Video Lecture M8: Matrix Inverses: Properties, Elementary Matrices, & Algorithms

Tom Roby

Outline & Objectives

- Analyze algebraic properties of matrix inversion.
- Reconceptualize row operations as multiplication by elementary matrices to create an algorithm for computing matrix inverses.

Algebraic properties of matrix inversion

Proposition

Suppose that A *and* B *are invertible mxs in* $\mathbb{R}^{n\times n}$. *Then*

1. A^{-1} *is invertible and* $(A^{-1})^{-1} = A$.
2. A^T *is invertible and* $(A^T)^{-1} = (A^{-1})^T$.
3. AB *is invertible, and* $(AB)^{-1} = B^{-1}A^{-1}$

Inverse of any product of matrices in $\mathbb{R}^{n\times n}$ *is the product of the inverses in the reverse order:* $(A_1A_2\cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_1^{-1}$.

Elementary Matrices

Definition

An *elementary matrix* $E \in \mathbb{R}^{n \times n}$ *is one obtained from* I_n *by a single row operation.*

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \; E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Multiplying E by A just applies corresponding row-op to A!

Every elementary matrix is invertible. (why?)

Theorem

$A \in \mathbb{R}^{n \times n}$ *is invertible* $\iff$ A *is row-equivalent to* I_n*. Then any sequence of row ops that takes* A *to* I_n *also takes* I_n *to* A^{-1}.

Row-reduction algorithm for computing matrix inverses

Form "super-augmented" matrix $[A \mid I_n]$ and row-reduce until A is in RREF. If get $[I_n \mid B]$, then $B = A^{-1}$. If get $[\text{NOT } I_n | B]$, then A is not invertible.

$$\left[\begin{array}{rrr|rrr} 1 & -3 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -1/2 & -3/2 & -3/4 \\ 0 & 1 & 0 & -1/2 & -1/2 & -1/4 \\ 0 & 0 & 1 & -1/2 & -1/2 & 1/4 \end{array}\right]$$

CHECK!