, we have*

$$|\chi^J_{\mu_J}(H_J)| \lesssim N^{|\Sigma_J^+|}, \text{ for } H \in P_J.$$

Proof. Since $|\mu| \lesssim N$, we have $|\mu_J| \lesssim N$. By Lemma 6.1, $\mu_J \in \Lambda_J$. Using the isometry property $|\mathrm{Ad}_u H_J^1| = |H_J^1|$ of the adjoint map, and the fact that $|H_J^1| \lesssim N^{-1}$, we have

$$\left|\frac{\prod_{\alpha\in\Sigma_J^+}\alpha(H_J^1)}{\delta^J(H_J^1)}\right| \lesssim 1, \ \left|\int_{U_J} e^{\mu_J(\mathrm{Ad}_u H_J^1)}\, du\right| \lesssim 1.$$

Then the result follows from Lemma 6.3 and 6.5. ☐

Remark 6.7. *For any regular element μ_J in Λ_J, $|\chi^J_{\mu_J}|$ is bounded by the dimension $|d_{\mu_J}|$, which yields the above estimate by an application of the Weyl dimension formula. The essence of the above lemma is to treat non-regular μ_J, which is usually a subtle issue in high-rank analysis.*

7. Proof of Theorem 1.1

We are ready to prove Theorem 1.1.

Proof. Let Δ denote the Laplace-Beltrami operator. Then for $\mu \in \Lambda^+$,

$$\Delta\chi_\mu = -(|\mu|^2 - |\rho|^2)\chi_\mu.$$

Choose $N \sim |\mu|$. By Weyl's integration formula (2.1), we write $\|\chi_\mu\|_{L^p(U)} = \|\chi_\mu|\delta|^{\frac{2}{p}}\|_{L^p(A)}$. Using Lemma 3.3 the barycentric-semiclassical subdivision, we have

$$\|\chi_\mu\|_{L^p(U)} \lesssim \sum_{J,I\subset\{0,\dots,r\},\ |I|=r,\ J\subset I} \left\|\chi_\mu|\delta|^{\frac{2}{p}}\right\|_{L^p(P_{I,J})}.$$

Using (4.2), we have

$$\left|\chi_\mu(H)|\delta(H)|^{\frac{2}{p}}\right| \le \frac{|\delta_J|^{\frac{2}{p}}}{|W_J||\delta_I(H)|^{1-\frac{2}{p}}|\delta_{I,J}(H)|^{1-\frac{2}{p}}} \sum_{s\in W}\left|\chi^J_{(s\mu)_J}(H_J)\right|.$$

Part (i) is then a consequence of Lemma 3.5, Lemma 6.6, and the key Proposition 5.4. Now by the argument of TT^*, the estimate in part (ii) is equivalent to

$$\|\psi\|_{L^p(U)} \lesssim N^{d-r-\frac{2d}{p}}\|\psi\|_{L^{p'}(U)}.$$

For a joint eigenfunction ψ of spectral parameter μ, we have

$$\psi = \psi * (d_\mu\chi_\mu).$$

Then the above estimate is a consequence of Young's convolution inequality, part (i), and the dimension bound $d_\mu \lesssim N^{\frac{d-r}{2}}$ as from the Weyl dimension formula. ☐

Remark 7.1. *The exponent $\frac{d-r}{2} - \frac{d}{p}$ of N in part (i) is sharp, as can be seen by testing the character on a N^{-1}-sized neighborhood of the origin and choosing the spectral parameter μ away from the walls of the Weyl chamber such that $(\mu,\alpha) \gtrsim |\mu|$ for all $\alpha \in \Sigma^+$. Then it holds that $|\chi_\mu| \gtrsim N^{\frac{d-r}{2}}$ on this neighborhood thanks to an application of Lemma 6.5 (letting $J = \{1,\dots,r\}$). As an N^{-1}-neighborhood is of volume $\sim N^{-d}$, this shows sharpness of the exponent.*

8. The Schrödinger kernel

To treat Strichartz estimates for the Schrödinger equation, we now study the Schrödinger propagator $e^{it\Delta}$. As is standard from Littlewood-Paley theory on compact manifolds as developed in [9], for the purpose of Strichartz estimates it suffices to consider the spectrally localized or say mollified version of the Schrödinger propagator. Fix a large positive number N. Let $\phi(-N^{-2}\Delta)$ be a spectral projector for the Laplace-Beltrami operator Δ on U associated to the standard metric as induced from the Killing form. We define the mollified Schrödinger kernel $\mathscr{K}_N(t,x)$ ($t\in\mathbb{R}$, $x\in U$) as follows

$$f*\mathscr{K}_N(t,\cdot):=\phi(-N^{-2}\Delta)e^{it\Delta}f.$$

Then expressing it as a Fourier series, we have that

$$\mathscr{K}_N(t,\exp H)=\sum_{\mu\in\Lambda^+}\phi\left(\frac{|\mu|^2-|\rho|^2}{N^2}\right)e^{-it(|\mu|^2-|\rho|^2)}d_\mu\chi_\mu(H),\ \text{ for } H\in\mathfrak{t},\tag{8.1}$$

where

$$d_\mu=\frac{\prod_{\alpha\in\Sigma^+}(\mu,\alpha)}{\prod_{\alpha\in\Sigma^+}(\rho,\alpha)},\tag{8.2}$$

is the Weyl formula for the dimension of representation and $\chi_\mu(H)$ is the character associated to μ. As is the case for the Weyl character formula, the above dimension formula makes sense for any $\mu\in i\mathfrak{t}^*$ and in particular for any $\mu\in\Lambda$.

We begin to derive important formulas for the Schrödinger kernel similar to those for the characters. We first recall a lemma from [31] that expresses $\mathscr{K}_N$ as a sum over the whole weight lattice Λ.

Lemma 8.1.

$$\mathscr{K}_N(t,\exp H)=\frac{1}{|W|}\sum_{\mu\in\Lambda}\phi\left(\frac{|\mu|^2-|\rho|^2}{N^2}\right)e^{-it(|\mu|^2-|\rho|^2)}d_\mu\chi_\mu(H).$$

Proof. This is a direct consequence of Lemma 4.1, the fact that $|s\mu|=|\mu|$ for any $s\in W$ and $\mu\in i\mathfrak{t}^*$, and the formula (8.2) of d_μ. ☐

We also recall a standard lemma of root system theory.

Lemma 8.2. *For $\mu\in i\mathfrak{t}^*$ and $s\in W$, $d_\mu=\det s\ d_{s\mu}$.*

Now we have:

Lemma 8.3. *For any $H\in\mathfrak{t}$ and $t\in\mathbb{R}$, we have*

$$\mathscr{K}_N(t,\exp H)=\frac{1}{|W_J|\delta_I(H)\delta_{I,J}(H)}\cdot\mathscr{K}_N^J(t,H)\tag{8.3}$$

where

$$\mathscr{K}_N^J(t,H)=\sum_{\mu\in\Lambda}e^{\mu(H_J^\perp)-it(|\mu|^2-|\rho|^2)}\phi\left(\frac{|\mu|^2-|\rho|^2}{N^2}\right)d_\mu\chi_{\mu_J}^J(H_J).$$

Proof. Using Lemma 8.1 and (4.2), we have

$$\begin{aligned}\mathscr{K}_N=\frac{1}{|W_J|\delta_I(H)\delta_{I,J}(H)}\cdot\frac{1}{|W|}\sum_{s\in W}\sum_{\mu\in\Lambda}&\phi\left(\frac{|\mu|^2-|\rho|^2}{N^2}\right)e^{-it(|\mu|^2-|\rho|^2)}d_\mu\\ &\cdot\det s\ e^{(s\mu)(H_J^\perp)}\chi_{(s\mu)_J}^J(H_J).\end{aligned}$$

Note that

$$s\Lambda=\Lambda$$

for any $s\in W$, then (8.3) holds by an application of Lemma 8.2 and the fact that $|s\mu|=|\mu|$ for any $s\in W$ and $\mu\in i\mathfrak{t}^*$. ☐

Now we wish to incorporate information in Section 6 to refine the above formula. By Lemma 6.1 and 6.2, let ${}^J\Gamma$ be the preimage of Γ_J under the isomorphism $\mathrm{Proj}_{V_J} : {}^J\Lambda \xrightarrow{\sim} \mathrm{Proj}_{V_J}(\Lambda)$. Then we can write

$$\Lambda = \bigsqcup_{\mu \in {}^J\Lambda/{}^J\Gamma} \left(\mu + {}^J\Gamma + {}^J\Lambda^\perp\right),$$

with

$$|{}^J\Lambda/{}^J\Gamma| \lesssim 1. \tag{8.4}$$

Using Lemma 6.3, Lemma 6.5 and the above decomposition of Λ, we get the following formula for $\mathscr{K}_N^J$.

Lemma 8.4. *Let*

$$a(t,\mu,H) := e^{-it(|\mu|^2-|\rho|^2)+\mu(H_J^\perp)+(\mu_J+\rho_J)(H_J^0)} \cdot \frac{\prod_{\alpha\in\Sigma_J^+}\alpha(H_J^1)}{\delta^J(H_J^1)},$$

$$P(\mu,\lambda_1,\lambda_2,H) := \phi\left(\frac{|\mu+\lambda_1+\lambda_2|^2-|\rho|^2}{N^2}\right) d_{\mu+\lambda_1+\lambda_2} \frac{\prod_{\alpha\in\Sigma_J^+}(\alpha,\mu_J+(\lambda_1)_J)}{\prod_{\alpha\in\Sigma_J^+}(\alpha,\rho_J)} \cdot \int_{U_J} e^{(\mu_J+(\lambda_1)_J)(\mathrm{Ad}_u H_J^1)}\, du,$$

$$\kappa_N^J(\mu,t,H) := \sum_{\lambda_1\in {}^J\Gamma,\ \lambda_2\in{}^J\Lambda^\perp} e^{(\lambda_1+\lambda_2)(H_J^\perp)-it(|\lambda_1+\lambda_2|^2+2(\mu,\lambda_1+\lambda_2))} P(\mu,\lambda_1,\lambda_2,H).$$

Then

$$\mathscr{K}_N^J(t,H) = \sum_{\mu\in{}^J\Lambda/{}^J\Gamma} a(t,\mu,H)\cdot \kappa_N^J(\mu,t,H). \tag{8.5}$$

The above $\kappa_N^J(\mu,t,H)$ is of the form of a Weyl type exponential sum. We will treat it in later sections in two different ways, one by Weyl differencing, one by Poisson summation, oscillatory integrals and Kloosterman and Salié sums.

