Video Lecture F6: Factoring Linear Transformations

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Outline & Objectives

- Define the matrix $[T]$ of any linear transformation $T : V \to W$ relative to ordered bases $\mathcal{B}$ (for V) and $\mathcal{C}$ (for W).

- Analyze diagonalization of a matrix A as the result of computing an ideal basis for the linear transformation $\vec{x} \mapsto A\vec{x}$ and changing coordinates relative to that basis.

The matrix of a linear transformation

Definition (Matrix of $T : V \to W$ relative to $\mathcal{B}$ and $\mathcal{C}$)

Let $T : V \to W$ be a lin transf, and let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ and $\mathcal{C} = \{\vec{c}_1, \ldots, \vec{c}_m\}$ be ordered bases for V and W, (resp). Then each $T(\vec{b}_i) = a_{1i}\vec{c}_1 + \cdots + a_{mi}\vec{c}_m$ (uniquely). Define the matrix of T relative to $\mathcal{B}$ and $\mathcal{C}$ by $[T] = [a_{ij}]$ (a $|\mathcal{C}| \times |\mathcal{B}|$ matrix).

Let $D : \mathbb{P}_3 \to \mathbb{P}_2$ by $D(f) = f'$. Let
$\mathcal{B} = \{1, 1+t, 1+t+t^2, 1+t+t^2+t^3\}$,
$\mathcal{C} = \{1, 1+t, 1+t+t^2\}$. Then $[D] = \begin{bmatrix} 0 & 1 & -1 & -1 \\ & & & \\ & & & \end{bmatrix}$

The matrix of a linear operator

Definition (Matrix of $T : V \to V$ relative to $\mathcal{B}$)

Let $T : V \to V$ be a lin transf, and let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be an ordered basis for V. Call the above matrix $[T]$ the matrix of T relative to $\mathcal{B}$ or the $\mathcal{B}$-matrix of T, written $[T]_{\mathcal{B}}$.

Let $T : \mathbb{P}_3 \to \mathbb{P}_3$ by $D(f) = tf'$. Let $\mathcal{E} = \{1, t, t^2, t^3\}$. Find $[T]_{\mathcal{E}}$.

Theorem (Similar matrices can represent same LT in $\mathbb{R}^n$)

Suppose $A = PCP^{-1}$, where $P = [\vec{b}_1 \ \vec{b}_2 \ \ldots \ \vec{b}_n]$. Let $T : \vec{x} \to A\vec{x}$ and $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$. Then $C = [T]_{\mathcal{B}}$ is the $\mathcal{B}$-matrix of T.

Proof: $P = P_{\mathcal{B}}$, the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{E}$.
So $[T]_{\mathcal{B}} = \left[[T(\vec{b}_1)]_{\mathcal{B}} \ \cdots \ [T(\vec{b}_n)]_{\mathcal{B}} \right] = \left[[A\vec{b}_1]_{\mathcal{B}} \ \cdots \ [A\vec{b}_n]_{\mathcal{B}} \right]$
$= [P^{-1}A\vec{b}_1 \ \cdots \ P^{-1}A\vec{b}_n] = P^{-1}A[\vec{b}_1 \ \cdots \ \vec{b}_n] = P^{-1}AP.$ ∎

$$A = \begin{bmatrix} 13 & -15 \\ 10 & -12 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$