Abstract. In this note, we prove “scale-invariant” $L^p$-estimates for the Schrödinger kernel on compact semisimple groups for major arcs of the time variable and give two applications. The first application is to improve the range of exponent for scale-invariant Strichartz estimates on compact semisimple groups. For such a group $M$ of dimension $d$ and rank $r$, let $s$ be the largest among the numbers $2d_0/(d_0 - r_0)$, where $d_0, r_0$ are respectively the dimension and rank of a simple factor of $M$. We establish

$$
\|e^{it\Delta}f\|_{L^p(I \times M)} \lesssim \|f\|_{H^{d/2-(d+2)/p}(M)}
$$

for $p > 2 + 8(s - 1)/sr$ when $r \geq 2$. The second application is to prove some eigenfunction bounds for the Laplace-Beltrami operator on compact semisimple groups. For any eigenfunction $f$ of eigenvalue $-\lambda$, we establish

$$
\|f\|_{L^p(M)} \lesssim \lambda^{(d-2)/4-d/2p} \|f\|_{L^2(M)}
$$

for $p > 2sr/(sr - 4s + 4)$ when $r \geq 5$.

1. Introduction

We continue the study of Strichartz estimates for the Schrödinger flow on compact globally symmetric spaces after [14, 13] and refer to [13] for a summary of known results. In [13], the author proved the following scale-invariant Strichartz estimates for the Schrödinger flow on any compact globally symmetric space $M$ of dimension $d$ and rank $r$ equipped with the canonical Killing metric

$$
\|e^{it\Delta}f\|_{L^p(I \times M)} \lesssim \|f\|_{H^{d/2-(d+2)/p}(M)}
$$

for any $p \geq 2 + 8/r$. The proof adapts the framework of Bourgain [3] for proving similar estimates for tori. On one hand, it applies the Hardy-Littlewood method of decomposing the circle on which $t$ lives into major arcs and minor arcs, incorporating the following key $L^\infty$ “dispersive” estimates for the mollified Schrödinger kernel $\mathcal{K}_N$

$$
\|\mathcal{K}_N(t, \cdot)\|_{L^\infty(M)} \lesssim \frac{N^d}{\left(\sqrt{q} \left(1 + N \left|\frac{t}{T} - \frac{a}{q}\right|^2\right)\right)^r}
$$

on major arcs $\left|\frac{t}{T} - \frac{a}{q}\right| \lesssim \frac{1}{qN}$ centered at the fraction $a/q$ for $(a, q) = 1$ and $q < N$. Here $N^2$ is the scale of the localized spectrum and $T$ is a period for the Schrödinger kernel. On the other hand, the proof applies interpolation for the operator norm between $L^1 \to L^\infty$ and $L^2 \to L^2$, a method that traces back to the Stein-Tomas restriction theorem. In this note, we intend to improve the range of $p$ for compact semisimple Lie groups. A distinction between flat tori and compact semisimple Lie groups and more generally symmetric spaces of compact type is that joint eigenfunctions of invariant differential operators for the latter are concentrated on conjugate points while the characters on tori are uniform in size. This is behind the “scale-invariant” $L^p$-estimates enjoyed by such eigenfunctions $\psi$ on irreducible symmetric spaces $M$ of compact
type established by Marshall [10] (at least when the spectral parameter of $\psi$ varies in a fixed cone away from the walls of the Weyl chamber) as follows

$$\|\psi\|_{L^p(M)} \lesssim N^{d-\frac{q}{p}} \|\psi\|_{L^2(M)}, \quad \text{for any } p > \frac{2(d+r)}{d-r}. \quad (1.2)$$

In comparison, the only such scale-invariant estimates on tori is when $p = \infty$. In a similar vein, one expects scale-invariant $L^p(M)$-upgrades of (1.1) for symmetric spaces of compact type

$$\|\mathcal{K}_N(t, \cdot)\|_{L^p(M)} \lesssim \frac{N^{d-\frac{q}{p}}}{\left(\sqrt{q} \left(1 + N \left\|\frac{t}{T} - \frac{2}{q}\right\|^2\right)\right)^{\frac{1}{r}}} \quad (1.3)$$

This point has already been observed in [14, Proposition 7.28], where such estimates were proved for $p > 3$ for any compact semisimple Lie group. We will establish the following sharp refinements of this result.

**Theorem 1.1.** Suppose $M$ is a compact simply connected simple Lie group. Then for any $p > \frac{3d}{d-r}$, inequality (1.3) holds uniformly for $\left\|\frac{t}{T} - \frac{2}{q}\right\| \lesssim \frac{1}{qN}$.

We will prove this result by encapsulating the $L^\infty$-estimates established in [14] of the Schrödinger kernel restricted to different regions of an alcove in a maximal torus determined by how close the points are to the walls of the alcove, into $L^p$-estimates, with the help of sharp integral estimates for some weight functions on the alcove.

Next, in order to incorporate these $L^p$-estimates into Strichartz estimates which the author failed to do in [14], we replace the major-minor arc decomposition by the Farey dissection into major arcs only, observing that the contributions from the minor arcs would not enjoy the same $L^p$ scale invariance. By an interpolation between $L^p \to L^p$ ($p, p'$ are some finite conjugate exponents) and $L^2 \to L^2$, we are able to obtain the following improved scale-invariant Strichartz estimates on compact semisimple Lie groups.

**Theorem 1.2.** Let $M$ be a compact semisimple Lie group of dimension $d$ and rank $r \geq 2$. For each irreducible factor $M_0$ of $M$, set

$$s_0 = \frac{2d_0}{d_0 - r_0}$$

where $d_0, r_0$ are respectively the dimension and rank of $M_0$. Let $s$ be the largest among the $s_0$’s. Then

$$\|e^{it\Delta} f\|_{L^p(I \times M)} \lesssim \|f\|_{H^{d/2-(d+2)/p}(M)} \quad (1.4)$$

holds for any $p > 2 + \frac{8(s-1)}{sr}$.

This theorem seems to saturate the method of [3] for the setting of compact semisimple groups. It seems reasonable to conjecture that both Theorems 1.1 and 1.2 extend to symmetric spaces of compact type. As will be seen in the proof, a detailed analysis of the distribution of both the phase and size of spherical functions across an alcove in a maximal torus would be needed if one were to follow a similar line of argument for these extensions.

We will also add more evidence for the following conjecture.

**Conjecture 1.3.** Estimate (1.4) holds on any compact globally symmetric space of dimension $d$ and rank $r \geq 2$ for any $p > 2 + \frac{4}{3}$.

We will present another application of Theorem 1.1 to the problem of $L^p$ eigenfunction bounds for the Laplace-Beltrami operator on compact semisimple groups. Eigenfunction bounds on compact manifolds have been intensively studied in the literature. We do not aspire to give a broad survey but review some
fundamental results with an emphasis on globally symmetric spaces. Let $M$ be a compact manifold of dimension $d$, and let $f$ be an eigenfunction for the Laplace-Beltrami operator of eigenvalue $-N^2$. The fundamental result of Sogge [12] states

\begin{equation}
\|f\|_{L^p(M)} \lesssim N^{\gamma(d,p)} \|f\|_{L^2(M)}
\end{equation}

for

\[ \gamma(d,p) = \begin{cases} \frac{d-1}{2} - \frac{d}{p}, & \text{if } p \geq \frac{2(d+1)}{d-1}, \\ \frac{d-1}{2} \left(1 - \frac{1}{p}\right), & \text{if } 2 \leq p \leq \frac{2(d+1)}{d-1}. \end{cases} \]

These exponents were shown to be optimal by Sogge [12] on the standard spheres. We also have examples of improvement of the above exponents. On the square tori $M = \mathbb{T}^d$, we first have the result of Zygmund [15] where it was shown that (1.5) holds with $\gamma(2,4) = 0$. Then Bourgain [2] conjectured (1.5) should hold with $\gamma(2,p) = 0$ for all $p < \infty$, and with

\[ \gamma(d,p) = \frac{d-2}{2} - \frac{d}{p} \]

for $p > 2d/(d-2)$ when $d \geq 3$, with an $N^\varepsilon$-loss for $d = 3,4$. These conjectures for $p = \infty$ are indeed true, which are consequences of counting representations of integers as sums of squares, as observed in [2]. In a series of papers, Bourgain [2, 4], Bourgain and Demeter [5, 6, 7] established the conjectured estimates with an $\varepsilon$-loss for $p \geq 2(d-1)/(d-3)$ when $d \geq 4$. For a globally symmetric space of compact type, using sharp $L^p$ bounds for joint eigenfunctions of the full ring of invariant differential operators discovered by Sarnak [11] for $p = \infty$ and Marshall [10] for $p < \infty$ as in (1.2), one may establish

\[ \|f\|_{L^p(M)} \lesssim N^{\frac{d-2}{2} - \frac{d}{p} + \varepsilon} \|f\|_{L^2(M)} \]

on irreducible spaces $M$ for $p > 2(d+r)/(d-r)$, at least when the spectral parameter of $f$ lies in a fixed cone away from the walls of the Weyl chamber; see Theorem 3.3. These estimates resemble those on tori.

We add the following theorems to the existing literature, which will be established using Theorem 1.1 and the circle method as in [2].

**Theorem 1.4.** Let the assumptions be as in Theorem 1.2. Then we have the eigenfunction estimate

\begin{equation}
\|f\|_{L^p(M)} \lesssim N^{\frac{d-2}{2} - \frac{d}{p}} \|f\|_{L^2(M)}
\end{equation}

for any $p > \frac{2r}{sr-4s+4}$ when $r \geq 5$.

For a general compact globally symmetric space, using the $L^\infty$-estimate (1.1), we have the following result.

**Theorem 1.5.** Let $M$ be a compact globally symmetric space. Then (1.6) holds for any $p > 2 + \frac{8}{r-4}$ when $r \geq 5$.

We will provide evidence for the following conjecture.

**Conjecture 1.6.** Let $M$ be a compact globally symmetric space of rank $r \geq 2$. Then (1.6) holds for any $p > 2 + \frac{4}{4-r}$, with an $\varepsilon$-loss if $2 \leq r \leq 4$.

Notations. Throughout this note, $A \lesssim B$ means $A \leq CB$ for some positive constant $C$, $A \lesssim \varepsilon B$ means $A \leq C(\varepsilon)B$ for some function $C(\varepsilon)$ for any small enough positive $\varepsilon$, and $A \asymp B$ means $|A| \lesssim |B| \lesssim |A|$.