Some preliminary estimates are in order. We pick a $\mathbb{Z}$-basis $\{u_1,\dots,u_{|J|}\}$ of ${}^J\Gamma$, and a $\mathbb{Z}$-basis $\{u_{|J|+1},\dots,u_r\}$ of ${}^J\Lambda^\perp$ so that $\{(u_1)_J,\dots,(u_{|J|})_J\}$ is a basis of Γ_J. For $\lambda_1\in{}^J\Gamma$, $\lambda_2\in{}^J\Lambda^\perp$, we write

$$\lambda_1 = n_1u_1+\cdots+n_{|J|}u_{|J|},$$

$$(\lambda_1)_J = n_1(u_1)_J+\cdots+n_{|J|}(u_{|J|})_J,$$

$$\lambda_2 = n_{|J|+1}u_{|J|+1}+\cdots+n_ru_r,$$

for $n_1,\dots,n_r\in\mathbb{Z}$. Note that $P(\mu,\lambda_1,\lambda_2,H)$ makes sense if $n_1,\dots,n_r$ take values in $\mathbb{R}$. We write

$$P(\mu,n_1,\dots,n_r,H) := P(\mu,\lambda_1,\lambda_2,H).$$

Let ∂_{n_j} denote differential operator with respect to the variable $n_j\in\mathbb{R}$, and let D_{n_j} denote the forward difference operator with respect to the variable $n_j\in\mathbb{Z}$.

Lemma 8.5. *Let $H\in P_J$. We have*

$$|a(t,\mu,H)| \lesssim 1. \tag{8.6}$$

Let $\mathbf{m}=(m_1,\dots,m_r)\in(\mathbb{Z}_{\geq 0})^r$ *and let* $|\mathbf{m}|=\sum_j m_j$. *Then*

$$\left|\left(\prod_{j=1}^r \partial_{n_j}^{m_j}\right) P(\mu,n_1,\dots,n_r,H)\right| \lesssim_{\mathbf{m}} N^{|\Sigma^+|+|\Sigma_J^+|-|\mathbf{m}|} \text{ for all } n_j\in\mathbb{R},\ j=1,\dots,r, \tag{8.7}$$

and

$$\left|\left(\prod_{j=1}^{r} D_{n_j}^{m_j}\right) P(\mu, n_1, \dots, n_r, H)\right| \lesssim_{\mathbf{m}} N^{|\Sigma^+|+|\Sigma_J^+|-|\mathbf{m}|} \text{ for all } n_j \in \mathbb{Z},\ j = 1, \dots, r. \tag{8.8}$$

Proof. The first inequality follows directly from the fact that $|H_J^1| \lesssim N^{-1}$. The remaining estimates are standard, observing that the cutoff function ϕ results in the restraint $|n_j| \lesssim N$ $(j = 1, \dots, r)$, and that $|\mathrm{Ad}_u H_J^1| = |H_J^1| \lesssim N^{-1}$. □

9. Proof of Theorem 1.2

By rationality of the weight lattice under the Killing form, there exsits $\mathcal{T} \in 2\pi\mathbb{Q}$ such that

$$|\mu|^2 \in \frac{2\pi}{\mathcal{T}}\mathbb{Z}, \text{ for all } \mu \in \Lambda. \tag{9.1}$$

In particular, $\mathcal{T}$ is a period of the Schrödinger kernel as well as the function $\kappa_N^J(\mu, t, H)$. We have the following treatment of $\kappa_N^J(\mu, t, H)$.

Lemma 9.1. *For $H \in P_J$, it holds*

$$|\kappa_N^J(\mu, t, H)| \lesssim \frac{N^{|\Sigma^+|+|\Sigma_J^+|+r}}{\left(\sqrt{q}\left(1 + N\left\|\frac{t}{\mathcal{T}} - \frac{a}{q}\right\|^{\frac{1}{2}}\right)\right)^r}$$

for $\left\|\frac{t}{\mathcal{T}} - \frac{a}{q}\right\| \lesssim \frac{1}{qN}$. Here $\|\cdot\|$ denotes the distance from the nearest integer.

Proof. This is a multi-dimensional Weyl type inequality. Using (8.8), the estimate follows from an application of Weyl differencing. For details of proof, we refer to for example [4, Lemma 3.18] for one-dimensional Weyl inequality and [31, Lemma 7.4] for multi-dimensional Weyl type inequality; the key condition used for the latter is the non-degeneracy of the Killing form $|\cdot|^2$. □

Using (8.5), (8.4), and (8.6), we immediately have:

Lemma 9.2. *For $H \in P_J$, it holds*

$$|\mathscr{K}_N^J(t, H)| \lesssim \frac{N^{|\Sigma^+|+|\Sigma_J^+|+r}}{\left(\sqrt{q}\left(1 + N\left\|\frac{t}{\mathcal{T}} - \frac{a}{q}\right\|^{\frac{1}{2}}\right)\right)^r}$$

for $\left\|\frac{t}{\mathcal{T}} - \frac{a}{q}\right\| \lesssim \frac{1}{qN}$.

We are ready to prove Theorem 1.2.

Proof. By Weyl's integration formula as in (2.1), we have

$$\|\mathscr{K}_N(t, \cdot)\|_{L^p(U)} = \|\mathscr{K}_N(t, \cdot)|\delta|^{\frac{2}{p}}\|_{L^p(A)}.$$

Since $A = \bigcup_{J \subset I, |I| = r} P_{I,J}$, it suffices to prove $\|\mathscr{K}_N(t, \cdot)|\delta|^{\frac{2}{p}}\|_{L^p(P_{I,J})}$ has the desired bound for all I, J. Using (8.3), we have

$$|\mathscr{K}_N(t, H)| \cdot |\delta(H)|^{\frac{2}{p}} = \frac{|\delta^J(H)|^{\frac{2}{p}}}{|W_J| \cdot |\delta_I(H)|^{1-\frac{2}{p}} |\delta_{I,J}(H)|^{1-\frac{2}{p}}} \cdot |\mathscr{K}_N^J(t, H)|.$$

Then we have the desired estimate, combining Lemmas 3.2 and 9.2, inequality (3.3), and Proposition 5.4. □

Remark 9.3. *In light of Remark 5.6, it is clear from the proof that Theorem 1.2 may be generalized to any product of compact simple Lie groups. Let M be such a product and for each irreducible/simple factor M_0 of M, let*

$$s_0 := \frac{2d_0}{d_0 - r_0}$$

where d_0, r_0 are respectively the dimension and rank of M_0. Let s be the largest among these s_0's. Then for any $p > s$, inequality (1.10) *holds for $\left\| \frac{t}{\mathcal{T}} - \frac{a}{q} \right\| \lesssim \frac{1}{qN}$.*

10. Farey dissection

In this section, we review the circle method of Farey dissection, which will be used to derive Strichartz estimates and Laplace-Beltrami eigenfunction bounds. Let n be an integer and consider the Farey sequence

$$\left\{ \frac{a}{q},\ a \in \mathbb{Z}_{\geq 0},\ q \in \mathbb{Z}_{\geq 1},\ (a,q) = 1,\ a < q,\ q \leq n \right\}$$

of order n on the unit circle. For each three consecutive fractions $\frac{a_l}{q_l}$, $\frac{a}{q}$, $\frac{a_r}{q_r}$ in the sequence, consider the Farey arc

$$\mathcal{M}_{a,q} = \left[\frac{a_l + a}{q_l + q}, \frac{a + a_r}{q + q_r} \right]$$

around $\frac{a}{q}$. The Farey dissection $\bigsqcup_{a,q} \mathcal{M}_{a,q}$ of order n of the unit circle has the uniformity property that both $\left[\frac{a_l+a}{q_l+q}, \frac{a}{q}\right]$ and $\left[\frac{a}{q}, \frac{a+a_r}{q+q_r}\right]$ are of length $\asymp \frac{1}{qn}$; see for example Theorem 35 in [13]. We make a further dissection of the unit circle as follows, in order to make use of the kernel bound as in Lemma 9.2; such methods have been explored by Bourgain [3, 4]. Fix a large number N and let Q be dyadic integers, i.e. powers of 2 between 1 and N. Consider the Farey sequence of order $\lfloor N \rfloor$. For $Q \leq q < 2Q$, let M denote dyadic integers between Q and N, and we decompose the Farey arc into a disjoint union

$$\mathcal{M}_{a,q} = \bigsqcup_{Q \leq M \leq N, M \text{ dyadic}} \mathcal{M}_{a,q,M}$$

where $\mathcal{M}_{a,q,M}$ is an interval on the unit circle of the form $\left\| t - \frac{a}{q} \right\| \asymp \frac{1}{NM}$, except when M is the largest dyadic integer $\leq N$, $\mathcal{M}_{a,q,M}$ is defined as an interval of the form $\left\| t - \frac{a}{q} \right\| \lesssim \frac{1}{N^2}$. Let $\mathbb{1}_{Q,M}$ denote the indicator function of the subset

$$\mathcal{M}_{Q,M} = \bigsqcup_{0 \leq a < q,\ (a,q)=1,\ Q \leq q < 2Q} \mathcal{M}_{a,q,M}$$

of the unit circle, then we have a partition of unity

$$1 = \sum_{Q,M} \mathbb{1}_{Q,M} \left(\frac{t}{\mathcal{T}} \right)$$

on the circle $\mathbb{T} = \mathbb{R}/\mathcal{T}\mathbb{Z}$. Let $\widehat{\mathbb{1}_{Q,M}}$ denote the Fourier transform of $\mathbb{1}_{Q,M}\left(\frac{t}{\mathcal{T}}\right)$ on $\mathbb{T}$ such that

$$\mathbb{1}_{Q,M} \left(\frac{t}{\mathcal{T}} \right) = \sum_{n \in \frac{2\pi}{\mathcal{T}} \mathbb{Z}} \widehat{\mathbb{1}_{Q,M}}(n) e^{int},$$

then clearly

$$\|\widehat{\mathbb{1}_{Q,M}}\|_{l^\infty} \lesssim \|\mathbb{1}_{Q,M}(\cdot/\mathcal{T})\|_{L^1(\mathbb{T})} \lesssim \frac{Q^2}{NM}. \tag{10.1}$$

