Video Lecture M7: Matrix Inverses, Definitions & Examples

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Outline & Objectives

- Memorize the definition of the *inverse* A^{-1} of a matrix A and compute examples.
- Memorize the *formula* for the *inverse* A^{-1} of a matrix $A \in \mathbb{R}^{2\times 2}$, understand how it relates to the *determinant* of A.
- Utilize matrix inverses to solve corresponding systems of linear equations.

Inverse of square matrix

Definition

Call a square matrix $A \in \mathbb{R}^{n\times n}$ *invertible if there is a matrix* $B \in \mathbb{R}^{n\times n}$ *such that* $AB = I_n = BA$. *Write* $B = A^{-1}$.

$$\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -5/2 & 3/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5/2 & 3/2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}.$$

If A^{-1} exists, then it is *unique*: Say B, Q both inverses. Then $AB = I \implies Q = QI = Q(AB) = (QA)B = IB = B$.

Theorem (Formula for inverse of 2×2 matrix)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$ *if A is invertible.*

Check the formula for the above example

Call $ad - bc$ the *determinant of* A and write $\det A = ad - bc$. So for 2×2 matrices, A is invertible $\iff \det A \neq 0$.

Matrix inverses and solving systems of equations

Theorem (Invertible coeff. mxs always have unique solns)

If $A \in \mathbb{R}^{n\times n}$ is invertible, then for every $\vec{b} \in \mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$ in $\mathbb{R}^n$.

Rewrite $\begin{cases} 3x_1 + 2x_2 = -1 \\ 5x_1 + 4x_2 = 3 \end{cases}$ as $A\vec{x} = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Then $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5/2 & 3/2 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$. *CHECK!*

NB: $\vec{x} = A^{-1}\vec{b}$ is soln since $A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = \vec{b}$.