2. Preliminaries

2.1. Alcoves and Decompositions. We refer to [1] for information on affine Weyl groups and alcoves that we review in this section. Let $U$ be a compact simply connected simple Lie group with Lie algebra $\mathfrak{u}$. Let $t$
be a Cartan subalgebra of \( \mathfrak{u} \) and let \( \Sigma \subset \mathfrak{t}^* \) be the associated root system. Pick a positive system \( \Sigma^+ \subset \Sigma \) and the corresponding simple system \( \{\alpha_1, \ldots, \alpha_r\} \subset \Sigma^+ \), and let \( \alpha_0 \in -\Sigma^+ \) be the corresponding lowest root. Let

\[
A = \{ H \in \mathfrak{t} : \alpha_j(H)/i + 2\pi \delta_{0j} > 0 \ \forall j = 0, \ldots, r \}
\]

be the fundamental alcove. The Weyl group translates \( sA \) (\( s \) lies in the Weyl group \( W \)) of \( A \) are disjointly embedded in \( T \) and form the regular elements of \( T \), such that \( T \setminus \bigcup_{s \in W} sA \) is of zero measure in \( T \). In particular, for class functions \( f \) on \( U \), Weyl's integration formula can be written as

\[
\int_U f(u) \, du = \int_A f(\exp H)|\delta(H)|^2 \, dH
\]

where

\[
\delta(H) = \prod_{\alpha \in \Sigma^+} \left( e^{\frac{\alpha(H)}{2}} - e^{-\frac{\alpha(H)}{2}} \right).
\]

\( A \) is a simplex whose geometry may be described using the extended Dynkin diagram for \( \Sigma \). Each \( \alpha_j \) \( (j = 0, \ldots, r) \) corresponds to a node in the extended Dynkin diagram (Figure 1), and for each proper subset \( J \) of \( \{0, \ldots, r\} \), \( \{\alpha_j, j \in J\} \) is a simple system for a root subsystem \( \Sigma_J \) whose Dynkin diagram can be obtained from the extended Dynkin diagram of \( \Sigma \) by removing all the nodes not belonging to \( J \). For \( j = 0, \ldots, r \), let \( \tilde{s}_j : t \to t \) denote the reflection across the hyperplane \( \{H \in \mathfrak{t} : \alpha_j(H)/i + 2\pi \delta_{0j} = 0\} \). For each \( J \subset \{0, \ldots, r\} \), let \( \tilde{W}_J \) be the group generated by the reflections \( \{\tilde{s}_j, j \in J\} \). \( \tilde{W} = \tilde{W}_{\{0, \ldots, r\}} \) is the affine Weyl group. The facets of \( A \) correspond to proper subsets of \( \{0, \ldots, r\} \): for \( J \subsetneq \{0, \ldots, r\} \),

\[
A_J = \{ H \in A : \alpha_j(H)/i + 2\pi \delta_{0j} = 0 \ \forall j \in J, \ \alpha_j(H)/i + 2\pi \delta_{0j} > 0 \ \forall j \notin J \}
\]

is the corresponding \((r - |J|)\)-dimensional facet. We have \( A = \bigsqcup J A_J \). The stabilizer in \( \tilde{W} \) of any point of \( A_J \) coincides with \( \tilde{W}_J \). Let \( W_J \) denote the Weyl group associated to the root subsystem \( \Sigma_J \). \( W_J \) is isomorphic to \( W_J \) under the map \( \tilde{s} \mapsto \tilde{s} - \tilde{s}(0) \).
Consider a barycentric decomposition of $A$ as follows. For each vertex $A_I$ ($|I| = r$) of $A$, consider the convex hull $C_I$ of the barycenters of the facets $A_J$ of $A$ such that $J \subseteq I$, i.e., facets that contain $A_I$ in their boundary. Then $A = \bigcup_{|I|=r} C_I$. It is instructive to think of $C_I$ as part of the Weyl chamber with respect to the root system $\Sigma_I$. Set

$$\delta_I(H) = \prod_{\alpha \in \Sigma^+ \setminus \Sigma^+_I} \left( e^{\frac{i\alpha(H)}{2}} - e^{-\frac{-i\alpha(H)}{2}} \right).$$

We have the following lemma.

**Lemma 2.1.** $|\delta_I(H)|$ is bounded from below by a positive constant, uniformly for $H \in C_I$.

**Proof.** Write $C_I = \bigcup_{J \subseteq I} C_I \cap A_J$. For $H \in C_I \cap A_J$ for some $J \subseteq I$, the stabilizer in $W$ of $H$ is $W_J$. Set $\tilde{s}_{a,n} : t \to t$ to be the reflection across the hyperplane $\{H \in t : \alpha(H)/i + 2\pi n = 0\}$ for each $\alpha \in \Sigma$ and $n \in \mathbb{Z}$. For any $\alpha \in \Sigma \setminus \Sigma_I$ and $n \in \mathbb{Z}$, $\tilde{s}_{a,n}$ does not belong to $W_J$, since the only reflections in $W_J$ are those of the form $\tilde{s}_{a,n}$ for $\alpha \in \Sigma_J \subseteq \Sigma_I$. Thus for $\alpha \in \Sigma \setminus \Sigma_I$ and $H \in C_I \cap A_J$, $H$ cannot be fixed by $\tilde{s}_{a,n}$, in other words, $\alpha(H)/i \notin 2\pi \mathbb{Z}$. This implies the desired result. \hfill \Box

Then we consider an “$N^{-1}$-decomposition” for each $C_I$ as follows. Fix a large positive number $N$. For $I \subseteq \{0, \ldots, r\}$ such that $|I| = r$ and for each $J \subseteq I$, let

$$P_{I,J} = \{H \in C_I : \alpha_j(H)/i + 2\pi \delta_{0j} \leq N^{-1} \forall j \in J, \alpha_j(H)/i + 2\pi \delta_{0j} > N^{-1} \forall j \in I \setminus J\}.$$ Then $C_I = \bigcup_{J \subseteq I} P_{I,J}$. Let $t_j(H) = \alpha_j(H)/i + 2\pi \delta_{0j}$. Then $\{t_j, j \in I\}$ provide a coordinate system for each such $P_{I,J}$, and $P_{I,J}$ is contained in the set

$$\{H \in t : 0 \leq t_j(H) \leq N^{-1} \forall j \in J, N^{-1} < t_j(H) \leq C \forall j \in I \setminus J\}$$

for a uniform positive constant $C < 2\pi$.

For any proper subset $J$ of $\{0, \ldots, r\}$, consider $P_J = \bigcup_{I \subseteq J, |I| = r} P_{I,J}$. Then

$$P_J = \{H \in A : \alpha_j(H)/i + 2\pi \delta_{0j} \leq N^{-1} \forall j \in J, \alpha_j(H)/i + 2\pi \delta_{0j} \geq N^{-1} \forall j \notin J\}.$$ Set

$$\delta^J(H) = \prod_{\alpha \in \Sigma^+_J} \left( e^{\frac{i\alpha(H)}{2}} - e^{-\frac{-i\alpha(H)}{2}} \right).$$

Then clearly

$$|\delta^J(H)| \lesssim N^{-|\Sigma^+_J|}, \text{ for } H \in P_J.$$

### 2.2. Decomposition of the Characters and the Schrödinger Kernel.

Fix a large positive number $N$. Let $(\cdot, \cdot)$ denote the Killing form. The weight lattice reads $\Lambda = \left\{ \mu \in i\mathbb{R} : \frac{2(\mu, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \forall \alpha \in \Sigma \right\}$, and let

$$\Lambda^+ = \left\{ \mu \in i\mathbb{R} : \frac{2(\mu, \alpha)}{(\alpha, \alpha)} \geq 1 \forall \alpha \in \Sigma^+ \right\}$$

be the subset of strictly dominant weights. The mollified Schrödinger kernel $\mathcal{K}_N(t, x)$ as in $\varphi(-N^{-2}\Delta)e^{it\Delta}f = f \ast \mathcal{K}_N(t, \cdot)$ reads

$$\mathcal{K}_N(t, \exp H) = \sum_{\rho \in \Lambda^+} \varphi \left( \frac{|\mu|^2 - |\rho|^2}{N^2} \right) e^{-it(|\mu|^2 - |\rho|^2)} d_{\mu} \chi_{\mu}(H)$$

where

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha, \quad d_{\mu} = \frac{\prod_{\alpha \in \Sigma^+} (\mu, \alpha)}{\prod_{\alpha \in \Sigma^+} (\rho, \alpha)}, \quad \chi_{\mu}(H) = \frac{\sum_{s \in W} \det s e^{(s\mu)(H)}}{\sum_{s \in W} \det s e^{(s\rho)(H)}}.$$
are respectively the Weyl vector, the dimension and character for the irreducible representation of highest weight \( \mu - \rho \), and \( \varphi \) is a smooth bump function on \( \mathbb{R} \). We now study the behavior of \( \mathcal{K}_N \) near each facet of \( A \). For \( J \subseteq \{0, \ldots, r\} \), consider the subspace

\[
t_J = \bigoplus_{j \in J} \mathbb{R}H_{\alpha_j}
\]

of \( t \), where \( H_{\alpha_j} \in t \) is defined such that \( (H_{\alpha_j}, H) = \alpha_j(H)/i \) for all \( H \in t \). Let \( H_J \) denote the orthogonal projection of \( H \in t \) on \( t_J \). Let \( H_J^\perp = H - H_J \), which lies in the orthogonal complement \( t_J^\perp \) of \( t_J \) in \( t \). Also consider the subspace \( V_J \) of \( t^* \) spanned by the root subsystem \( \Sigma_J \), and let \( \mu_J \) denote the orthogonal projection of \( \mu \in \Lambda \) on \( V_J \). Let \( \Sigma_J^+ = \Sigma^+ \cap \Sigma_J \) be the positive system for \( \Sigma_J \) and let \( \Lambda_J \) be the weight lattice for \( \Sigma_J \). For each regular element \( \gamma \) in \( \Lambda_J \), let

\[
\chi_{\gamma}^J = \frac{\sum_{s_j \in W_J} \det s_J e^{s_j \gamma}}{\sum_{s_j \in W_J} \det s_J e^{s_j \rho_J}}
\]

be the associated character where \( \rho_J = \frac{1}{2} \sum_{\alpha \in \Sigma_J^+} \alpha \). Then the characters and the Schrödinger kernel can be rewritten as follows.