11. Proof of Theorem 1.3

Proof. Reducing to a finite cover, it suffices to prove it for the case of a compact simply connected semisimple Lie group $U = U_1 \times U_2 \times \cdots \times U_k$, where the U_i's are the simple components, equipped with the canonical Killing metrics. Consider the product Schrödinger kernel

$$\mathscr{K}_N = \prod_{i=1}^{k} \mathscr{K}_{N,i} \tag{11.1}$$

where

$$\mathscr{K}_{N,i}(t, H_i) = \sum_{\mu_i \in \Lambda_i^+} \phi_i\left(\frac{|\mu_i|^2 - |\rho_i|^2}{N^2}\right) e^{-it(|\mu_i|^2 - |\rho_i|^2)} d_{\mu_i} \chi_{\mu_i}(H_i)$$

is the kernel for the component U_i. By rationality of the weight lattices (see (9.1)), the component kernels $\mathscr{K}_{N,i}$ share a period in the time variable t, say $\mathcal{T}$, and we set $\mathbb{T} = \mathbb{R}/\mathcal{T}\mathbb{Z}$. Let Σ_i be the root system of rank r_i for U_i $(1 \leq i \leq k)$, then Theorem 1.2 implies

$$\|\mathscr{K}_N(t, \cdot)\|_{L^u(U)} = \prod_{i=1}^{k} \|\mathscr{K}_{N,i}(t, \cdot)\|_{L^u(U_i)} \lesssim \frac{N^{d - \frac{d}{u}}}{\left(\sqrt{q}\left(1 + N\left\|\frac{t}{\mathcal{T}} - \frac{a}{q}\right\|^{\frac{1}{2}}\right)\right)^r} \tag{11.2}$$

provided

$$u > s := \max\left\{\frac{2d_i}{d_i - r_i},\ i = 1, \ldots, k\right\}. \tag{11.3}$$

Here d_i is the dimension of U_i $(1 \leq i \leq k)$.

Using Farey dissection as reviewed in Section 10, we write

$$\mathscr{K}_N(t, x) = \sum_{Q,M} \mathscr{K}_{Q,M}(t, x),\ \mathscr{K}_{Q,M}(t, x) := \mathscr{K}_N(t, x) \cdot \mathbb{1}_{Q,M}\left(\frac{t}{\mathcal{T}}\right),$$

for $(t, x) \in \mathbb{T} \times U$. Let $F : \mathbb{T} \times U \to \mathbb{C}$ be a continuous function. Let $*$ denote the convolution on the product group $\mathbb{T} \times U$. By Young's inequality for unimodular groups, inequality (11.2), and the estimate

$$\|\mathbb{1}_{Q,M}\|_{L^u(\mathbb{T})} \lesssim \left(\frac{Q^2}{NM}\right)^{\frac{1}{u}},$$

we have for $u > s$

$$\begin{aligned} \|F * \mathscr{K}_{Q,M}\|_{L^{2u}(\mathbb{T}\times U)} &\leq \|\mathscr{K}_{Q,M}\|_{L^u(\mathbb{T}\times U)} \|F\|_{L^{(2u)'}(\mathbb{T}\times U)} \\ &\lesssim N^{d - \frac{d+1}{u} - \frac{r}{2}} M^{\frac{r}{2} - \frac{1}{u}} Q^{-\frac{r}{2} + \frac{2}{u}} \|F\|_{L^{(2u)'}(\mathbb{T}\times U)}. \end{aligned} \tag{11.4}$$

Here $2u$ and $(2u)'$ are conjugate exponents. On the other hand, as a class function on the compact Lie group $\mathbb{T} \times U$, $\mathscr{K}_{Q,M}$ has its Fourier transform $\widehat{\mathscr{K}_{Q,M}}(n, \mu)$ $((n, \mu) \in \mathbb{Z} \times \Lambda^+)$ computed as follows

$$\widehat{\mathscr{K}_{Q,M}}(n, \mu) = \phi(\mu, N)\widehat{\mathbb{1}_{Q,M}}(2\pi n/\mathcal{T} + |\mu|^2 - |\rho|^2),$$

where

$$\phi(\mu, N) = \prod_{i=1}^{k} \phi_i((|\mu_i|^2 - |\rho_i|^2)/N^2),\ |\mu|^2 - |\rho|^2 = \sum_{i=1}^{k} |\mu_i|^2 - |\rho_i|^2.$$

By (10.1), we have

$$|\widehat{\mathscr{K}_{Q,M}}(n, \mu)| \lesssim \frac{Q^2}{NM}.$$

As a consequence, we have

$$\|F * \mathscr{K}_{Q,M}\|_{L^2(\mathbb{T}\times U)} \lesssim \frac{Q^2}{NM}\|F\|_{L^2(\mathbb{T}\times U)}. \tag{11.5}$$

Interpolating (11.4) with (11.5) for $\frac{\theta}{2} + \frac{1-\theta}{2u} = \frac{1}{p}$, we get

$$\|F * \mathscr{K}_{Q,M}\|_{L^p(\mathbb{T}\times U)} \lesssim N^{\left(d - \frac{d+1}{u} - \frac{r}{2}\right)(1-\theta)-\theta} M^{\left(\frac{r}{2} - \frac{1}{u}\right)(1-\theta)-\theta} Q^{\left(-\frac{r}{2} + \frac{2}{u}\right)(1-\theta)+2\theta} \|F\|_{L^{p'}(\mathbb{T}\times U)}.$$

We require the exponent of Q satisfy

$$\left(-\frac{r}{2} + \frac{2}{u}\right)(1-\theta) + 2\theta < 0 \Leftrightarrow \theta < \frac{ru-4}{4u+ru-4},$$

which implies the exponent of M satisfies $\left(\frac{r}{2} - \frac{1}{u}\right)(1-\theta) - \theta > 0$. Summing over the dyadic integers M and Q, we get

$$\begin{aligned}\|F * \mathscr{K}_N\|_{L^p(\mathbb{T}\times U)} &\lesssim \sum_{1\le Q\le N}\sum_{Q\le M\le N} \|F * \mathscr{K}_{Q,M}\|_{L^p(\mathbb{T}\times U)} \\ &\lesssim N^{\left(d - \frac{d+2}{u}\right)(1-\theta)-2\theta}\|F\|_{L^{p'}(\mathbb{T}\times U)} = N^{d - \frac{2(d+2)}{p}}\|F\|_{L^{p'}(\mathbb{T}\times U)},\end{aligned}$$

provided

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{2u} < \frac{ru-4}{2(4u+ru-4)} + \frac{2}{4u+ru-4} \Leftrightarrow p > 2 + \frac{8(u-1)}{ur}$$

for some $u > s$. This implies Theorem 1.3, by a standard application of Littlewood-Paley theory and the TT^* argument. □

12. Proof of Theorem 1.6

We are ready to prove Theorem 1.6.

Proof. We inherit the notations in the proof of Theorem 1.3. Let f be an eigenfunction of eigenvalue $-N^2$. Then $N^2 = |\mu|^2 - |\rho|^2$ for some $\mu \in \Lambda^+$. Set

$$\mathcal{K}_N = \sum_{\mu\in\Lambda^+, |\mu|^2-|\rho|^2=N^2} d_\mu \chi_\mu.$$

Then it is clear that $f = f * \mathcal{K}_N$. By an argument of TT^*, it suffices to establish bounds of the form

$$\|f * \mathcal{K}_N\|_{L^p(U)} \lesssim N^{\frac{d-2}{2} - \frac{d}{p}}\|f\|_{L^{p'}(U)}.$$

Let $\mathscr{K}_N$ be again the Schrödinger kernel as in (8.1) and more generally as in (11.1), and here we assume that the cutoff functions ϕ_i satisfy $\prod_i \phi_i(y_i^2) = 1$ for all y_i such that $\sum_i y_i^2 = 1$. Then we may write

$$\mathcal{K}_N = \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} \mathscr{K}_N(t,\cdot)e^{itN^2}\, dt.$$

Using the Farey dissection, we decompose

$$\mathcal{K}_N = \sum_{Q,M} \mathcal{K}_{Q,M},$$

where

$$\mathcal{K}_{Q,M} = \int_{\mathcal{M}_{Q,M}} \mathscr{K}_{Q,M}(t,\cdot)e^{itN^2}\, d\left(\frac{t}{\mathcal{T}}\right).$$

By Theorem 1.2, Minkowski's integral inequality, and the estimate that the length of $\mathcal{M}_{Q,M}$ is $\lesssim \frac{Q^2}{NM}$, we have for $u > s$

$$\|\mathcal{K}_{Q,M}\|_{L^u(U)} \lesssim N^{d-\frac{d}{u}-\frac{r}{2}-1} M^{\frac{r}{2}-1} Q^{-\frac{r}{2}+2}, \tag{12.1}$$

which implies by Young's inequality

$$\|f * \mathcal{K}_{Q,M}\|_{L^{2u}(U)} \lesssim N^{d-\frac{d}{u}-\frac{r}{2}-1} M^{\frac{r}{2}-1} Q^{-\frac{r}{2}+2} \|f\|_{L^{(2u)'}(U)}. \tag{12.2}$$

On the other hand, the Fourier transform of $\mathcal{K}_{Q,M}$ on U equals

$$\widehat{\mathcal{K}_{Q,M}}(\mu) = \phi(\mu, N) \int_{\mathcal{M}_{Q,M}} e^{it(N^2-|\mu|^2+|\rho|^2)}\; d\left(\frac{t}{\mathcal{T}}\right), \text{ for all } \mu \in \Lambda^+.$$

Thus

$$|\widehat{\mathcal{K}_{Q,M}}(\mu)| \lesssim |\mathcal{M}_{Q,M}| \lesssim \frac{Q^2}{NM},$$

which implies

$$\|f * \mathcal{K}_{Q,M}\|_{L^2(U)} \lesssim \frac{Q^2}{NM} \|f\|_{L^2(U)}. \tag{12.3}$$

