Video Lecture G1: Inner Products and Distance in $\mathbb{R}^n$

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Outline & Objectives

- Define the *inner product* $\vec{u} \cdot \vec{v}$ of two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ and analyze its essential properties.
- Define the *length* of a vector $\vec{v} \in \mathbb{R}^n$ as $\sqrt{\vec{v} \cdot \vec{v}}$.
- Generalize the notion of perpendicularity to define two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ to be *orthogonal* when $\vec{u} \cdot \vec{v} = 0$.

Inner Products in $\mathbb{R}^n$

To give approximate solns to $A\vec{x} = \vec{b}$, we need a notion of distance!

$$\begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Definition (Inner product of two vectors in $\mathbb{R}^n$)

The inner product or dot product of two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ is:

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Proposition

For any $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}$:

(1) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

(2) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

(3) $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$

(4) $\vec{u} \cdot \vec{u} \geq 0$ *and* $\vec{u} \cdot \vec{u} = 0 \iff \vec{u} = \vec{0}$.

Length of vectors in $\mathbb{R}^n$

Definition (length of a vector)

Call $\|\vec{x}\| := \sqrt{\vec{x}\cdot\vec{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ *the length or norm of* $\vec{x} \in \mathbb{R}^n$. *A unit vector* $\vec{u}$ *has* $\|\vec{u}\| = 1$.

$\|(-2,4,1,2)\| = \sqrt{(-2)^2 + 4^2 + 1^2 + 2^2} = 5 \Rightarrow \frac{\vec{x}}{\|\vec{x}\|} = (\frac{-2}{5}, \frac{4}{5}, \frac{1}{5}, \frac{2}{4})$.

Definition

The distance between $\vec{u}, \vec{v} \in \mathbb{R}^n$ *is* $\mathrm{dist}(\vec{u},\vec{v}) := \|\vec{u} - \vec{v}\|$.

Definition (orthogonality of two vectors)

Call two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ *orthogonal* if $\vec{u}\cdot\vec{v} = 0$.

Theorem ("Pythagorean Theorem")

Two vectors $\vec{u}$ *and* $\vec{v}$ *in* $\mathbb{R}^n$ *are orthogonal if and only if* $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$.