**Lemma 2.2.** For \( H \in t \) and any regular element \( \mu \) in \( \Lambda \), we have

\[
\chi_{\mu}(H) = \frac{1}{|W_J| \cdot \prod_{\alpha \in \Sigma^+ \setminus \Sigma_J^+} \left( e^{\frac{\alpha(H)}{2}} - e^{-\frac{\alpha(H)}{2}} \right)} \sum_{s \in W} \det s e^{s \mu(H_J)} \chi_{\mu(s\mu)}^J(H_J).
\]

As a consequence, we have

\[
\mathcal{K}_N(t, \exp H) = \frac{1}{|W_J| \cdot \prod_{\alpha \in \Sigma^+ \setminus \Sigma_J^+} \left( e^{\frac{\alpha(H)}{2}} - e^{-\frac{\alpha(H)}{2}} \right)} \cdot \mathcal{K}_N^J(t, H)
\]

where

\[
\mathcal{K}_N^J(t, H) = \sum_{\mu \in \Lambda} e^{\mu(H_J - it(|\mu|^2 - |\rho|^2))} \varphi \left( \frac{|\mu|^2 - |\rho|^2}{N^2} \right) d_{\mu} \chi_{\mu(s\mu)}^J(H_J).
\]

For a proof, we refer to equations (7-56)–(7-60) in [14]. Now by rationality of the weight lattice \( \Lambda \), let \( T \) be a positive number such that \( |\mu|^2 - |\rho|^2 \in \frac{2}{\pi^2} \mathbb{Z} \) for all \( \mu \in \Lambda \). The following key estimates are from Proposition 7.23 in [14]. Note that if \( H \in P_J \), then the root subsystem \( \Phi_H \) in Section 7E of [14] can be chosen to be \( \Sigma_J \).

**Lemma 2.3.** It holds

\[
|\mathcal{K}_N^J(t, H)| \lesssim \frac{N^{d - |\Sigma^+ \setminus \Sigma_J^+|}}{\left( \sqrt{q} \left( 1 + N \left\| \frac{t}{q} - \frac{\rho}{q} \right\|^\frac{1}{d} \right) \right)^d}
\]

uniformly for \( \left\| \frac{t}{q} - \frac{\rho}{q} \right\| \lesssim \frac{1}{q^N} \) and \( H \in P_J \). Here \( \| \cdot \| \) denotes the distance from the nearest integer.

Also note the following character bound, which is a direct consequence of the Weyl dimension formula.

**Lemma 2.4.** For any regular element \( \mu \) in \( \Lambda \) and \( J \subseteq \{0, \ldots, r\} \), \( |\chi_{\mu}^J| \lesssim |\mu_J|^{|\Sigma_J^+|} \).

We also record a decomposition of the weight lattice \( \Lambda \); see Lemma 14 in [13].

**Lemma 2.5.** We have \( \Lambda = J\Lambda \bigoplus J\Lambda^\perp \), where \( J\Lambda \) is a lattice of rank \( |J| \) and \( J\Lambda^\perp \) is a lattice of rank \( r - |J| \) such that the following holds. Let \( V_J^\perp \) denote the orthogonal complement of \( V_J \) in \( t^* \), and let...
Then \( \text{Proj}_{V^*} : \text{it}^* \rightarrow \text{it}^* \) denote the orthogonal projection of \( \text{it}^* \) on \( V^*_j \). Then \( \text{Proj}_{V^*_j} \Lambda \) is isomorphic to \( \Lambda^\perp \) as a \(|J|\)-dimensional lattice, while \( \text{Proj}_{V^*_j} \Lambda = 0 \).

2.3. Farey Dissection. Let \( n \) be an integer and consider the Farey sequence

\[
\{ \frac{a}{q}, a \geq 0, q \geq 1, (a, q) = 1, q \leq n \}
\]

of order \( n \) on the unit circle. For each three consecutive fractions \( \frac{a}{q}, \frac{a}{q}, \frac{a}{q} \) in the sequence, consider the Farey arc

\[
\mathcal{M}_{a,q} = \left[ \frac{a + a}{q + q}, \frac{a + a}{q + q} \right]
\]

around \( \frac{a}{q} \). The Farey dissection \( \bigcup_{a,q} \mathcal{M}_{a,q} \) of order \( n \) of the unit circle has the uniformity property that both \( \left[ \frac{a + a}{q + q}, \frac{a + a}{q + q} \right] \) and \( \left[ \frac{a + a}{q + q}, \frac{a + a}{q + q} \right] \) are of length \( \geq \frac{1}{q^2} \); see for example Theorem 35 in [8]. We make a further dissection of the unit circle as follows, in order to make use of the kernel bound as in Lemma 2.3; such methods have been explored by Bourgain [2, 3]. Fix a large number \( N \) and let \( Q \) be dyadic integers between 1 and \( N \). Consider the Farey sequence of order \( |N| \). For \( Q \leq q < 2Q \), we decompose the Farey arc into a disjoint union

\[
\mathcal{M}_{a,q} = \bigcup_{Q \leq M \leq N, M \text{ dyadic}} \mathcal{M}_{a,q,M}
\]

where \( \mathcal{M}_{a,q,M} \) is an interval of length \( \geq \frac{1}{NM} \) such that \( \|t - \frac{a}{q}\| \approx \frac{1}{NM} \) for any \( t \in \mathcal{M}_{a,q,M} \), except when \( M \) is the largest dyadic integer \( \leq N \), \( \mathcal{M}_{a,q,M} \) is defined by \( \|t - \frac{a}{q}\| \approx \frac{1}{N} \). Let \( \mathbb{1}_{Q,M} \) denote the indicator function of the subset

\[
\mathcal{M}_{Q,M} = \bigcup_{Q \leq q < 2Q, (a,q)=1} \mathcal{M}_{a,q,M}
\]

of the unit circle, then we have a partition of unity

\[
1 = \sum_{Q,M} \mathbb{1}_{Q,M}
\]

Let \( \widehat{\mathbb{1}_{Q,M}} : \mathbb{Z} \rightarrow \mathbb{C} \) denote the Fourier series of \( \mathbb{1}_{Q,M} \), then clearly

\[
\| \widehat{\mathbb{1}_{Q,M}} \|_{L^\infty} \lesssim \frac{Q^2}{NM}.
\]

2.4. \( L^p \) Norm of the Weight Functions. Let \( I \) be a subset of \( \{0, \ldots, r\} \) with \( |I| = r \) and \( J \) be a subset of \( I \). Let

\[
\delta_{I,J} = \prod_{\alpha \in \Sigma^+_J \setminus \Sigma^+_I} \left( e^{\frac{\alpha(H)}{2}} - e^{-\frac{\alpha(H)}{2}} \right).
\]

We obtain sharp \( L^p \)-estimate for \( 1/\delta_{I,J} \) in this section. We need the following key lemma.

**Lemma 2.6.** Assume \( t_j(H) = \alpha_j(H)/i > 0 \) for each \( j = 1, \ldots, r \). (This defines the Weyl chamber.) Suppose \( \{s_j(H), j = 1, \ldots, r\} \) is the rearrangement of \( \{t_j(H), j = 1, \ldots, r\} \) such that \( s_1(H) \leq s_2(H) \leq \cdots \leq s_r(H) \). Then

\[
\prod_{\alpha \in \Sigma^+} \frac{\alpha(H)}{i} \gtrsim s_r^{p_r}(H)s_{r-1}^{p_{r-1}}(H) \cdots s_1^{p_1}(H)
\]

for some positive integral exponents \( p_r > p_{r-1} > \cdots > p_1 = 1 \) such that \( p_r + \cdots + p_1 = |\Sigma^+| \).

We prove this lemma in the appendix.
Corollary 2.7. Inherit the assumptions in the above lemma. Consider the subsystem $\Sigma_J$ of $\Sigma$ for some $J \subset \{1, \ldots, r\}$. We assume furthermore $0 < t_j(H) \leq N^{-1}$ for all $j \in J$, while $t_j(H) > N^{-1}$ for all $j \in \{1, \ldots, r\} \setminus J$. Then
\begin{equation}
\prod_{\alpha \in \Sigma^+ \setminus \Sigma^+_J} \frac{\alpha(H)}{t} \asymp s_r^q(H)s_{r-1}^{q-1}(H) \cdots s_{|J|+1}^{q_j+1}(H)
\end{equation}
for some nonnegative integral exponents $q_r, q_{r-1}, \ldots, q_{|J|+1}$ with $q_r + q_{r-1} + \cdots + q_{|J|+1} = |\Sigma^+| - |\Sigma^+_J|$, such that
\begin{equation}
q_r + q_{r-1} + \cdots + q_{|J|+1} \geq \frac{|\Sigma^+| \cdot (r - j)}{r}, \text{ for all } j = r - 1, r - 2, \ldots, |J|,
\end{equation}
in which equality holds if and only if $j = 0 = |J|$.

Proof. Let $\prod_{\alpha \in \Sigma^+ \setminus \Sigma^+_J} \frac{\alpha(H)}{t}$ be a polynomial in the $t_j$'s. Let $n_j$ be the number of linear terms in the polynomial that contain some of the variables $s_r, s_{r-1}, \ldots, s_j$ ($j = 1, \ldots, r$). Then (2.8) may be restated as $n_j \geq p_r + p_{r-1} + \cdots + p_j$ for any $j = 1, \ldots, r$. Since $p_j > p_{j-1}$ ($j = 2, \ldots, r$) and $\sum_{j=1}^{r} p_j = |\Sigma^+|$, we have $n_j \geq \frac{|\Sigma^+|(r - j + 1)}{r}$ for any $j = 2, \ldots, r$. Let $q_r = n_r$ and $q_j = n_j - n_{j+1}$ for $j = |J| + 1, \ldots, r - 1$, then they satisfy (2.10). The reason (2.9) holds is that the assumptions on the $t_j(H)$'s imply that $\{s_r, s_{r-1}, \ldots, s_{|J|+1}\}$ is the same set as $\{t_j, j \in \{1, \ldots, r\} \setminus J\}$.

Corollary 2.8. For any nonempty proper subset $J$ of $\{1, \ldots, r\}$, we have $\frac{|J|}{|\Sigma^+_J|} > \frac{r}{|\Sigma^+|}$.

Proof. Let $\{j_1, j_2, \ldots, j_s\}$ be any permutation of $\{1, \ldots, r\}$ such that $j_k \in J$ for $k = 1, 2, \ldots, |J|$. Let $n_k$ be the number of linear terms in $\prod_{\alpha \in \Sigma^+ \setminus \Sigma^+_J} \frac{\alpha(H)}{t}$ that contain some of the variables $t_{j_1}, t_{j_2}, \ldots, t_{j_k}$ ($k = 1, \ldots, r$). Then $|\Sigma^+_J| = |\Sigma^+| - n_{|J|+1}$. We may pretend each of $t_{j_1}, t_{j_2}, \ldots, t_{|J|+1}$ is larger than any of $t_{j_1}, t_{j_2}, \ldots, t_{|J|}$; as argued in the proof of the previous corollary, (2.8) implies $n_{|J|+1} > \frac{|\Sigma^+|(r - |J|)}{r}$, which gives the desired result.