Interpolating (12.2) with (12.3) for $\frac{\theta}{2} + \frac{1-\theta}{2u} = \frac{1}{p}$, we get

$$\|f * \mathcal{K}_{Q,M}\|_{L^p(U)} \lesssim N^{\left(d-\frac{d}{u}-\frac{r}{2}-1\right)(1-\theta)-\theta} M^{\left(\frac{r}{2}-1\right)(1-\theta)-\theta} Q^{\left(-\frac{r}{2}+2\right)(1-\theta)+2\theta} \|f\|_{L^{p'}(U)}.$$

We require the exponent of Q be negative, i.e.,

$$\left(-\frac{r}{2}+2\right)(1-\theta) + 2\theta < 0 \Leftrightarrow \theta < \frac{r-4}{r},$$

which implies that the exponent $\left(\frac{r}{2}-1\right)(1-\theta)-\theta$ of M is positive. Summing over the dyadic integers M and Q, we have

$$\|f * \mathcal{K}_\lambda\|_{L^p(U)} \lesssim N^{\left(d-\frac{d}{u}-2\right)(1-\theta)-2\theta} \|f\|_{L^{p'}(U)} = N^{d-2-\frac{2d}{p}} \|f\|_{L^{p'}(U)},$$

provided

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{2u} < \frac{r-4}{2r} + \frac{2}{ru} \Leftrightarrow p > \frac{2ur}{ur-4u+4}$$

for some $u > s$. This finishes the proof. □

13. Proof of Theorem 1.7

We provide another approach to the Weyl type exponential sum $\kappa_N^J(\mu, t, H)$ in Lemma 8.4 to prove Theorem 1.7. It is based on Bourgain's work [3] (see also [6]) and involves the Poisson summation formula, and bounds on oscillatory integrals and Kloosterman and Salié sums.

Proof. We inherit the notations as in the proof of Theorem 1.6. We will show that in addition to the estimate (12.1) on $\mathcal{K}_{Q,M}$, we also have for $u > s$

$$\|\mathcal{K}_{Q,M}\|_{L^u(U)} \lesssim_\varepsilon N^{d\left(1-\frac{1}{u}\right)-\frac{r}{2}-1+\varepsilon} M^{\frac{r}{2}-1} Q^{-\frac{r}{2}+\frac{3}{2}},$$

which albeit an N^ε loss lowers the power of Q by 1/2. By the same interpolation argument in the remaining proof of Theorem 1.6, this then yields

$$\|f * \mathcal{K}_\lambda\|_{L^p(U)} \lesssim_\varepsilon N^{d-2-\frac{2d}{p}+\varepsilon} \|f\|_{L^{p'}(U)}$$

for any $p > \frac{2s(r+1)}{sr-3s+4}$ and $r \geq 4$. Let

$$\frac{t}{\mathcal{T}} = \frac{a}{q} + \gamma.$$

We wish to perform Poisson summation to the Weyl type sum $\kappa_N^J(\mu, t, H)$ in Lemma 8.4. Using the notations there we write

$$(\lambda_1 + \lambda_2)(H_J^\perp) = 2\pi i(n_1 x_1 + \cdots + n_r x_r),$$

for some $(x_1, \ldots, x_r) \in \mathbb{R}^r$ depending on $H_J^\perp$, and

$$|\lambda_1 + \lambda_2|^2 = \frac{2\pi}{\mathcal{T}} \sum_{1 \leq i,j \leq n} a_{ij} n_i n_j$$

for some integral positive-definite symmetric matrix (a_{ij}) out of the Killing form, and

$$2(\mu, \lambda_1 + \lambda_2) = \frac{2\pi}{\mathcal{T}}(n_1 b_1 + \cdots + n_r b_r)$$

for some $(b_1, \ldots, b_r) \in \mathbb{Z}^r$ depending on $\mu \in {}^J\Lambda/{}^J\Gamma$. Now put

$$n_j = r_j q + k_j, \ \ k_j = 0, 1, \ldots, q-1.$$

Using the standard notation $e^{2\pi i \cdot} = e(\cdot)$, we get

$$\begin{aligned}\kappa_N^J(\mu, t, H) = \sum_{\substack{k_j = 0,1,\ldots,q-1 \\ j=1,\ldots,r}} \Bigg[& e\left(-\frac{a}{q}\left(\sum_{i,j} a_{ij} k_i k_j + \sum_j k_j b_j\right)\right) \sum_{\substack{r_j \in \mathbb{Z} \\ j=1,\ldots,r}} \\ & e\left(\sum_j (r_j q + k_j) x_j - \gamma\left(\sum_{ij} a_{ij}(r_i q + k_i)(r_j q + k_j) + \sum_j (r_j q + k_j) b_j\right)\right) \\ & \cdot P(\mu, r_1 q + k_1, \ldots, r_r q + k_r, H)\Bigg].\end{aligned}$$

Apply Poisson summation in r dimensions and change of variables, we have

$$\begin{aligned}\kappa_N^J(\mu, t, H) = \sum_{\substack{m_j \in \mathbb{Z} \\ j=1,\ldots,r}} & \underbrace{\left\{\frac{1}{q} \sum_{\substack{k_j = 0,1,\ldots,q-1 \\ j=1,\ldots,r}} e\left(-\left(\frac{a}{q}\sum_{i,j} a_{ij} k_i k_j + \frac{1}{q}\sum_j k_j (a b_j + m_j)\right)\right)\right\}}_{=:S(a, a\mathbf{b}+\mathbf{m}; q)} \\ & \underbrace{\left\{\int_{\mathbb{R}^r} e\left(\sum_j y_j (x_j + m_j/q) - \gamma\left(\sum_{ij} a_{ij} y_i y_j + \sum_j y_j b_j\right)\right) P(\mu, y_1, \ldots, y_r, H)\ dy_1 \cdots dy_r\right\}}_{=:J(\mathbf{x}, \gamma, \mathbf{m}; q)}.\end{aligned}$$

Here we have adopted the notation of a bold case to denote row vectors: $\mathbf{b} := (b_1, \ldots, b_r)$, $\mathbf{m} := (m_1, \ldots, m_r)$. Now using (8.7) and applying the standard asymptotics of the oscillatory integral $J(\mathbf{x}, \gamma, \mathbf{m}; q)$ [28, Chapter VIII section 5.1], we get

$$|J(\mathbf{x}, \gamma, \mathbf{m}; q)| \lesssim N^{|\Sigma^+| + |\Sigma_J^+|} \min\{N^r, |\gamma|^{-\frac{r}{2}}\}. \tag{13.1}$$

Moreover, for each $\varepsilon > 0$, if any of the m_j satisfies $|x_j q + m_j| \geq N^\varepsilon$, then an integration by parts shows that

$$|J(\mathbf{x}, \gamma, \mathbf{m}; q)| \lesssim_{M,\varepsilon} N^{-M}$$

for all $M > 0$. This will produce a negligible contribution and we may now assume that in the summation $\sum_{m_j}$ only at most N^{ε} values of m_j have to be considered for each $j = 1, \ldots, r$.

Now for $H \in P_{I,J}$, we may write

$$\begin{aligned}\mathcal{K}_{Q,M}(H) &= \frac{1}{|W_J|\delta_I\delta_{I,J}}\sum_{\mu} a(t,\mu,H)\int_{\mathcal{M}_{Q,M}} \kappa_N^J(\mu,t,H)e^{itN^2}\ d\left(\frac{t}{\mathcal{T}}\right)\\ &= \frac{1}{|W_J|\delta_I\delta_{I,J}}\sum_{\mu} a(t,\mu,H)\underbrace{\int_{\mathcal{M}_{Q,M}} \sum_{\substack{m_j\in\mathbb{Z}\\ j=1,\ldots,r}} S(a,a\mathbf{b}+\mathbf{m};q)e\left((a/q)n_0\right)J(\mathbf{x},\gamma,\mathbf{m};q)e\left(\gamma n_0\right)\ d\left(\frac{t}{\mathcal{T}}\right)}_{=:\kappa_{Q,M}(\mu,H)}\end{aligned}$$

where $n_0 = (\mathcal{T}/2\pi)N^2 \in \mathbb{Z}$. In performing the integral over $\mathcal{M}_{Q,M}$, we first sum over a ($0 \le a < q$, $(a,q)=1$), then sum over q ($Q \le q < 2Q$), and last integrate over γ. It is checked that the expression

$$\mathscr{S}(q,n_0) := \sum_{a,\ (a,q)=1} S(a,a\mathbf{b}+\mathbf{m};q)e((a/q)n_0) \tag{13.2}$$

is multiplicative in q, i.e.,

$$\mathscr{S}(q_1q_2,n_0) = \mathscr{S}(q_1,n_0)\cdot\mathscr{S}(q_2,n_0),\ \text{for } \gcd(q_1,q_2)=1.$$

By adapting the method of Weyl differencing [18, Theorem 8.1] to high dimensions (using the key non-degeneracy of the matrix $\mathbf{A} = (a_{ij})$; compare with Lemma 7.4 of [31] for the slightly different version with smooth cutoff), we may establish

$$|S(a,a\mathbf{b}+\mathbf{m};q)| \lesssim_{\varepsilon} q^{-r/2+\varepsilon}, \tag{13.3}$$

which gives the crude estimate

$$|\mathscr{S}(q,n_0)| \lesssim_{\varepsilon} q^{-r/2+1+\varepsilon}. \tag{13.4}$$

