Video Lecture M9: The Invertible Matrix Theorem

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Outline & Objectives

- Analyze the reasoning that connects different statements that are all equivalent to a matrix $A \in \mathbb{R}^{n \times n}$ being invertible, thereby reviewing a number of crucial course concepts.
- Apply these in examples.

The Invertible Matrix Theorem

Theorem

For any square matrix $A \in \mathbb{R}^{n \times n}$ TFAE (The Following Are Equiv.):

1. A *is invertible.*
2. A *is row-equivalent to* I_n.
3. A *has* n *pivot positions.*
4. $A\vec{x} = \vec{0}$ *has only the trivial soln.* $\vec{x} = \vec{0}$.
5. *Columns of* A *are linearly independent.*
6. $A \mapsto A\vec{x}$ *is one-to-one.*
7. $\forall\, \vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ *has* ≤ 1 *soln.*
8. $\forall\, \vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ *has* ≥ 1 *soln.*
9. $A \mapsto A\vec{x}$ *is onto.*
10. *Columns of* A *span* $\mathbb{R}^n$.
11. $\exists\, C \in \mathbb{R}^{n \times n}$ *such that* $CA = I_n$.
12. $\exists\, D \in \mathbb{R}^{n \times n}$ *such that* $AD = I_n$.
13. A^T *is invertible.*
14. . . . *More to come!*

Applications of the Invertible Matrix Theorem

Suppose M is a 4×4 matrix and $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is not a linear combination of the columns of M. Could M be one-to-one? Can you say anything about the set of columns of M? What if M were 4×3?

Definition

Call a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ invertible if $\exists\, S : \mathbb{R}^n \to \mathbb{R}^n$ linear such that $S(T(\vec{x})) = \vec{x} = T(S(\vec{x}))$.

Theorem

Let A be the standard matrix for the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. Then T is invertible (with inverse S) $\iff$ A is invertible (with $S\vec{x} = A^{-1}\vec{x}$).