We are ready to prove the following estimate.

Proposition 2.9. For $I \subset \{0, \ldots, r\}$, $|I| = r$, $J \subset I$, we have
\begin{equation}
\left\| \frac{1}{\delta_{I,J}} \right\|_{L^p(P_{I,J})} \lesssim N^{|\Sigma^+| - |\Sigma^+_J| - \frac{1}{p}}, \text{ provided } p > \frac{2r}{d-r} = \frac{r}{|\Sigma^+|},
\end{equation}
with
\begin{equation}
\delta_{I,J} \asymp \prod_{\alpha \in \Sigma^+ \setminus \Sigma^+_J} \frac{\alpha(H)}{t} \asymp s_r^q(H)s_{r-1}^{q-1}(H) \cdots s_{|J|+1}^{q_j+1}(H),
\end{equation}
using (2.9). We estimate
\begin{equation}
\int_{P_{I,J}} \left\| \frac{1}{\delta_{I,J}} \right\|^p dH \lesssim \sum_{\text{permutation of } \{1, \ldots, r\} \setminus J} \int_{\{0 \leq t_j \leq N^{-1}, \ j \in J, \ N^{-1} < t_{n_{|J|+1}} \leq \cdots \leq t_{n_r} \leq C, \ \sum_{j=1}^{r} t_{j}^{q_j+1}dt_1 \cdots dt_r\}} \lesssim N^{-|J|+(q_r + q_{r-1} + \cdots + q_{|J|+1})p - (r - |J|)} = N(|\Sigma^+| - |\Sigma^+_J|)p - r.
\end{equation}
We have evaluated the above integral in an iterated manner first with respect to $t_{n_r}$, then $t_{n_{r-1}}$, and so on all the way to $t_{n_{|J|+1}}$, and have used $-(q_r + q_{r-1} + \cdots + q_{|J|+1})p + r - j < 0$ for each $j = |J|, \ldots, r - 1$, which
is a consequence of (2.10) and the assumption \( p > \frac{2r}{d - r} = \frac{r}{|\Sigma^+|} \). This proves the desired \( L^p \)-bound for this case. An inspection also reveals in this case
\[
\left\| \frac{1}{\delta_{I, J}} \right\|_{L^p(P_{I, J})} \lesssim \varepsilon N^{-|\Sigma^+_J| + \varepsilon}, \quad \text{provided } p \leq \frac{r}{|\Sigma^+|}.
\]

Case \( I \neq \{1, \ldots, r\} \), or equivalently \( 0 \in I \). The new technicality for this case is that \( \Sigma_I \) is not necessarily irreducible. By removing the node in the extended Dynkin diagram not belonging to \( I \), we obtain the Dynkin diagram for the root system \( \Sigma_I \). Checking Figure 1, \( \Sigma_I \) may be irreducible, or a product of two or three irreducible root systems. If \( \Sigma_I \) is irreducible, we may obtain the desired result following a similar argument as the case \( I = \{1, \ldots, r\} \) above. We here demonstrate the necessary modifications for the argument when \( \Sigma_I \) is a product of two irreducibles, and the case of three irreducibles may be treated similarly. Suppose \( \Sigma_I = \Sigma_{I_1} \cup \Sigma_{I_2} \) where \( \Sigma_{I_1} \) and \( \Sigma_{I_2} \) are nonempty, irreducible and orthogonal to each other, with \( I = I_1 \cup I_2 \). Let \( J_i = I_i \cap J \) (i = 1, 2), then \( J = J_1 \cup J_2 \). The polygon \( P_{I, J} \) is now the orthogonal product \( P_{I_1, J_1} \times P_{I_2, J_2} \), with coordinate functions \( \{t_j, j \in I_1\} \cup \{t_j, j \in I_2\} \), satisfying the restraints as in (2.2) respectively. With the positive systems also decomposed as \( \Sigma^+_I = \Sigma^+_I \cup \Sigma^+_I \), \( \Sigma^+_J = \Sigma^+_I \cup \Sigma^+_I \), we have \( \delta_{I, J} = \delta_{I_1, J_1} \cdot \delta_{I_2, J_2} \).

Apply the established result for irreducible root systems, we obtain for \( i = 1, 2 \)
\[
\left\| \frac{1}{\delta_{I, J_i}} \right\|_{L^p(P_{I_i, J_i})} \lesssim \varepsilon \left\{\begin{array}{ll}
N^{t_1 - |\Sigma^+_I| - \frac{r}{p}}, & \text{provided } p > \frac{r}{|\Sigma^+_I|}, \\
N^{-|\Sigma^+_I| + \varepsilon}, & \text{provided } p \leq \frac{r}{|\Sigma^+_I|}.
\end{array}\right.
\]

Here \( r_i = |I_i| \) is the rank of \( \Sigma_{I_i} \) (i = 1, 2). By Corollary 2.8, we may assume
\[
\frac{r}{|\Sigma^+|} < \frac{r_1}{|\Sigma^+_I|} \leq \frac{r_2}{|\Sigma^+_I|}.
\]

Then
\[
\left\| \frac{1}{\delta_{I, J}} \right\|_{L^p(P_{I, J})} = \left\| \frac{1}{\delta_{I_1, J_1}} \right\|_{L^p(P_{I_1, J_1})} \left\| \frac{1}{\delta_{I_2, J_2}} \right\|_{L^p(P_{I_2, J_2})} \lesssim \varepsilon \left\{\begin{array}{ll}
N^{t_1 - |\Sigma^+_I| - \frac{r}{p}}, & \text{if } p > \frac{r_2}{|\Sigma^+_I|}, \\
N^{t_1 - |\Sigma^+_I| - \frac{r}{p} + \varepsilon}, & \text{if } \frac{r_1}{|\Sigma^+_I|} < p \leq \frac{r_2}{|\Sigma^+_I|}, \\
N^{-|\Sigma^+_I| + 2\varepsilon}, & \text{if } \frac{r}{|\Sigma^+|} < p \leq \frac{r_1}{|\Sigma^+_I|}.
\end{array}\right.
\]

Note that the exponents of \( N \) on the right side is a piecewise linear function of \( \frac{1}{p} \), denoted \( e_1(\frac{1}{p}) \), in the range \( 0 < \frac{1}{p} < \frac{|\Sigma^+|}{r} \) with at most \( \varepsilon \)-sized discontinuities. \( e_1 \) is also a convex function modulo the \( \varepsilon \)-discontinuities. Comparing with the linear function \( e_2(\frac{1}{p}) = |\Sigma^+| - |\Sigma^+_I| - \frac{r}{p} \) of \( \frac{1}{p} \), we see for \( p \) large enough, since \( |\Sigma^+| > |\Sigma^+_I| \) (\( \Sigma_I \nsubseteq \Sigma \) since \( \Sigma_I \) is reducible), it holds \( e_1(\frac{1}{p}) < e_2(\frac{1}{p}) \); on the other hand, it is also clear that \( e_1(\frac{1}{p}) < e_2(\frac{1}{p}) \) for \( p = \frac{r}{|\Sigma^+|} + \eta \) for any small positive \( \eta \) if we choose the above \( \varepsilon \) small enough. By convexity (modulo \( \varepsilon \)-discontinuities) of \( e_1 \) and linearity of \( e_2 \), we get \( e_1(\frac{1}{p}) < e_2(\frac{1}{p}) \) for all \( p \geq \frac{r}{|\Sigma^+|} + \eta \), which yields the desired estimate. \( \square \)

3. \( L^p \) dispersive estimates and applications

3.1. \( L^p \) Dispersive Estimates on Major Arcs. We are ready to prove Theorem 1.1.

Proof. By Weyl’s integration formula as in (2.1), we have
\[
\|\mathcal{K}_N(t, \cdot)\|_{L^p(U)} = \|\mathcal{K}_N(t, \cdot)\delta I \|_{L^p(A)}.
\]
Since \( A = \bigcup_{J \subseteq I, |J|=r} P_{I,J} \), it suffices to prove \( \| \mathcal{K}_N(t, \cdot) \mathcal{N}(t) \|_{L^p(P_{I,J})} \) has the desired bound for all \( I, J \). Using (2.6), we have
\[
|\mathcal{K}_N(t, H)| \cdot |\delta(H)|^{\frac{2}{q}} = \frac{|\delta'(H)|^{\frac{2}{q}}}{|W_{I,J}| \cdot |\delta(H)|^{1-\frac{2}{q}} |\delta_{I,J}(H)|^{1-\frac{2}{q}} \cdot |\mathcal{K}_N'(t, H)|.
\]
Then we have the desired estimate, combining Lemmas 2.1 and 2.3, estimate (2.3), and Proposition 2.9. \( \square \)

### 3.2. Improved Strichartz Estimates on Compact Semisimple Groups

We are ready to prove Theorem 1.2.

**Proof.** Reducing to a finite cover, it suffices to prove it for the case of a compact simply connected semisimple Lie group \( U = U_1 \times U_2 \times \cdots \times U_k \), where the \( U_i \)'s are the simple components, equipped with the canonical Killing metrics. Consider the product Schrödinger kernel \( \mathcal{K}_N = \prod_{i=1}^{k} \mathcal{K}_{N,i} \) where
\[
\mathcal{K}_{N,i}(t, H_i) = \sum_{\mu_i \in \Lambda_i^\mu} \varphi_i \left( \frac{\mu_i^2 - |\rho_i|^2}{N^2} \right) e^{-it(|\mu_i|^2 - |\rho_i|^2)} d_{\mu_i} \rho_{\mu_i}(H_i)
\]
is the kernel for the component \( U_i \). The component kernels \( \mathcal{K}_{N,i} \) share a period in the time variable \( t \), say \( T \), and we set \( \mathbb{T} = \mathbb{R} / \mathbb{T}Z \). Let \( \Sigma_i \) be the root system of rank \( r_i \) for \( U_i \) \((1 \leq i \leq k)\), then Proposition 1.1 implies
\[
\| \mathcal{K}_N(t, \cdot) \|_{L^u(U)} = \prod_{i=1}^{k} \| \mathcal{K}_{N,i}(t, \cdot) \|_{L^u(U_i)} \lesssim \frac{N^{d-d_q}}{\left( 1 + N \| \frac{\mu}{\| \cdot \|} \|^{\frac{2}{q}} \right)^{\frac{1}{2}}}
\]
provided
\[
u > s = \max \left\{ \frac{2d_i}{d_i - r_i}, \ i = 1, \ldots, k \right\}.
\]
Here \( d_i \) is the dimension of \( U_i \) \((1 \leq i \leq k)\).