This ε might be eliminated for some cases, but we will not need it. For a more refined estimate, we now assume q is a large enough prime, to involve the Kloosterman and Salié sums. We complete the square:

$$\frac{a}{q}\sum_{i,j} a_{ij}k_ik_j + \frac{1}{q}\sum_{j} k_j(ab_j+m_j) \equiv \frac{a}{q}\sum_{i,j} a_{ij}(k_i+l_i)(k_j+l_j) - \frac{a}{q}\sum_{i,j} a_{ij}l_il_j,\ \pmod 1$$

for some $\mathbf{l} = (l_1,\ldots,l_r) \in \mathbb{Z}^r$. A calculation then shows

$$\mathbf{l} = 2^*(\mathbf{b}+a^*\mathbf{m})\mathbf{A}^*$$

satisfies the above equation. Here a^* and $\mathbf{A}^*$ are inverses of a and $\mathbf{A}$ respectively over the residue field $\mathbb{F}_q$, which always exist when q is large enough. We also have

$$S(a,0;q) = \left(\frac{a}{q}\right)^r S(1,0;q)$$

where $\left(\frac{a}{q}\right)$ is the Legendre symbol; this may be established by diagonalizing the quadratic form associated to the non-degenerate matrix $\mathbf{A}$ in $\mathbb{F}_q$, thus reducing to a similar identity for one-dimensional Gauss sums. (13.2) now becomes

$$\mathscr{S}(q,n_0) = e\left(\frac{2^*\mathbf{m}\mathbf{A}^*\mathbf{b}^T}{q}\right)S(1,0;q)\sum_{a=1}^{q-1}\left(\frac{a}{q}\right)^r e\left(\frac{a(4^*\mathbf{b}\mathbf{A}^*\mathbf{b}^T+n_0)}{q}+\frac{a^*(4^*\mathbf{m}\mathbf{A}^*\mathbf{m}^T)}{q}\right).$$

We recall for prime q the Weil bound for Kloosterman sums

$$\left|\sum_{a=1}^{q-1} e\left(\frac{am}{q}+\frac{a^*n}{q}\right)\right| \leq 2\sqrt{\gcd(m,n,q)}\sqrt{q},$$

and the standard bound for Salié sums

$$\left|\sum_{a=1}^{q-1} \left(\frac{a}{q}\right) e\left(\frac{am}{q}+\frac{a^*n}{q}\right)\right| \leq 2\sqrt{q},$$

then we arrive at the bound

$$|\mathscr{S}(q,n_0)| \lesssim_\varepsilon q^{-r/2+1/2+\varepsilon}\sqrt{\gcd(n_1,q)},$$

where $n_1 = 4^*\mathbf{b}\mathbf{A}^*\mathbf{b}^T + n_0 \lesssim N^2$. Now for any positive integer q, write uniquely $q = q_1q_2q_3$, where q_1 is squarefree and coprime to n_1, q_2 is squarefree and divides n_1, q_3 is the product of prime powers of exponent at least two, and q_1, q_2, q_3 are coprime to each other. Note that we may further write uniquely $q_3 = q_4^2 q_5$ where q_5 is a squarefree divisor of q_4. Combine the above bound with the estimate (13.4) and the multiplicativity of $\mathscr{S}(q,n_0)$, we may estimate

$$|\mathscr{S}(q,n_0)| \lesssim_\varepsilon \frac{q^{-r/2+1+\varepsilon}}{q_1}.$$

Using (13.1), we arrive at

$$\begin{aligned}|\kappa_{Q,M}(\mu,H)| &\lesssim_\varepsilon N^{|\Sigma^+|+|\Sigma_J^+|+\varepsilon}(NM)^{r/2-1}\sum_{Q\leq q<2Q}\frac{q^{-r/2+1+\varepsilon}}{q_1}\\ &\lesssim_\varepsilon N^{|\Sigma^+|+|\Sigma_J^+|+\varepsilon}(NM)^{r/2-1}Q^{-r/2+1}\sum_{q_1<2Q,\ q_2|n_1,\ q_4\leq\left(\frac{Q}{q_1q_2}\right)^{1/2},\ q_5|q_4}\frac{1}{\sqrt{q_1}}.\end{aligned}$$

Apply the divisor bound $d(n) \lesssim_\varepsilon n^\varepsilon$, we then get

$$\begin{aligned}|\kappa_{Q,M}(\mu,H)| &\lesssim_\varepsilon N^{|\Sigma^+|+|\Sigma_J^+|+\varepsilon}(NM)^{r/2-1}Q^{-r/2+1}\sum_{q_1<2Q,\ q_2|n_1}\frac{1}{\sqrt{q_1}}\left(\frac{Q}{q_1q_2}\right)^{1/2+\varepsilon}\\ &\lesssim_\varepsilon N^{|\Sigma^+|+|\Sigma_J^+|+r/2-1+\varepsilon}M^{r/2-1}Q^{-r/2+3/2},\end{aligned}$$

which holds true uniformly for $H \in P_{I,J}$. Finally, using Weyl's integration formula and the barycentric-semiclassical subdivision again, we write

$$\|\mathcal{K}_{Q,M}\|_{L^u(U)} = \|\mathcal{K}_{Q,M}|\delta|^{\frac{2}{u}}\|_{L^u(A)} \lesssim \sum_{J\subset I}\|\mathcal{K}_{Q,M}|\delta|^{\frac{2}{u}}\|_{L^u(P_{I,J})}.$$

For $H \in P_{I,J}$, using $|a(t,\mu,H)| \lesssim 1$, we get

$$|\mathcal{K}_{Q,M}(H)|\cdot|\delta(H)|^{\frac{2}{u}} \lesssim \frac{|\delta^J(H)|^{\frac{2}{u}}}{|\delta_I(H)|^{1-\frac{2}{u}}|\delta_{I,J}(H)|^{1-\frac{2}{u}}}\cdot|\kappa_{Q,M}(\mu,H)|,$$

then we may use Lemma 3.2, inequality (3.3), and Remark 5.6 to conclude for any $u > s$

$$\|\mathcal{K}_{Q,M}\|_{L^u(U)} \lesssim_\varepsilon N^{d\left(1-\frac{1}{u}\right)-\frac{r}{2}-1+\varepsilon}M^{\frac{r}{2}-1}Q^{-\frac{r}{2}+\frac{3}{2}}.$$

□

14. Discussion on the optimal range

14.1. **On Strichartz estimates.** We now provide evidence for Conjecture 1.5 that the Strichartz estimate (1.11) should hold on any compact Lie group of rank $r \geq 2$ with canonical metrics for any $p > 2 + \frac{4}{d}$, which is the largest possible range except the endpoint, by studying class functions. As will be seen, Strichartz type estimates on tori become crucial. We first recall the following Strichartz estimates on tori, established by Bourgain and Demeter in [7, Theorem 2.4 and Remark 2.5] (see also [31] for the ε-removal).

Theorem 14.1. *Let $(\cdot,\cdot)$ be a positive-definite quadratic form with integral coefficients in r variables. Let B be a bounded domain in $\mathbb{R}^r$ and let I be a bounded interval. Then*

$$\Big\|\sum_{\xi\in\mathbb{Z}^r,\ |\xi|\leq N} a_\xi e^{it(\xi,\xi)+i(\xi,x)}\Big\|_{L^p((I\times B,\ dt\ dx))} \lesssim N^{\frac{r}{2}-\frac{r+2}{p}}\|a_\xi\|_{l^2(\mathbb{Z}^r)}$$

for all $p>\frac{2(r+2)}{r}$.

As a corollary, we have the following estimates for restriction to lower dimensional subsets.

Corollary 14.2. *Let $(\cdot,\cdot)$ be a positive-definite quadratic form with integral coefficients in r variables. Let $s=0,1,\ldots,r$. Pick any s-dimensional affine subspace $\mathbb{R}^s$ of $\mathbb{R}^r$ and let B_s be a bounded region in this $\mathbb{R}^s$. Then we have*

$$\Big\|\sum_{\xi\in\mathbb{Z}^r,\ |\xi|\leq N} a_\xi e^{it(\xi,\xi)+i(\xi,x_s)}\Big\|_{L^p(I\times B_s,\ dt\ dx_s)} \lesssim N^{\frac{r}{2}-\frac{s+2}{p}}\|a_\xi\|_{l^2(\mathbb{Z}^r)}$$

for any $p>\frac{2(r+2)}{r}$.

Proof. By integrity of the inner product $(\cdot,\cdot)$, we can pick vectors $v_1,\ldots,v_{r-s}\in\mathbb{R}^r$ such that $(\xi,v_j)\in\mathbb{Z}$ for any $\xi\in\mathbb{Z}^r$ and that $\bigoplus_{j=1}^{r-s}\mathbb{R}v_j$ is transversal to B_s in $\mathbb{R}^r$. Let $I_{r-s}:=\{\sum_{j=1}^{r-s}s_jv_j:\ s_j\in[0,1]\}$ and consider $B:=B_s\times I_{r-s}$. Then we have

$$\Big\|\sum_{|\xi|\leq N} a_\xi e^{it(\xi,\xi)+i(\xi,x_s)}\Big\|_{L^p(I\times B_s,\ dt\ dx_s)} \leq \Big\|\sum_{|\xi|\leq N} a_\xi e^{it(\xi,\xi)+i(\xi,x_s+\sum_{j=1}^{r-s}s_jv_j)}\Big\|_{L^p((I\times B_s,\ dt\ dx_s),L^\infty([0,1]^{r-s},\prod_{j=1}^{r-s}ds_j))}$$

Apply Bernstein's inequality on tori to the variables $s_1,\ldots,s_{r-s}$, the above is then bounded by

$$\begin{aligned}
&\lesssim N^{\frac{r-s}{p}}\Big\|\sum_{\xi\in\mathbb{Z}^r,\ |\xi|\leq N} a_\xi e^{it(\xi,\xi)+i(\xi,x_s+\sum_{j=1}^{r-s}s_jv_j)}\Big\|_{L^p((I\times B_s,\ dt\ dx_s),L^p([0,1]^{r-s},\prod_{j=1}^{r-s}ds_j))}\\
&\lesssim N^{\frac{r-s}{p}}\Big\|\sum_{\xi\in\mathbb{Z}^r,\ |\xi|\leq N} a_\xi e^{it(\xi,\xi)+i(\xi,x)}\Big\|_{L^p(I\times B,\ dt\ dx)} \lesssim N^{\frac{r}{2}-\frac{s+2}{p}}\|a_\xi\|_{l^2(\mathbb{Z}^r)},
\end{aligned}$$

for all $p>\frac{2(r+2)}{r}$, using Theorem 14.1. $\square$

As will be seen, the above Strichartz type estimates on tori would not be enough to derive the optimal range for compact Lie groups. We conjecture the following Strichartz estimates on tori for mixed Lebesgue norms.

