Video Lecture F8: Diagonalization of $\mathbb{R}$ Symmetric Matrices

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Outline & Objectives

- Combine the
  - geometric work we've done on orthogonality and projections with the
  - algebraic work on eigenvalues and change-of-basis

  to analyze the following fundamental result:
  Every symmetric matrix $A \in \mathbb{R}^{n \times n}$ is orthogonally diagonalizable: $A = PDP^{-1}$, where P is an orthogonal matrix.

- Give an example of how to compute this in practice.

Symmetric Matrices & orthogonal diagonalization

Which are *symmetric* ($A = A^T$)? If not, make them so!

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Which matrices are *orthogonal* ($U^T = U^{-1}$)? If not...

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad P = \frac{1}{5} \cdot \begin{bmatrix} 3 & 4 & 3 \\ -4 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Recall: A matrix $A \in \mathbb{R}^{n \times n}$ is diagonalizable (i.e., similar to a diagonal matrix) $\iff$ A has n lin indep eigenvectors, $\{\vec{v}_i\}_{i=1}^n$. Then A *factors* as $A = PDP^{-1}$, where D is diagonal and P is invertible. In fact, $P = [\vec{v}_1 \ \vec{v}_2 \ \ldots \ \vec{v}_n]$, and $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

Definition

Call a matrix $A \in \mathbb{R}^{n \times n}$ orthogonally diagonalizable if $A = PDP^{-1}$ for some orthogonal matrix P. (So $A = PDP^T$ also.)

Spectral Theorem for Real Symmetric Matrices

Proposition

$A \in \mathbb{R}^{n\times n}$ is orthogonally diagonalizable $\implies$ A is symmetric.

Pf: $A = PDP^T \implies A^T = (PDP^T)^T = P^{TT}D^TP^T = PDP^T = A$.

Lemma

If A is a (real) symmetric matrix, then any two eigenvectors from different eigenspaces are orthogonal.

Proof: Let $\vec{v}$ correspond to λ and $\vec{w}$ to μ with $\lambda \neq \mu$. Want to show that $\lambda(v\cdot w) = \mu(v \cdot w)$, forcing $v \cdot w = 0 \ldots$ ■

Theorem (Spectral Theorem for $\mathbb{R}$ symmetric matrics)

Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Then
(1) A has n real eigenvalues (counting multiplicities);
(2) $\dim E_\lambda = \operatorname{mult}\lambda$ (as root of $\chi(\lambda)$);
(3) For $\lambda \neq \mu$ in $\operatorname{Spec} A$, $E_\lambda \perp E_\mu$;
(4) A is orthogonally diagonalizable.

An Example of Orthogonal Diagonalization

$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$. *Q: What's an obvious eigenvalue/vector pair?*

$(1, 1, 1)$ is an eigenvector with eigenvalue 6.

$$\chi(\lambda) = \lambda^3 - 12\lambda^2 + 45\lambda - 54 = (\lambda - 6)(\lambda - 3)^2.$$

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

Q: Are the columns of P orthogonal? NO! What should we do? Apply Gram-Schmidt to the columns of P as needed. . . Now divide each column through as appropriate to normalize. Get

$A = QDQ^{-1} = QDQ^T$, where $Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$

CHECK: Q is an orthogonal matrix.