Using the Farey dissection in Section 2.3, we write \( \mathcal{K}_N = \sum_{Q,M} \mathcal{K}_{Q,M} \) where \( \mathcal{K}_{Q,M}(t, x) = \mathcal{K}_N(t, x) \cdot \mathbb{1}_{Q,M}(t) \), for \((t, x) \in \mathbb{T} \times U \). Let \( F : \mathbb{T} \times U \to \mathbb{C} \) be a continuous function. Let \( \ast \) denote the convolution on the product group \( \mathbb{T} \times U \). By Young’s inequality for unimodular groups, inequality (3.1), and the estimate
\[
\| \mathbb{1}_{Q,M} \|_{L^u(\mathbb{T})} \lesssim \left( \frac{Q^2}{MN} \right)^{\frac{1}{2}},
\]
we have for \( u > s \)
\[
\| F \ast \mathcal{K}_{Q,M} \|_{L^{2u}(\mathbb{T} \times U)} \lesssim \| \mathcal{K}_{Q,M} \|_{L^u(\mathbb{T} \times U)} \| F \|_{L^{(2u)'}(\mathbb{T} \times U)}
\]
\[
\lesssim N^{d-d_q - \frac{d+1}{2} + \frac{1}{2}} M^{\frac{1}{2}} Q^{-2} \| F \|_{L^{(2u)'}(\mathbb{T} \times U)}.
\]
Here \( 2u \) and \((2u)\)' are conjugate exponents. On the other hand, the Fourier series \( \mathcal{K}_{Q,M}(n, \mu) \in M_{d_\mu} \) \((n, \mu) \in \mathbb{Z} \times \Lambda^+ \) of \( \mathcal{K}_{Q,M} \) on the compact Lie group \( \mathbb{T} \times U \), where \( M_{d_\mu} \) is the space of \( d_\mu \) by \( d_\mu \) matrices, equals
\[
\mathcal{K}_{Q,M}(n, \mu) = \varphi(\mu, N) \mathbb{1}_{Q,M}(2\pi n/T + |\mu|^2) \cdot \text{Id}_{d_\mu},
\]
where
\[
\varphi(\mu, N) = \prod_{i=1}^{k} \varphi_i((|\mu_i|^2 - |\rho_i|^2)/N^2), \ |\mu|^2 = |\mu|^2 - |\rho|^2 = \sum_{i=1}^{k} |\mu_i|^2 - |\rho_i|^2,
\]
and $\text{Id}_{d_r}$ is the identity matrix in $M_{d_r}$. Let $\| \cdot \|_{HS}$ denote the Hilbert-Schmidt norm of square matrices. Using the Plancherel identity and estimate (2.7), we have

$$\|F \ast \mathcal{K}_{Q,M}\|_{L^2(\mathbb{T}^r \times U)} = \left\| \sqrt{d_r} \| \mathcal{F} \mathcal{K}_{Q,M} \|_{HS} \right\|_{L^2(\mathbb{Z} \times \Lambda^+)} \lesssim \frac{Q^2}{NM} \|F\|_{L^2(\mathbb{T}^r \times U)},$$

(3.3)

Interpolating (3.2) with (3.3) for $\frac{\theta}{2} + \frac{1-\theta}{2u} = \frac{1}{p}$, we get

$$\|F \ast \mathcal{K}_{Q,M}\|_{L^p(\mathbb{T}^r \times U)} \lesssim N^{(d-\frac{d+2}{2})\frac{1}{2}(1-\theta) - \frac{1}{\theta}} M^{(\frac{d+2}{2})\frac{1}{2}(1-\theta) - \frac{1}{\theta}} (\frac{r_0 + \frac{1}{2}}{1-\theta}) \|F\|_{L^p(\mathbb{T}^r \times U)}.$$ 

We require the exponent of $Q$ satisfy

$$\left(-\frac{r}{2} + \frac{2}{u}\right)(1-\theta) + 2\theta < 0 \Leftrightarrow \theta < \frac{ru - 4}{4u + ru - 4},$$

which implies the exponent of $M$ satisfies $\left(\frac{r}{2} - \frac{1}{u}\right)(1-\theta) - \theta > 0$. Summing over $M$ and $Q$, we get

$$\|F \ast \mathcal{K}_N\|_{L^p(\mathbb{T}^r \times U)} \lesssim \sum_{1 \leq Q \leq N} \sum_{1 \leq M \leq N} \|F \ast \mathcal{K}_{Q,M}\|_{L^p(\mathbb{T}^r \times U)} \lesssim N^{\frac{d-\frac{d+2}{2}}{2}(1-\theta) - \frac{1}{\theta}} \|F\|_{L^p(\mathbb{T}^r \times U)},$$

provided

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{2u} < \frac{ru - 4}{2(4u + ru - 4)} + \frac{2}{4u + ru - 4} \Leftrightarrow p > 2 + \frac{8(u - 1)}{ru}$$

for some $u > s$. This implies Theorem 1.2, by an application of the product Littlewood-Paley theory and the $TT^*$ argument.

We conjecture that Strichartz estimate (1.4) holds on any compact globally symmetric space of rank $\geq 2$ with canonical metrics for any $p > 2 + \frac{4}{d}$, which is the largest possible range (except the endpoint). Some evidence was gained in [13] where the author used the following conjectured mixed norm Strichartz estimates on tori and their restriction-to-hyperplane versions, and Proposition 2.9 (with $|J| = 0, 1$) for $SU(r + 1)$, to deduce the full-range Strichartz estimates on $SU(r + 1)$ (except the endpoint) for class functions.

**Conjecture 3.1.** Let $\mathbb{R}^r$ be equipped with an inner product $(\cdot, \cdot)$ and let $|\cdot|$ denote the corresponding norm. Let $\Gamma$ be a rank-$r$ rational lattice in $\mathbb{R}^r$. Let $B$ be a bounded domain in $\mathbb{R}^r$ and $B^{r-1}$ be a bounded domain in some hyperplane in $\mathbb{R}^r$. Let $I$ be a bounded interval. Then

$$\left\| \sum_{\mu \in \Gamma, |\mu| \leq N} a_\mu e^{it(\mu, \mu) + i(\mu, x)} \right\|_{L^p(I, L^q(B))} \lesssim N^\frac{d}{2} - \frac{d-2}{p} \|a_\mu\|_{L^2(\Gamma)}$$

for all pairs $p, q \geq 2$ with $\frac{d}{2} - \frac{2}{p} - \frac{d}{p} > 0$. We also have

$$\left\| \sum_{\mu \in \Gamma, |\mu| \leq N} a_\mu e^{it(\mu, \mu) + i(\mu, x)} \right\|_{L^p(I, L^q(B^{r-1}))} \lesssim N^\frac{d}{2} - \frac{d-2}{p} \|a_\mu\|_{L^2(\Gamma)}$$

for all pairs $p, q \geq 2$ with $\frac{d}{2} - \frac{2}{p} - \frac{d-1}{q} > 0$.

With now the knowledge of Proposition 2.9 for all root systems, the argument in [13] extends to all compact Lie groups to yield the following result.
Proposition 3.2. The above conjecture implies (1.4) for class functions on any compact Lie group with canonical metrics for any $p > 2 + \frac{4}{d}$.

3.3. Eigenfunction Bounds for the Laplace-Beltrami Operator. We are ready to prove Theorem 1.4.

Proof. We inherit the notations in the proof of Theorem 1.2. Let $f$ be an eigenfunction of eigenvalue $-\lambda$. Then $\lambda = |\mu|^2$ for some $\mu \in \Lambda^+$. Set

$$K_\lambda = \sum_{\mu \in \Lambda^+, |\mu|^2 = -\lambda} d_\lambda \chi_\lambda,$$

Then it is clear $f = f \ast K_\lambda$. By an argument of $TT^*$, it suffices to establish bounds of the form

$$\|f \ast K_\lambda\|_{L^p(U)} \lesssim \lambda^{d/2 - d/p} \|f\|_{L^{p'}(U)}.$$ 

Let $N = \lambda^{1/2}$ and let $\mathcal{X}_N$ be again the Schrödinger kernel as in (2.4), where we assume the bump function satisfies $\varphi(y) = 1$ for all $|y| \leq 1$. We may write

$$K_\lambda = \frac{1}{T} \int_0^T \mathcal{X}_N(t, \cdot) e^{it\lambda} dt.$$ 

Using the Farey dissection, we decompose

$$K_\lambda = \sum_{Q, M} K_{Q, M},$$

where

$$K_{Q, M} = \int_{M_{Q, M}} \mathcal{X}_{Q, M}(t, \cdot) e^{it\lambda} dt \left( t \right).$$

By Theorem 1.1, Minkowski’s integral inequality, and the estimate that the length of $M_{Q, M}$ is $\lesssim \frac{Q^2}{NM}$, we have for $u > s$

$$\|K_{Q, M}\|_{L^u(U)} \leq N^{d - \frac{d}{2} - 1} M^{\frac{d}{2} - 1} Q^{-\frac{d}{2} + 2},$$

which implies by Young’s inequality

(3.4)  

$$\|f \ast K_{Q, M}\|_{L^{2u}(U)} \lesssim N^{d - \frac{d}{2} - 1} M^{\frac{d}{2} - 1} Q^{-\frac{d}{2} + 2} \|f\|_{L^{2u'}(U)}.$$ 

On the other hand, the Fourier series of $K_{Q, M}$ on $U$ equals

$$\widehat{K_{Q, M}}(\mu) = \left( \varphi(\mu, N) \int_{M_{Q, M}} e^{i(t - |\mu|^2) } dt \right) \cdot \text{Id}_{d_u},$$

for all $\mu \in \Lambda^+$, which implies

(3.5)  

$$\|f \ast K_{Q, M}\|_{L^2(U)} \lesssim \|\sqrt{d_\mu} \| \widehat{K_{Q, M}} \|_{HS} \|_{L^2(\Lambda^+)} \lesssim \frac{Q^2}{NM} \|H\|_{HS} \|_{L^2(\Lambda^+)} = \frac{Q^2}{NM} \|f\|_{L^2(U)}.$$ 

Interpolating (3.4) with (3.5) for $\frac{2}{q} + \frac{1}{2u'} = \frac{1}{p}$, we get

$$\|f \ast K_{Q, M}\|_{L^p(U)} \lesssim N^{(d - \frac{d}{2} - 1)(1 - \theta) - \theta} M^{(\frac{d}{2} - 1)(1 - \theta) - \theta} Q^{(-\frac{d}{2} + 2)(1 - \theta) + 2\theta} \|f\|_{L^{p'}(U)}.$$ 

We require the exponent of $Q$ be negative, i.e.,

$$\left(-\frac{r}{2} + 2\right)(1 - \theta) + 2\theta < 0 \Leftrightarrow \theta < \frac{r - 4}{r},$$

which implies the exponent $(\frac{d}{2} - 1)(1 - \theta) - \theta$ of $M$ is positive. Summing over $M$ and $Q$, we have

$$\|f \ast K_\lambda\|_{L^p(U)} \lesssim N^{(d - \frac{d}{2} - 2)(1 - \theta) - 2\theta} \|f\|_{L^{p'}(U)} = N^{d - 2 + 2\theta} \|f\|_{L^{p'}(U)},$$

which completes the proof.
provided
\[ \frac{1}{p} = \frac{\theta}{2} + \frac{1 - \theta}{2u} < \frac{r - 4}{2r} + \frac{2}{ru} \iff p > \frac{2ur}{ur - 4u + 4} \]
for some \( u > s \). This finishes the proof. \( \square \)

If we let \( s \to \infty \), then \( 2 + \frac{8(s - 1)}{sr^2 - 4s + 4} \to 2 + \frac{8}{r^2 - 4} \). Indeed, using the \( L^\infty \)-estimate (1.1) for the Schrödinger kernel on a general compact globally symmetric space \( M \) as established in [13], and following a similar argument, we are able to generalize the above theorem partially to compact globally symmetric spaces of high rank, to yield Theorem 1.5.