Conjecture 14.3. *We have*

$$\Big\|\sum_{\xi\in\mathbb{Z}^r,\ |\xi|\leq N} a_\xi e^{it(\xi,\xi)+i(\xi,x)}\Big\|_{L^p((I,dt),L^q(B,dx))} \lesssim N^{\frac{r}{2}-\frac{2}{p}-\frac{r}{q}}\|a_\xi\|_{l^2(\mathbb{Z}^r)} \tag{14.1}$$

for all pairs $p,q\geq 2$ with $\frac{r}{2}-\frac{2}{p}-\frac{r}{q}>0$.

The above exponent of N is of course based on a scale-invariance consideration. On Euclidean spaces, Strichartz estimates for mixed Lebesgue norms as the above are indeed true, as proved in [12] and [20]. Arguing exactly as in the proof of Corollary 14.2, we also arrive at the following lemma.

Lemma 14.4. *Let $s = 0, 1, \dots, r$. Pick any s-dimensional affine subspace $\mathbb{R}^s$ of $\mathbb{R}^r$ and let B_s be a bounded region in this $\mathbb{R}^s$. Then the above conjecture implies that*

$$\left\| \sum_{\xi\in\mathbb{Z}^r,\ |\xi|\le N} a_\xi e^{it(\xi,\xi)+i(\xi,x_s)} \right\|_{L^p((I,dt),L^q(B_s,dx_s))} \lesssim N^{\frac{r}{2}-\frac{2}{p}-\frac{s}{q}} \|a_\xi\|_{l^2(\mathbb{Z}^r)} \tag{14.2}$$

for all pairs $p, q \ge 2$ with $\frac{r}{2} - \frac{2}{p} - \frac{r}{q} > 0$.

Theorem 14.1 was proved by the so-called decoupling theory, which is a Euclidean Fourier analytic tool developed in recent years. Although it has proved very successful in estimating exponential sums, the mixed-norm setting as in (14.1) seems to present new difficulties and is interesting by itself. We now provide the following solid evidence for Conjecture 1.5.

Theorem 14.5. *(i) The Strichartz estimate (1.11) holds for class functions on compact Lie groups for any $p > 2 + \frac{4}{r}$. (ii) Moreover, if we assume Conjecture 14.3 to hold, then (1.11) holds for class functions on compact Lie groups for any $p > 2 + \frac{4}{d}$ and $r \ge 2$.*

Proof. For the sake of simplicity of exposition, we assume that U is a compact simply connected simple Lie group. The general case may be established by slightly adapting the argument. By Schur's orthogonality relations, it is well known that with respect to the normalized Haar measure on U,

$$\|\chi_\mu\|_{L^2(U)} = 1,\ \forall \mu \in \Lambda^+.$$

Let $L^2_\sharp(U)$ denote the set of class functions in $L^2(U)$. Then $L^2_\sharp(U) \cong l^2(\Lambda^+)$, by

$$L^2_\sharp(U) \ni f = \sum_{\mu\in\Lambda^+} a_\mu \chi_\mu \mapsto (a_\mu)_{\mu\in\Lambda^+} \in l^2(\Lambda^+).$$

Apply Weyl's integration formula (2.1), then (1.11) is reduced to

$$\left\| \sum_{\mu\in\Lambda^+,|\mu|\le N} e^{-it|\mu|^2} a_\mu \chi_\mu |\delta|^{\frac{2}{p}} \right\|_{L^p(I\times A)} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}} \|a_\mu\|_{l^2(\Lambda^+)}. \tag{14.3}$$

Since $A = \bigsqcup_J P_J$, it suffices to prove the above estimate replacing A by each P_J. In the following, for a_μ initially defined for $\mu \in \Lambda^+$, we let $a_{s\mu} := a_\mu,\ \forall \mu \in \Lambda^+,\ s \in W$. The fact that

$$|\Sigma^+| = \frac{d-r}{2}$$

will be used in various places.

Case 1. $J = \varnothing$. We first treat part (i). Using the Weyl character formula

$$\chi_\mu(H) = \frac{\sum_{s\in W} \det s\ e^{(s\mu)(H)}}{\delta(H)}$$

and (3.5), we have

$$\left\| \sum_{\mu\in\Lambda^+,|\mu|\le N} e^{-it|\mu|^2} a_\mu \chi_\mu |\delta|^{\frac{2}{p}} \right\|_{L^p(I\times P_\varnothing)} \lesssim N^{|\Sigma^+|\left(1-\frac{2}{p}\right)} \sum_{s\in W} \left\| \sum_{\mu\in s\Lambda^+,|\mu|\le N} e^{-it|\mu|^2+\mu(H)} a_\mu \right\|_{L^p(I\times P_\varnothing)}.$$

Here we have used $|s\mu| = |\mu|$, $\forall s \in W$, $\mu \in \Lambda$. If we apply Theorem 14.1 to the sum on the right inside of $\|\cdot\|_{L^p}$, then we have that for any $p > 2 + \frac{4}{r}$

$$\begin{aligned}\Bigg\| \sum_{\mu \in \Lambda^+, |\mu| \le N} e^{-it|\mu|^2} a_\mu \chi_\mu |\delta|^{\frac{2}{p}} \Bigg\|_{L^p(I \times P_\varnothing)} &\lesssim N^{|\Sigma^+|\left(1-\frac{2}{p}\right)+\frac{r}{2}-\frac{r+2}{p}} \|a_\mu\|_{l^2(\Lambda^+)} \\ &\lesssim N^{\frac{d}{2}-\frac{d+2}{p}} \|a_\mu\|_{l^2(\Lambda^+)}.\end{aligned}$$

For part (ii), we need to exploit L^p estimates of the weight functions as in Proposition 5.4. Write $P_\varnothing = \bigcup_{|I|=r} P_{I,\varnothing}$. Writing $\delta = \delta_I \cdot \delta_{I,J}$ and using the character formula again, we estimate using (3.2)

$$\begin{aligned}&\Bigg\| \sum_{\mu \in \Lambda^+, |\mu| \le N} e^{-it|\mu|^2} a_\mu \chi_\mu |\delta|^{\frac{2}{p}} \Bigg\|_{L^p(I \times P_{I,\varnothing})} \\ &\lesssim \sum_{s \in W} \Bigg\| \sum_{\mu \in s\Lambda^+, |\mu| \le N} e^{-it|\mu|^2 + \mu(H)} a_\mu \Bigg\|_{L^p(I, L^q(P_{I,\varnothing}))} \cdot \Bigg\| \frac{1}{|\delta_{I,\varnothing}|^{1-\frac{2}{p}}} \Bigg\|_{L^u(P_{I,\varnothing})}.\end{aligned}$$

Here

$$\frac{1}{u} = \frac{1}{p} - \frac{1}{q}.$$

Using the conjectured (14.1) and Proposition 5.4, the above is bounded by

$$\lesssim N^{\frac{r}{2}-\frac{2}{p}-\frac{r}{q}+\frac{d-r}{2}\left(1-\frac{2}{p}\right)-\frac{r}{u}} \|a_\mu\|_{l^2} = N^{\frac{d}{2}-\frac{d+2}{p}} \|a_\mu\|_{l^2}$$

provided the conditions hold

$$2 \le p \le q,\ \frac{r}{2} - \frac{2}{p} - \frac{r}{q} \ge 0,\ \frac{d-r}{2}\left(1 - \frac{2}{p}\right) - r\left(\frac{1}{p} - \frac{1}{q}\right) > 0.$$

An inspection of the above inequalities in the $\left(\frac{1}{p}, \frac{1}{q}\right)$ plane shows that any $p > 2 + \frac{4}{d}$ is admissible.

Case 2. $|J| \ge 1$. We first treat part (i). Apply formula (4.2), we have

$$\Bigg| \sum_{\mu \in \Lambda^+, |\mu| \le N} e^{-it|\mu|^2} a_\mu \chi_\mu |\delta|^{\frac{2}{p}} \Bigg| = \frac{|\delta^J|^{\frac{2}{p}}}{|W_J| \cdot |\delta_I|^{1-\frac{2}{p}} |\delta_{I,J}|^{1-\frac{2}{p}}} \sum_{s \in W} \Bigg| \sum_{\mu \in s\Lambda^+, |\mu| \le N} e^{-it|\mu|^2 + \mu(H_J^\perp)} a_\mu \chi_{\mu_J}^J(H_J) \Bigg|. \tag{14.4}$$

We wish to apply Corollary 14.2. Let

$$\mathfrak{e}_J := \{X \in \mathfrak{t}_J :\ 0 \le \alpha_j(X)/2\pi i + \delta_{0j} \le N^{-1}\ \forall j \in J\}. \tag{14.5}$$

For $X \in \mathfrak{e}_J$, let

$$\mathfrak{e}_J^\perp(X) := \{Y \in \mathfrak{t}_J^\perp :\ X + Y \in P_J\}.$$

Then we may express the polytope P_J as

$$P_J = \{X + Y :\ Y \in \mathfrak{e}_J^\perp(X),\ X \in \mathfrak{e}_J\}.$$

We decompose the Lebesgue measure on $\mathfrak{t}$ into the product of the measure on $\mathfrak{t}_J^\perp$ and that on $\mathfrak{t}_J$. Fubini's theorem gives

$$\begin{aligned}
&\left\|\sum_{\mu\in s\Lambda^+,|\mu|\leq N} e^{-it|\mu|^2+\mu(H_J^\perp)}a_\mu\chi_{\mu_J}^J(H_J)\right\|_{L^p(I\times P_J)} \\
&=\left\|\left\|\sum_{\mu\in s\Lambda^+,|\mu|\leq N} e^{-it|\mu|^2+\mu(Y)}a_\mu\chi_{\mu_J}^J(X)\right\|_{L^p(I\times\mathfrak{e}_J^\perp(X),\ dt\ dY)}\right\|_{L^p(\mathfrak{e}_J,\ dX)}.
\end{aligned} \tag{14.6}$$

