Video Lecture D4: Cramer's Rule

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Outline & Objectives

- Derive a simple theoretical formula (called *Cramer's Rule*) for computing solutions to an $n \times n$ system, $A\vec{x} = \vec{b}$, via determinants.
- Use this to prove a simple rule for computing the inverse of an invertible matrix A in terms of its determinant and a matrix of cofactors called the *adjugate*.

Cramer's Rule

For $A \in \mathbb{R}^{n\times n}$ and $\vec{b} \in \mathbb{R}^n$ set $A_i(\vec{b}) := [\vec{a}_1 \cdots \vec{b} \cdots \vec{a}_n]$.

Theorem (Cramer's Rule)

Let $A \in \mathbb{R}^{n\times n}$ be invertible and $\vec{b} \in \mathbb{R}^n$. Then the unique solution of $A\vec{x} = \vec{b}$ is given by $x_i = \dfrac{\det A_i(\vec{b})}{\det A}$ for $1 \le i \le n$.

$$\begin{cases} 3x_1 + 2x_2 = -1 \\ 5x_1 + 4x_2 = 3 \end{cases} \text{ has } x_1 = \frac{\begin{vmatrix} -1 & 2 \\ 3 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix}},\ x_2 = \frac{\begin{vmatrix} 3 & -1 \\ 5 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix}}, \text{ i.e., } \begin{bmatrix} -5 \\ 7 \end{bmatrix}.$$

Proof: $AI_i(\vec{x}) = [\vec{a}_1 \cdots \vec{a}_n][\vec{e}_1 \cdots \vec{x} \cdots \vec{e}_n] = [\vec{a}_1 \cdots (A\vec{x}) \cdots \vec{a}_n] = [\vec{a}_1 \cdots \vec{b} \cdots \vec{a}_n] = A_i(\vec{b})$. So $(\det A)(\det I_i(\vec{x})) = \det A_i(b)$. Just need to check that $\vec{x}_i = \det I_i(\vec{x})$. ■

Computing inverses via determinants

For $A \in \mathbb{R}^{n\times n}$, define the the *adjugate* of A by

$$\operatorname{adj} A = [C_{ij}]^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}, \text{ where } C_{k\ell} = (-1)^{k+\ell}|A_{k\ell}|.$$

For $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ -1 & 2 & 3 \end{bmatrix}$, $\operatorname{adj} A = \begin{bmatrix} -5 & 0 & 5 \\ -4 & 6 & -4 \\ 1 & -4 & 1 \end{bmatrix}$.

Theorem

For any invertible $A \in \mathbb{R}^{n\times n}$, $A^{-1} = \frac{1}{\det A} \operatorname{adj} A$.

Proof: $AY = I \Leftrightarrow [\vec{a}_1 \cdots \vec{a}_n][\cdots \vec{y} \cdots] = [\cdots \vec{e}_j \cdots] \Leftrightarrow A\vec{y} = \vec{e}_j$.
So jth col $\vec{y}$ of A^{-1} is given coordinatewise by $y_i = \frac{|A_i(\vec{e}_j)|}{|A|}$. Now check that $|A_i(\vec{e}_j)| = C_{ji}$, so $A^{-1} = \left[\frac{C_{ji}}{|A|}\right] = \frac{1}{\det A} \operatorname{adj} A$. ■

CHECK: What should we get when multiply $A(\operatorname{adj} A)$ above?