One should compare these results with consequences of Marshall’s eigenfunction bounds [10] for the full ring of invariant differential operators.

**Theorem 3.3.** Let \( M \) be a symmetric space of compact type of rank \( r \geq 2 \) whose universal cover is a product \( M_1 \times M_2 \times \cdots \times M_k \) of irreducible spaces. Let \( d_i, r_i \) be respectively the dimension and rank of \( M_i \) and let \( v = \max \left\{ \frac{2(d_i + r_i)}{d_i - r_i}, \ i = 1, \ldots, k \right\} \). Suppose \( f \) is an eigenfunction of the Laplace-Beltrami operator with eigenvalue \(-\lambda\) and that the spectral parameter of \( f \) lies in a fixed cone centered at the origin and away from the walls of the Weyl chamber. Then

\[ \|f\|_{L^p(M)} \lesssim_{\varepsilon} \lambda^{d_{\varepsilon'} - \frac{d}{p} + \varepsilon} \|f\|_{L^2(M)} \]
for any \( p > v \).

**Proof.** (Sketch) Let \( \Lambda^+ \) denote the set of dominant weights which correspond to irreducible spherical representations \( \pi_\mu \ (\mu \in \Lambda^+) \). We may write the spherical Fourier series of \( f \)

\[ f = \sum_{\mu \in \Lambda^+, |\mu|^2 - |\rho|^2 = \lambda} a_\mu f_\mu \]
where \( f_\mu \) is a joint eigenfunction for the full ring of invariant differential operators of spectral parameter \( \mu \in \Lambda^+ \) lying in a fixed cone away from the walls of the Weyl chamber. By Theorem 1.1 in [10], we have

\[ \|f_\mu\|_{L^p(M)} \lesssim |\mu|^{\frac{d}{2} - \varepsilon} \|f_\mu\|_{L^2(M)} \lesssim \lambda^{d_{\varepsilon'} - \frac{d}{p} + \varepsilon} \|f_\mu\|_{L^2(M)} \]
for all \( p > v \). Then

\[
\left\| \sum_{\mu \in \Lambda^+, |\mu|^2 - |\rho|^2 = \lambda} a_\mu f_\mu \right\|_{L^p(M)} \lesssim \left( \#\{\mu \in \Lambda^+: |\mu|^2 - |\rho|^2 = \lambda\} \right)^{\frac{1}{2}} \left( \sum_{\mu \in \Lambda^+, |\mu|^2 - |\rho|^2 = \lambda} |a_\mu|^2 \|f_\mu\|^2_{L^p(M)} \right)^{\frac{1}{2}} \\
\lesssim \varepsilon \lambda^{d_{\varepsilon'} - \frac{d}{2} + \varepsilon} \left( \sum_{\mu \in \Lambda^+, |\mu|^2 - |\rho|^2 = \lambda} |a_\mu|^2 \|f_\mu\|^2_{L^p(M)} \right)^{\frac{1}{2}} \\
\lesssim \varepsilon \lambda^{d_{\varepsilon'} - \frac{d}{2} + \varepsilon} \|f\|_{L^2(M)}.
\]

Here we used a standard counting estimate on integral solutions to a positive definite integral form (see for example Lemma 15 in [13]), and Minkowski’s integral inequality. \( \square \)

**Remark 3.4.** In the counting estimate \#\{\mu \in \Lambda^+: |\mu|^2 - |\rho|^2 = \lambda\} \lesssim \varepsilon \lambda^{d_{\varepsilon'} - \frac{d}{2} + \varepsilon} used above, it is possible to remove the \( \varepsilon \)-loss for \( r \geq 5 \), as is the case for counting the number of representations of an integer as a sum of \( r \) squares.

Observe that ranges of \( p \) in Theorems 1.4 and 1.5 are larger than that in Theorem 3.3 typically when the space has a large number of irreducible factors. We now provide some evidence of Conjecture 1.6 by
showing how this conjecture specialized for class functions on compact Lie groups could be deduced from the conjectured eigenfunction bounds on tori by Bourgain [2] and their restriction-to-hyperplane versions as follows.

**Conjecture 3.5.** Inherit the assumptions in Conjecture 3.1. For \( r \geq 3 \), Bourgain [2] conjectured

\[
\left\| \sum_{\mu \in \Gamma, |\mu| = N} a_{\mu} e^{i\mu \cdot x} \right\|_{L^p(B)} \lesssim \varepsilon N^{\frac{r-2}{r} - \frac{d}{2}} \|a_{\mu}\|_{L^2(\Gamma)}
\]

for any \( p > \frac{2r}{r-2} \), and \( \varepsilon \) can be removed for \( r \geq 5 \). We also conjecture for \( r \geq 3 \)

\[
\left\| \sum_{\mu \in \Gamma, |\mu| = N} a_{\mu} e^{i\mu \cdot x} \right\|_{L^p(B)} \lesssim \varepsilon N^{\frac{r-2}{r} - \frac{d}{2} + \varepsilon} \|a_{\mu}\|_{L^2(\Gamma)}
\]

for any \( p > \frac{2(r-1)}{r-2} \), and \( \varepsilon \) can be removed for \( r \geq 5 \).

The above two inequalities indeed hold when \( r = 2 \) and \( p = \infty \), since \#\{\mu \in \Gamma : |\mu| = N\} \lesssim N^\varepsilon \) when \( \Gamma \) is a rank-2 rational lattice (see for example Lemma 15 in [13]).

**Proposition 3.6.** Conjecture 3.5 implies Conjecture 1.6 for class eigenfunctions on compact Lie groups of rank \( r \geq 3 \), with an \( \varepsilon \)-loss. For \( r = 2 \), Conjecture 1.6 holds for class eigenfunctions on compact Lie groups.

**Proof.** We follow closely a line of arguments in [13] on the discussion of Strichartz estimates. We sketch the proof when the compact group \( U \) is simple and simply connected. The general case follows by taking a finite cover and modifying the proof for the covering group case group. Now class eigenfunctions \( f \) of eigenvalue \(-\lambda = -N^2\) can be expressed as

\[
f = \sum_{\mu \in \Lambda^+, |\mu|^2 = N^2} a_{\mu} \chi_{\mu}.
\]

Using Weyl's integration formula (2.1), inequality (1.6) with an \( \varepsilon \)-loss reads

\[
\left\| \sum_{\mu \in \Lambda^+, |\mu|^2 = N^2} a_{\mu} \chi_{\mu} |\delta|^\frac{d}{2} \right\|_{L^p(A)} \lesssim \varepsilon N^{\frac{d-2}{2} - \varepsilon} \|a_{\mu}\|_{L^2(\Lambda^+)}.
\]

Recalling the decomposition \( A = \bigcup_{I \subset I, |I| = r} P_{I,J} \), the above estimate reduces to those replacing \( A \) by each \( P_{I,J} \). We set \( a_{\mu I} = a_{\mu} \) for \( \mu \in \Lambda^+, s \in W \).

**Case 1.** \( J = \emptyset \). Writing

\[
\sum_{\mu \in \Lambda^+, |\mu|^2 = N^2} a_{\mu} \chi_{\mu} |\delta|^\frac{d}{2} = \sum_{s \in W} \det s \sum_{\mu \in \Lambda^+, |\mu|^2 = N^2 + |\rho|^2} a_{\mu} e^{i\rho} \frac{1}{|\delta_{I,\emptyset}|^{1 - \frac{d}{2}}},
\]

and noting \( |\delta_I| \approx 1 \) uniformly on \( P_{I,\emptyset} \) by Lemma 2.1, we estimate for \( \frac{1}{p} = \frac{1}{w} + \frac{4}{7} \)

\[
\left\| \sum_{\mu \in \Lambda^+, |\mu|^2 = N^2} a_{\mu} \chi_{\mu} |\delta|^\frac{d}{2} \right\|_{L^p(P_{I,\emptyset})} \lesssim \sum_{s \in W} \left\| \sum_{\mu \in \Lambda^+, |\mu|^2 = N^2 + |\rho|^2} a_{\mu} e^{i\rho} \right\|_{L^w(P_{I,\emptyset})} \frac{1}{|\delta_{I,\emptyset}|^{1 - \frac{d}{2}}} \lesssim \varepsilon N^{\frac{d-2}{2} - \frac{d}{2} + \varepsilon} \|a_{\mu}\|_{L^2(\Lambda^+)} = N^{\frac{d-2}{2} + \varepsilon} \|a_{\mu}\|_{L^2(\Lambda^+)},
\]
using conjectured estimate (3.6) and Proposition 2.9, provided the necessary conditions hold

\[(3.8) \quad u > \frac{2r}{r-2} \quad (u = \infty \text{ if } r = 2), \quad \left(1 - \frac{2}{p}\right) \left(\frac{1}{p} - \frac{1}{u}\right) > \frac{2r}{d-r}, \quad u \geq p \geq 2.\]

An inspection shows any \( p > \frac{2d}{d-2} \) is admissible.