Note that $\mathfrak{e}_J^\perp(X)$ is a bounded region in the $(r-|J|)$-dimensional subspace $\mathfrak{t}_J^\perp$ of $\mathfrak{t}$. We apply Corollary 14.2 to obtain for $p>2+\frac{4}{r}$ that

$$\left\|\sum_{\mu\in s\Lambda^+,|\mu|\leq N} e^{-it|\mu|^2+\mu(H_J^\perp)}a_\mu\chi_{\mu_J}^J(H_J)\right\|_{L^p(I\times P_J)}\lesssim N^{\frac{r}{2}-\frac{r-|J|+2}{p}}\left\|\left\|a_\mu\chi_{\mu_J}^J(X)\right\|_{l^2(\mathbb{Z}^r)}\right\|_{L^p(\mathfrak{e}_J,\ dX)}.$$

Now Lemma 6.6 gives $|\chi_{\mu_J}^J(X)|\lesssim N^{|\Sigma_J^+|}$ for any $X\in\mathfrak{e}_J$; observing that the measure of $\mathfrak{e}_J$ in $\mathfrak{t}_J$ is $\asymp N^{-|J|}$, then the above is bounded by $\lesssim N^{|\Sigma_J^+|+\frac{r}{2}-\frac{r+2}{p}}\|a_\mu\|_{l^2}$. Combined with Lemma 3.5, this then implies that (14.4) is bounded by

$$\lesssim N^{|\Sigma_J^+|+\frac{r}{2}-\frac{r+2}{p}-|\Sigma_J^+|\frac{2}{p}+(|\Sigma^+|-|\Sigma_J^+|)\left(1-\frac{2}{p}\right)}\|a_\mu\|_{l^2}=N^{\frac{d}{2}-\frac{d+2}{p}}\|a_\mu\|_{l^2}$$

for all $p>2+\frac{4}{r}$. For part (ii), write $P_J=\bigcup_{I\supset J,|I|=r}P_{I,J}$. For $X\in\mathfrak{e}_J$, let

$$\mathfrak{e}_{I,J}^\perp(X):=\{Y\in\mathfrak{t}_J^\perp:\ X+Y\in P_{I,J}\}.$$

Then

$$P_{I,J}=\{X+Y:\ Y\in\mathfrak{e}_{I,J}^\perp(X),\ X\in\mathfrak{e}_J\}.$$

A similar Fubini identity as (14.6) holds, replacing P_J by $P_{I,J}$ and $\mathfrak{e}_J^\perp(X)$ by $\mathfrak{e}_{I,J}^\perp(X)$. Using this Fubini identity, (14.4), (3.2) and (3.3), we estimate

$$\begin{aligned}
&\left\|\sum_{\mu\in\Lambda^+,|\mu|\leq N} e^{-it|\mu|^2}a_\mu\chi_\mu|\delta|^{\frac{2}{p}}\right\|_{L^p(I\times P_{I,J})} \\
&\lesssim\sum_{s\in W}N^{-|\Sigma_J^+|\frac{2}{p}}\left\|\left\|\sum_{\mu\in s\Lambda^+,|\mu|\leq N} e^{-it|\mu|^2+\mu(Y)}a_\mu\chi_{\mu_J}^J(X)\right\|_{L^p(I,L^q(\mathfrak{e}_{I,J}^\perp(X)))}\cdot\left\|\frac{1}{|\delta_{I,J}|^{1-\frac{2}{p}}}\right\|_{L^u(\mathfrak{e}_J^\perp(X))}\right\|_{L^p(\mathfrak{e}_J)}.
\end{aligned}$$

Here $\frac{1}{u}=\frac{1}{p}-\frac{1}{q}$. Since $\mathfrak{e}_{I,J}^\perp(X)$ is a bounded region in the $(r-|J|)$-dimensional subspace $\mathfrak{t}_J^\perp$ of $\mathfrak{t}$, assuming the conjectured estimates (14.2), we have that the above is bounded by

$$\sum_{s\in W}N^{-|\Sigma_J^+|\frac{2}{p}+\frac{r}{2}-\frac{2}{p}-\frac{r-|J|}{q}}\left\|\left\|a_\mu\chi_{\mu^J}^J(X)\right\|_{l_\mu^2}\cdot\left\|\frac{1}{|\delta_{I,J}|^{1-\frac{2}{p}}}\right\|_{L^u(\mathfrak{e}_{I,J}^\perp(X))}\right\|_{L^p(\mathfrak{e}_J)}.$$

By again Lemma 6.6, the above is bounded by

$$\lesssim N^{|\Sigma_J^+|\left(1-\frac{2}{p}\right)+\frac{r}{2}-\frac{2}{p}-\frac{r-|J|}{q}}\|a_\mu\|_{l^2}\cdot\left\|\left\|\frac{1}{|\delta_{I,J}|^{1-\frac{2}{p}}}\right\|_{L^u(\mathfrak{e}_{I,J}^\perp(X))}\right\|_{L^p(\mathfrak{e}_J)}$$

which is bounded via Hölder's inequality by

$$\lesssim N^{|\Sigma_J^+|(1-\frac{2}{p})+\frac{r}{2}-\frac{2}{p}-\frac{r-|J|}{q}}\|a_\mu\|_{l^2}\cdot\left\|\left\|\frac{1}{|\delta_{I,J}|^{1-\frac{2}{p}}}\right\|_{L^u(\mathfrak{e}_{I,J}^\perp(X))}\right\|_{L^u(\mathfrak{e}_J)}\cdot\|1\|_{L^q(\mathfrak{e}_J)}$$

$$\lesssim N^{|\Sigma_J^+|(1-\frac{2}{p})+\frac{r}{2}-\frac{2}{p}-\frac{r-|J|}{q}-\frac{|J|}{q}}\|a_\mu\|_{l^2}\cdot\left\|\frac{1}{|\delta_{I,J}|^{1-\frac{2}{p}}}\right\|_{L^u(P_J)},$$

which is then bounded via Proposition 5.4 by

$$\lesssim N^{|\Sigma_J^+|(1-\frac{2}{p})+\frac{r}{2}-\frac{2}{p}-\frac{r-|J|}{q}-\frac{|J|}{q}+(|\Sigma^+|-|\Sigma_J^+|)(1-\frac{2}{p})-\frac{r}{u}}\|a_\mu\|_{l^2}=N^{\frac{d}{2}-\frac{d+2}{p}}\|a_\mu\|_{l^2}.$$

In the application of (14.2) and Proposition 5.4, we have assumed the following conditions

$$2\le p\le q,\ \frac{r}{2}-\frac{2}{p}-\frac{r}{q}\ge 0,\ \frac{d-r}{2}\left(1-\frac{2}{p}\right)-r\left(\frac{1}{p}-\frac{1}{q}\right)>0.$$

These coincide with those obtained in Case 1 and any $p>2+\frac{4}{d}$ is still admissible.

□

14.2. **On eigenfunction bounds.** Following a similar line of treatment as for Strichartz estimates, we provide evidence of Conjecture 1.8 by showing how this conjecture specialized to class functions on compact Lie groups could be deduced from the following conjectured eigenfunction bounds on tori by Bourgain [3].

Conjecture 14.6. *Let B be a bounded region in $\mathbb{R}^r$. For $r\ge 3$, it holds that*

$$\left\|\sum_{\xi\in\mathbb{Z}^r,\ |\xi|=N}a_\xi e^{i(\xi,x)}\right\|_{L^p(B,dx)}\lesssim N^{\frac{r-2}{2}-\frac{r}{p}}\|a_\xi\|_{l^2(\mathbb{Z}^r)}\tag{14.7}$$

for any $p>\frac{2r}{r-2}$, with an N^ε loss if $r=3,4$.

Similar to the discussion of Strichartz estimates, the above conjecture implies the following estimate for restriction to lower dimensional subsets, by an argument similar to the proof of Corollary 14.2.

Lemma 14.7. *Let $s=0,1,\dots,r$. Pick any s-dimensional affine subspace $\mathbb{R}^s$ of $\mathbb{R}^r$ and let B_s be a bounded region in this $\mathbb{R}^s$. Then (14.7) implies*

$$\left\|\sum_{\xi\in\mathbb{Z}^r,\ |\xi|=N}a_\xi e^{i(\xi,x_s)}\right\|_{L^p(B_s,dx_s)}\lesssim N^{\frac{r-2}{2}-\frac{s}{p}}\|a_\xi\|_{l^2(\mathbb{Z}^r)}$$

for any $p>\frac{2r}{r-2}$, with an N^ε loss if $r=3,4$.

Remark 14.8. *The following $r=2$ version of the above estimates indeed holds. We have for $s=0,1,2$*

$$\left\|\sum_{\xi\in\mathbb{Z}^2,\ |\xi|=N}a_\xi e^{i(\xi,x_s)}\right\|_{L^\infty(B_s,dx_s)}\lesssim_\varepsilon N^\varepsilon\|a_\xi\|_{l^2(\mathbb{Z}^r)},$$

by an application of the counting estimate $\#\{\mu\in\mathbb{Z}^2:\ |\mu|=N\}\lesssim_\varepsilon N^\varepsilon$ (see Lemma 8 in [1]).

We are ready to provide the following solid evidence for Conjecture 1.8.

Theorem 14.9. *For rank $r=2$, Conjecture 1.8 holds for class eigenfunctions on compact Lie groups. For $r\ge 3$, Conjecture 14.6 implies Conjecture 1.8 for class eigenfunctions on compact Lie groups.*

Proof. The proof is similar to that of Theorem 14.5 and we present it in detail for the sake of completeness. Class eigenfunctions f of eigenvalue $-N^2+|\rho|^2$ can be expressed as

$$f=\sum_{\mu\in\Lambda^+,\ |\mu|=N} a_\mu\chi_\mu.$$

Using Weyl's integration formula (2.1), inequality (1.13) reads

$$\left\|\sum_{\mu\in\Lambda^+,\ |\mu|=N} a_\mu\chi_\mu|\delta|^{\frac{2}{p}}\right\|_{L^p(A)}\lesssim_\varepsilon N^{\frac{d-2}{2}-\frac{d}{p}+\varepsilon}\|a_\mu\|_{l^2(\Lambda^+)}.$$

Recalling the decomposition $A=\bigcup_{J\subset I,|I|=r}P_{I,J}$, the above estimate reduces to those replacing A by each $P_{I,J}$.