**Case 2.** \(|J| = 1\). For this case \( t_J \) is a line and \( t_J^+ \) is a hyperplane in \( t \). Set

\[E_J = \{H \in t_J : 0 \leq \alpha_j(H)/i + 2\pi \delta_{i,j} \leq N^{-1} \ \forall j \in J\}\]

and \( E_{t_j,J}^+(H_J) = \{H \in t_j^+ : H + H_J \in P_{t_j,J}\} \) for any \( H_J \in E_J \). Using (2.5), we write

\[
\sum_{\mu \in \Lambda^+, \|\mu\|^2 = N^2} a_{\mu} \chi_{\mu} |\delta|^2 \left| \frac{\delta_J}{|W_J| \cdot \delta_J} \right| \sum_{s \in W} \det s \sum_{\mu \in \Lambda^+, \|\mu\|^2 = N^2 + |\mu|^2} a_{\mu} e^{i(H_j^+)} \chi_{\mu,J}^+(H_J) \left| \delta_j^+ \right|^2 \delta_{I,J} |\delta|^{-\frac{2}{p}}
\]

and estimate for \( \frac{1}{p} = \frac{1}{u} + \frac{1}{v} \) using \( \delta_J \approx 1 \) and \( |\delta_j| \lesssim N^{-|\Sigma^+_J|} = N^{-1} \) on \( P_{t_j,J} \)

\[
\left\| \sum_{\mu \in \Lambda^+, \|\mu\|^2 = N^2} a_{\mu} \chi_{\mu} [\delta] \right\|_{L^p(P_{t_j,J})} \lesssim \sum_{s \in W} N^{-\frac{2}{p}} \left\| \sum_{\mu \in \Lambda^+, \|\mu\|^2 = N^2 + |\mu|^2} a_{\mu} e^{i(H_j^+)} \chi_{\mu,J}^+(H_J) \right\|_{L^\infty(P_{t_j,J})} \left\| \frac{1}{|\delta_J|^{1-\frac{2}{p}}} \right\|_{L^p(P_{t_j,J})}
\]

\[
\lesssim \sum_{s \in W} N^{-\frac{2}{p}} \left\| \sum_{\mu \in \Lambda^+, \|\mu\|^2 = N^2 + |\mu|^2} a_{\mu} e^{i(H_j^+)} \chi_{\mu,J}^+(H_J) \right\|_{L^\infty(E_{t_j,J}^+(H_J))} \left\| \frac{1}{|\delta_{I,J}|^{1-\frac{2}{p}}} \right\|_{L^p(E_{t_j,J}^+(H_J))}
\]

\[
\lesssim N^{\frac{r-2}{r-1} - \frac{1}{u} + \frac{1}{v} + (\frac{d-r}{p} - 1)(1-\frac{2}{p}) - \frac{r}{2} + \epsilon} \|a_{\mu}\|_{L^2(\Lambda^+)} = N^{\frac{d-2}{p} - \frac{d}{p} + \epsilon} \|a_{\mu}\|_{L^2(\Lambda^+)}.
\]

Here we have used the conjectured estimate (3.7), the estimate \( |\chi_{\mu,J}^+| \lesssim N \) from Lemma 2.4, the length of \( E_J \) being \( \approx N^{-1} \), and Proposition 2.9. We need to check that the necessary conditions hold

\[u > \frac{2(r-1)}{r-2} \quad (u = \infty \text{ if } r = 2), \quad \left(1 - \frac{2}{p}\right) \left(\frac{1}{p} - \frac{1}{u}\right) > \frac{2r}{d-r}, \quad u \geq p \geq 2.\]

These are less strict than those in (3.8); since \( p > \frac{2d}{d-2} \) is admissible for (3.8), it is also admissible here.

**Case 3.** \(|J| \geq 2\). According to the decomposition \( \Lambda = J_1 \Lambda \bigoplus J_2 \Lambda^\perp \) in Lemma 2.5, we write \( \mu = J_1 \mu + J_2 \mu^\perp \) for \( \mu \in \Lambda \). Write

\[
\sum_{\mu \in \Lambda^+, \|\mu\|^2 = N^2} a_{\mu} \chi_{\mu} [\delta]^2 \left| \frac{\delta_J}{|W_J| \cdot \delta_J} \right| \sum_{s \in W} \det s \sum_{\mu \in \Lambda^+, \mu + J_2 \mu^\perp \lesssim N} e^{i(H_j^+)} \sum_{\mu = J_1 \mu + J_2 \mu^\perp, \|\mu\|^2 = N^2 + |\mu|^2} a_{\mu} \chi_{\mu,J}^+(H_J) \left| \delta_j^+ \right|^2 \delta_{I,J} |\delta|^{-\frac{2}{p}}.
\]

and observe

\[
\left| \frac{|\delta_j|^{1-\frac{2}{p}}}{|\delta_{I,J}|^{1-\frac{2}{p}}} \right| \lesssim N^{-2|\Sigma^+_J|} + (1-\frac{2}{p})(|\Sigma^+_J| - |\Sigma^+_J|) = N^{1-\frac{2}{p}}|\Sigma^+_J| - |\Sigma^+_J|.
\]
We may now estimate for any \( p > 2 \)
\[
\left\| \sum_{\mu \in \Lambda^+, \|\mu\|^2 = N^2} a_{\mu} \chi_{\mu} \delta^{\frac{1}{p}} \right\|_{L^p(P_{l,j})} \lesssim \sum_{s \in W} \sum_{j \mu^+ \in \Lambda^+, \|j \mu^+\| \leq N} \sum_{j \mu \in \Lambda^+, \|j \mu\| = \|\mu\|} a_{\mu} \chi_{j \mu^+} \chi_{j \mu},
\]
\[
\lesssim \sum_{s \in W} N^{(1 - \frac{2}{p})|\Sigma^+| - |\Sigma^+_j|} \left( \sum_{j \mu \in \Lambda^+, \|j \mu\| = \|\mu\|} a_{\mu} \chi_{j \mu^+} \chi_{j \mu} \right)^2 \lesssim \|H_j\|_{L^p(E_j)} \left( \sum_{j \mu \in \Lambda^+, \|j \mu\| = \|\mu\|} a_{\mu} \chi_{j \mu^+} \chi_{j \mu} \right)^\frac{1}{2},
\]
where we used Bernstein’s inequality on the torus defined by the weight lattice \( j \Lambda^\perp \) in \( t^\perp_j \). We estimate for \( |j \mu^+| \lesssim N \) (see Lemma 15 in [13])
\[
\# \{ j \mu \in j \Lambda : |j \mu + j \mu^+| = N^2 + |\rho|^2 \} \lesssim N|\Sigma^+| - 2 + \epsilon,
\]
which implies
\[
\sum_{j \mu^+ \in \Lambda^+, \|j \mu^+\| \leq N^2} \sum_{j \mu \in \Lambda^+, \|j \mu\| = \|\mu\|} a_{\mu} \chi_{j \mu^+} \chi_{j \mu} \leq \epsilon \left( \sum_{j \mu \in \Lambda^+, \|j \mu\| = \|\mu\|} a_{\mu} \chi_{j \mu^+} \chi_{j \mu} \right)^2 \lesssim \|H_j\|_{L^p(E_j)} \left( \sum_{j \mu \in \Lambda^+, \|j \mu\| = \|\mu\|} a_{\mu} \chi_{j \mu^+} \chi_{j \mu} \right)^\frac{1}{2}.
\]
Here we used Lemma 2.4. Combine the above estimates with \( \|1\|_{L^p(E_j)} \lesssim N^{-\frac{2}{p}} \), we get
\[
\left\| \sum_{\mu \in \Lambda^+, \|\mu\|^2 = N^2} a_{\mu} \chi_{\mu} \delta^{\frac{1}{p}} \right\|_{L^p(P_{l,j})} \lesssim \epsilon \left( \sum_{\mu \in \Lambda^+, \|\mu\|^2 = N^2} a_{\mu} \chi_{\mu} \delta^{\frac{1}{p}} \right)^\frac{1}{2},
\]
\[
\lesssim N^{(1 - \frac{2}{p})|\Sigma^+| + |\Sigma^+_j| + (r - |J|) \left( \frac{1}{2} \right) + \frac{1}{2} (|\Sigma^-| + |\Sigma^+_j|) - \frac{2}{p} + \epsilon} \|a_{\mu}\|_{L^p(\Lambda^\perp)} \lesssim N^{\frac{2}{p} - 2 + \epsilon} \|a_{\mu}\|_{L^p(\Lambda^\perp)}.
\]

\[ \square \]

Appendix: Proof of Lemma 2.6

We prove this lemma case by case for all irreducible root systems. We assume \( t_1, t_2, \ldots, t_r \) are aligned according to the labels in the Dynkin diagrams in Figure 1. We will need explicit constructions of all root systems, which we refer to for example Appendices in [9].

Case \( A_r \). We prove (2.8) for \( (p_1, \ldots, p_1) = (r, 2, \ldots, 1) \). We calculate
\[
\Pi_{\alpha \in \Sigma^+} \frac{\alpha(H)}{i} = \Pi_{1 \leq i < j \leq n+1} \left( t_i + t_{i+1} + \cdots + t_{j-1} \right).
\]
Suppose \( t_1 = s_m \) for some \( m = 1, \ldots, r \). The induction hypothesis reads
\[
\Pi_{2 \leq i < j \leq n+1} \left( t_i + \cdots + t_{j-1} \right) \geq s_r^{-1} \cdots s_{m+1} s_{m-1} \cdots s_1.
\]
Also it is clear that \( \Pi_{2 \leq j \leq n+1} \left( t_1 + \cdots + t_{j-1} \right) \geq s_r \cdots s_{m+1} s_{m-1} \cdots s_1 \), which combined with the inductive hypothesis yields the desired result.
We prove (2.8) for \((p_r, \ldots, p_1) = (2r - 1, 2r - 3, \ldots, 1)\). A calculation for \(B_r\) shows
\[
\prod_{\alpha \in \Sigma^+} \frac{\alpha(H)}{i} = \prod_{1 \leq i \leq r} (t_i + t_{i+1} + \cdots + t_r) \cdot \prod_{1 \leq i < j \leq r} (t_i + t_{i+1} + \cdots + t_{j-1}) \\
\cdot \prod_{1 \leq i < j \leq r} (t_i + t_{i+1} + \cdots + t_{j-1} + 2t_j + \cdots + 2t_r).
\]
But \(t_i + t_{i+1} + \cdots + t_{j-1} + 2t_j + \cdots + 2t_r \approx t_i + t_{i+1} + \cdots + t_n\), which implies
\[
\prod_{\alpha \in \Sigma^+} \frac{\alpha(H)}{i} \approx \prod_{1 \leq i \leq r} (t_i + t_{i+1} + \cdots + t_r)^n \cdot \prod_{1 \leq i < j \leq r} (t_i + t_{i+1} + \cdots + t_{j-1}).
\]
A similar computation for \(C_r\) yields exactly the same estimate as above. Now suppose \(t_r = s_m\) for some \(m = 1, \ldots, r\). Since \(\{\alpha_j, j = 1, \ldots, r - 1\}\) make up a simple system for the root system \(A_{r-1}\), by the above result for Case \(A_r\), we have
\[
\prod_{1 \leq i < j \leq r} (t_i + t_{i+1} + \cdots + t_{j-1}) \gtrsim s_r^{r-1} s_{r-1} \cdots s_{m+1} \cdot s_m \cdots s_1.
\]
Also it is clear that
\[
\prod_{1 \leq i \leq r} (t_i + \cdots + t_r)^{r+1-i} \gtrsim s_r^{r-1} s_{r-1} \cdots s_{m+1} \cdot s_m \cdots s_1.
\]
The above two inequalities yield \(\prod_{\alpha \in \Sigma^+} \frac{\alpha(H)}{i} \gtrsim s_r^{2r-1}(H)s_{r-1}^2(H) \cdots s_1(H)\).