Case 1. $J=\varnothing$. Write

$$\left|\sum_{\mu\in\Lambda^+,|\mu|=N} a_\mu\chi_\mu|\delta|^{\frac{2}{p}}\right|=\left|\frac{1}{|\delta_I|^{1-\frac{2}{p}}|\delta_{I,\varnothing}|^{1-\frac{2}{p}}}\sum_{s\in W}\det s\sum_{\mu\in s\Lambda^+,|\mu|=N}a_\mu e^{\mu}\right|.$$

Using (3.2), we estimate for

$$\frac{1}{p}=\frac{1}{u}+\frac{1}{v}$$

that

$$\begin{aligned}\left\|\sum_{\mu\in\Lambda^+,|\mu|=N} a_\mu\chi_\mu|\delta|^{\frac{2}{p}}\right\|_{L^p(P_{I,\varnothing})}&\lesssim\sum_{s\in W}\left\|\sum_{\mu\in s\Lambda^+,|\mu|=N}a_\mu e^{\mu}\right\|_{L^u(P_{I,\varnothing})}\left\|\frac{1}{|\delta_{I,\varnothing}|^{1-\frac{2}{p}}}\right\|_{L^v(P_{I,\varnothing})}\\&\lesssim_\varepsilon N^{\frac{r-2}{2}-\frac{r}{u}+\frac{d-r}{2}\cdot\left(1-\frac{2}{p}\right)-\frac{r}{v}+\varepsilon}\|a_\mu\|_{l^2(\Lambda^+)}=N^{\frac{d-2}{2}-\frac{d}{p}+\varepsilon}\|a_\mu\|_{l^2(\Lambda^+)},\end{aligned}$$

where we also used the conjectured estimate (14.7) (and Remark 14.8) and Proposition 5.4, provided the following necessary conditions hold

$$u>\frac{2r}{r-2}\ (u=\infty\text{ if }r=2),\ \left(1-\frac{2}{p}\right)/\left(\frac{1}{p}-\frac{1}{u}\right)>\frac{2r}{d-r},\ u\geq p\geq 2. \tag{14.8}$$

An inspection shows any $p>\frac{2d}{d-2}$ is admissible.

Case 2. $|J|\geq 1$. Using (4.2), we write

$$\left|\sum_{\mu\in\Lambda^+,|\mu|=N} a_\mu\chi_\mu|\delta|^{\frac{2}{p}}\right|=\frac{|\delta^J|^{\frac{2}{p}}}{|W_J|\cdot|\delta_I|^{1-\frac{2}{p}}|\delta_{I,J}|^{1-\frac{2}{p}}}\left|\sum_{s\in W}\det s\sum_{\mu\in s\Lambda^+,|\mu|=N}a_\mu e^{\mu(H_J^\perp)}\chi^J_{\mu_J}(H_J)\right|.$$

For $\frac{1}{p} = \frac{1}{u} + \frac{1}{v}$, we estimate using (3.2) and (3.3) that

$$\begin{aligned}
&\left\| \sum_{\mu \in \Lambda^+, |\mu|=N} a_\mu \chi_\mu |\delta|^{\frac{2}{p}} \right\|_{L^p(P_{I,J})} \\
&\lesssim N^{-|\Sigma_J^+|\frac{2}{p}} \sum_{s \in W} \left\| \sum_{\mu \in s\Lambda^+, |\mu|=N} a_\mu e^{\mu(H_J^\perp)} \chi^J_{\mu_J}(H_J) \right\|_{L^u(P_{I,J})} \left\| \frac{1}{|\delta_{I,J}|^{1-\frac{2}{p}}} \right\|_{L^v(P_{I,J})} \\
&\lesssim N^{-|\Sigma_J^+|\frac{2}{p}} \sum_{s \in W} \left\| \left\| \sum_{\mu \in s\Lambda^+, |\mu|=N} a_\mu e^{\mu(Y)} \chi^J_{\mu_J}(X) \right\|_{L^u(\mathfrak{e}_{I,J}^\perp(X),\ dY)} \right\|_{L^u(\mathfrak{e}_J,\ dX)} \left\| \frac{1}{|\delta_{I,J}|^{1-\frac{2}{p}}} \right\|_{L^v(P_{I,J})} \\
&\lesssim_\varepsilon N^{-|\Sigma_J^+|\frac{2}{p} + \frac{r-2}{2} - \frac{r-|J|}{u} - \frac{|J|}{u} + |\Sigma_J^+| + \left(|\Sigma^+| - |\Sigma_J^+|\right)(1-\frac{2}{p}) - \frac{r}{v} + \varepsilon} \|a_\mu\|_{l^2(\Lambda^+)} = N^{\frac{d-2}{2} - \frac{d}{p} + \varepsilon} \|a_\mu\|_{l^2(\Lambda^+)}.
\end{aligned}$$

Here we have used Lemma 14.7, Lemma 6.6, Proposition 5.4, and the estimate $\|1\|_{L^u(\mathfrak{e}_J,\ dX)} \lesssim N^{-\frac{|J|}{u}}$. In applying these estimates, we assumed the following conditions to hold

$$u > \frac{2r}{r-2} \ (u = \infty \text{ if } r = 2), \ \left(1 - \frac{2}{p}\right) / \left(\frac{1}{p} - \frac{1}{u}\right) > \frac{2r}{d-r}, \ u \geq p \geq 2.$$

These are the same conditions as in Case 1 and any $p > \frac{2d}{d-2}$ is admissible.

□

References

[1] Blomer, V., and Pohl, A. The sup-norm problem on the Siegel modular space of rank two. *Amer. J. Math. 138*, 4 (2016), 999–1027.

[2] Bourbaki, N. *Lie groups and Lie algebras. Chapters 4–6*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley.

[3] Bourgain, J. Eigenfunction bounds for the Laplacian on the n-torus. *Internat. Math. Res. Notices*, 3 (1993), 61–66.

[4] Bourgain, J. Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations. *Geom. Funct. Anal. 3*, 2 (1993), 107–156.

[5] Bourgain, J. Moment inequalities for trigonometric polynomials with spectrum in curved hypersurfaces. *Israel J. Math. 193*, 1 (2013), 441–458.

[6] Bourgain, J., and Demeter, C. Improved estimates for the discrete Fourier restriction to the higher dimensional sphere. *Illinois J. Math. 57*, 1 (2013), 213–227.

[7] Bourgain, J., and Demeter, C. New bounds for the discrete Fourier restriction to the sphere in 4D and 5D. *Int. Math. Res. Not. IMRN*, 11 (2015), 3150–3184.

[8] Bourgain, J., and Demeter, C. The proof of the l^2 decoupling conjecture. *Ann. of Math. (2) 182*, 1 (2015), 351–389.

[9] Burq, N., Gérard, P., and Tzvetkov, N. Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds. *Amer. J. Math. 126*, 3 (2004), 569–605.

[10] Demeter, C. *Fourier restriction, decoupling, and applications*, vol. 184 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2020.

[11] Dooley, A. H. Norms of characters and lacunarity for compact Lie groups. *J. Functional Analysis 32*, 2 (1979), 254–267.

[12] Ginibre, J., and Velo, G. Generalized Strichartz inequalities for the wave equation. *J. Funct. Anal. 133*, 1 (1995), 50–68.

[13] Hardy, G. H., and Wright, E. M. *An introduction to the theory of numbers*, sixth ed. Oxford University Press, Oxford, 2008. Revised by D. R. Heath-Brown and J. H. Silverman, With a foreword by Andrew Wiles.

[14] Harish-Chandra. Differential operators on a semisimple Lie algebra. *Amer. J. Math. 79* (1957), 87–120.

[15] Hassell, A., and Tacy, M. Improvement of eigenfunction estimates on manifolds of nonpositive curvature. *Forum Math. 27*, 3 (2015), 1435–1451.

[16] Herr, S. The quintic nonlinear Schrödinger equation on three-dimensional Zoll manifolds. *Amer. J. Math. 135*, 5 (2013), 1271–1290.

[17] Humphreys, J. E. *Reflection groups and Coxeter groups*, vol. 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[18] Iwaniec, H., and Kowalski, E. *Analytic number theory*, vol. 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[19] Iwaniec, H., and Sarnak, P. L^∞ norms of eigenfunctions of arithmetic surfaces. *Ann. of Math. (2) 141*, 2 (1995), 301–320.

[20] Keel, M., and Tao, T. Endpoint Strichartz estimates. *Amer. J. Math. 120*, 5 (1998), 955–980.

[21] Marshall, S. L^p norms of higher rank eigenfunctions and bounds for spherical functions. *J. Eur. Math. Soc. 18*, 7 (2016), 1437–1493.

[22] Marshall, S. Upper bounds for Maass forms on semisimple groups, 2017. arXiv:1405.7033.

[23] Sarnak, P. Letter to C. Morawetz on bounds for eigenfunctions on symmetric spaces. `http://publications.ias.edu/sarnak`.

[24] Shi, J. Y. Alcoves corresponding to an affine Weyl group. *J. London Math. Soc. (2) 35*, 1 (1987), 42–55.

[25] Sogge, C. D. Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds. *J. Funct. Anal. 77*, 1 (1988), 123–138.

[26] Staffilani, G., and Tataru, D. Strichartz estimates for a Schrödinger operator with nonsmooth coefficients. *Comm. Partial Differential Equations 27*, 7-8 (2002), 1337–1372.

[27] Stanton, R. J., and Tomas, P. A. Polyhedral summability of Fourier series on compact Lie groups. *Amer. J. Math. 100*, 3 (1978), 477–493.

[28] Stein, E. M. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, vol. 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[29] Tomas, P. A. A restriction theorem for the Fourier transform. *Bull. Amer. Math. Soc. 81* (1975), 477–478.

[30] Varadarajan, V. S. *Lie groups, Lie algebras, and their representations*, vol. 102 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1984. Reprint of the 1974 edition.

[31] Zhang, Y. Strichartz estimates for the Schrödinger flow on compact Lie groups. *Anal. PDE 13*, 4 (2020), 1173–1219.

[32] Zygmund, A. On Fourier coefficients and transforms of functions of two variables. *Studia Math. 50* (1974), 189–201.

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