Case \(D_r\). We prove (2.8) for \((p_r, \ldots, p_1) = (2r - 2, 2r - 4, \ldots, 6, 3, 2, 1)\). A computation shows
\[
\prod_{\alpha \in \Sigma^+} \frac{\alpha(H)}{i} \approx \prod_{1 \leq i \leq r} (t_i + t_{i+1} + \cdots + t_r)^{r-i} \cdot \prod_{1 \leq i \leq r-1} (t_i + t_{i+1} + \cdots + t_{r-1} + t_r) \\
\cdot \prod_{1 \leq i < j \leq r} (t_i + t_{i+1} + \cdots + t_{j-1}).
\]
Suppose \(t_r = s_m\) for some \(m = 1, \ldots, r\). Since \(\{\alpha_j, j = 1, \ldots, r - 1\}\) make up a simple system for the root system \(A_{r-1}\), by the above result for Case \(A_r\), we have
\[
\prod_{1 \leq i < j \leq r} (t_i + t_{i+1} + \cdots + t_{j-1}) \gtrsim s_r^{r-1} s_{r-1} \cdots s_{m+1} \cdot s_m \cdots s_1.
\]
Now suppose \(t_{r-1} = s_n\) for some \(n = 1, \ldots, r\) and \(n \neq m\). Then
\[
t_i + t_{i+1} + \cdots + t_{r-2} + t_r \gtrsim \max\{s_{r-i}, s_m\}, \text{ for any } i = 1, \ldots, r - n,
\]
and
\[
t_i + t_{i+1} + \cdots + t_{r-2} + t_r \gtrsim \max\{s_{r-i}, s_m\}, \text{ for any } i = r - n + 1, \ldots, r - 1.
\]
Assume \(n \geq m\). We first have
\[
\prod_{1 \leq i \leq r-1} (t_i + t_{i+1} + \cdots + t_{r-2} + t_r) \gtrsim s_r s_{r-1} \cdots s_{n+1} \cdot s_{n-1} \cdots s_{m+1} \cdot s_m.
\]
We also have
\[
\prod_{1 \leq i \leq r-2} (t_i + t_{i+1} + \cdots + t_r)^{r-i-1} \gtrsim s_r^{r-2} s_{r-3} \cdots s_{n+1} \cdot s_{n-2} + n - 3 + \cdots + 1.
\]
The above two inequalities combined with (3.9) yield
\[
\prod_{\alpha \in \Sigma^+} \frac{\alpha(H)}{i} \gtrsim s_r^{2r-2} s_{r-1}^2 \cdots s_4^2 \cdot s_3^2 \cdot s_1.
\]
Assume $n < m$. We first have
\[
\prod_{1 \leq i \leq r-1} (t_i + t_{i+1} + \cdots + t_r) \geq s_r s_{r-1} \cdots s_{m+1} \cdot s_m^{m-1}.
\]
Also clearly
\[
\prod_{1 \leq i \leq r-2} (t_i + t_{i+1} + \cdots + t_r)^{r-i-1} \geq s_r^{r-2} s_{r-1}^{r-3} \cdots s_{m+1}^{m-1} \cdot s_m^{m-2+m-3+\cdots+1}.
\]
The above two inequalities combined with (3.9) also imply the desired estimate.

**Case $E_6$.** We prove (2.8) for $(p_6, \ldots, p_1) = (15, 9, 6, 3, 2, 1)$. An explicit computation yields
\[
\prod_{\alpha \in \Sigma^+} \frac{\alpha(H)}{i} \asymp (t_1 + \cdots + t_6)^6 (t_1 + \cdots + t_5)^5 (t_2 \cdots + t_6)^3 (t_1 + t_3 + \cdots + t_6)(t_2 + \cdots + t_5)^2
\]
\[
\cdot (t_2 + t_4 + t_6)^3 (t_1 + \cdots + t_4)(t_1 + t_3 + t_4 + t_6)(t_3 + \cdots + t_6)
\]
\[
\cdot (t_4 + t_5 + t_6)(t_3 + t_4 + t_5)(t_2 + t_3 + t_4)(t_1 + t_3 + t_4)(t_2 + t_4 + t_5)
\]
\[
\cdot (t_3 + t_4)(t_4 + t_5)(t_5 + t_6)(t_4 + t_2)(t_1 + t_3)(t_1 t_2) \cdots t_6.
\]
Let $n_j$ be the number of linear terms in the above expression that contain some of $s_6, s_5, \ldots, s_j$ ($j = 1, \ldots, 6$). An inspection shows that $n_6 \geq 15, n_5 \geq 24, n_4 \geq 30, n_3 \geq 33, n_2 = 35$. This implies the desired result.

**Case $E_7$.** We prove (2.8) for $(p_7, \ldots, p_1) = (27, 15, 9, 6, 3, 2, 1)$. Note that the Dynkin diagram of $E_7$ is that of $E_6$ adding the 7th node, an explicit computation shows
\[
\prod_{\alpha \in \Sigma^+} \frac{\alpha(H)}{i} = \prod_{\alpha \in \Sigma^+ \setminus \Sigma^+_{\{1, \ldots, 6\}}} \frac{\alpha(H)}{i} \cdot \prod_{\alpha \in \Sigma^+_{\{1, \ldots, 6\}}} \frac{\alpha(H)}{i},
\]
where
\[
\prod_{\alpha \in \Sigma^+ \setminus \Sigma^+_{\{1, \ldots, 6\}}} \frac{\alpha(H)}{i} \asymp (t_1 + \cdots + t_7)^{16} (t_2 + \cdots + t_7)^4 (t_1 + t_3 + \cdots + t_7)
\]
\[
\cdot (t_2 + t_4 + \cdots + t_7) \cdot \prod_{3 \leq j \leq 7} (t_j + \cdots + t_7)
\]
and $\prod_{\alpha \in \Sigma^+_{\{1, \ldots, 6\}}} \frac{\alpha(H)}{i}$ is the expression for $E_6$. By counting the number of linear terms in the above expression that contain a specific number of the $t_j$'s, we get a crude estimate
\[
\prod_{\alpha \in \Sigma^+ \setminus \Sigma^+_{\{1, \ldots, 6\}}} \frac{\alpha(H)}{i} \geq \max\{s_7, t_7\}^{16} \max\{s_6, t_7\}^{16} \max\{s_5, t_7\}^{16} \prod_{4 \leq i \leq 1} \max\{s_i, t_7\}.
\]
Assume $t_7 = s_7$. By Case $E_6$,
\[
\prod_{\alpha \in \Sigma^+_{\{1, \ldots, 6\}}} \frac{\alpha(H)}{i} \geq \frac{15}{6} \cdot \frac{9}{5} \cdot \frac{6}{4} \cdot s_3^3 s_2^2 s_1.
\]
By the above estimate, $\prod_{\alpha \in \Sigma^+ \setminus \Sigma^+_{\{1, \ldots, 6\}}} \frac{\alpha(H)}{i} \geq s_7^7$, hence we have the desired estimate.

Assume $t_7 = s_6$. By Case $E_6$,
\[
\prod_{\alpha \in \Sigma^+_{\{1, \ldots, 6\}}} \frac{\alpha(H)}{i} \geq \frac{15}{7} \cdot \frac{9}{5} \cdot \frac{6}{4} \cdot s_3^3 s_2^2 s_1.
\]
By the above estimate, $\prod_{\alpha \in \Sigma^+ \setminus \Sigma^+_{\{1, \ldots, 6\}}} \frac{\alpha(H)}{i} \geq s_7^6 s_6^{11}$, this also implies the desired estimate.

Assume $t_7 = s_5, s_4, \ldots, s_1$. Similarly argued as above.
Case $E_8$. We prove (2.8) for $(p_8, \ldots, p_1) = (57, 27, 15, 9, 6, 3, 2, 1)$. Similar to the computation for $E_7$, we have

$$\prod_{\alpha \in \Sigma^+} \frac{\alpha(H)}{i} = \prod_{\alpha \in \Sigma^+ \setminus \Sigma_2^{1, \ldots, 7}} \frac{\alpha(H)}{i} \cdot \prod_{\alpha \in \Sigma_1^{1, \ldots, 7}} \frac{\alpha(H)}{i},$$

where

$$\prod_{\alpha \in \Sigma^+ \setminus \Sigma_2^{1, \ldots, 7}} \frac{\alpha(H)}{i} \approx (t_1 + \cdots + t_8)^{44} (t_2 + \cdots + t_8)^5 (t_1 + t_3 + \cdots + t_8) \cdot (t_2 + t_4 + \cdots + t_8) \prod_{3 \leq j \leq 8} (t_j + \cdots + t_8).$$

Arguing exactly as Case $E_7$, we get the desired result.

Case $F_4$. We prove (2.8) for $(p_4, p_3, p_2, p_1) = (14, 6, 3, 1)$. A computation shows

$$\prod_{\alpha \in \Sigma^+} \frac{\alpha(H)}{i} \approx t_1 t_3 t_5 t_6 (t_1 + t_2) (t_2 + t_3)^2 (t_3 + t_4)^3 (t_1 + t_2 + t_3 + t_4)^9.$$ 

Let $n_j$ be the number of linear terms in the above expression that contain some of $s_4, s_3, \ldots, s_j$ ($j = 1, \ldots, 4$). An inspection shows $n_4 \geq 14, n_3 \geq 20, n_2 = 23$. This implies the desired result.

Case $G_2$. (2.8) holds trivially for $(p_2, p_1) = (5, 1)$.

References


