Video Lecture F5: Diagonalization of Square Matrices

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Outline & Objectives

- Leverage knowledge of eigentheory and similarity to *diagonalize* certain $A \in \mathbb{R}^{n\times n}$ and give conditions when A is *diagonalizable* (i.e., similar to a diagonal matrix).

Diagonalization and Eigenvalues

Theorem (Diagonalization Theorem)

A matrix $A \in \mathbb{R}^{n\times n}$ is diagonalizable (i.e., similar to a diagonal matrix) $\iff$ A has n linearly independent eigenvectors, $\{\vec{v}_i\}_{i=1}^n$. Then A factors as $A = PDP^{-1}$, where D is diagonal and P is invertible. In fact, $P = [\vec{v}_1\ \vec{v}_2\ \dots\ \vec{v}_n]$, and $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$.

Key: $A = PDP^{-1} \iff AP = PD \iff A[\vec{v}_1\ \vec{v}_2\ \dots\ \vec{v}_n] = [\vec{v}_1\ \vec{v}_2\ \dots\ \vec{v}_n]\operatorname{diag}(\lambda_i) \iff [A\vec{v}_1\ A\vec{v}_2\ \dots\ A\vec{v}_n] = [\lambda_1\vec{v}_1\ \lambda_2\vec{v}_2\ \dots\ \lambda_n\vec{v}_n] \iff A\vec{v}_i = \lambda_i\vec{v}_i\ \forall\, i \in [n]$. ■

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/7 & 1/7 \\ 5/7 & -2/7 \end{bmatrix} = PDP^{-1}.$$

Corollary

If $A \in \mathbb{R}^{n\times n}$ has n distinct eigenvalues, then A is diagonalizable.

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 5 & -1 & 0 \\ 3 & 2 & 4 \end{bmatrix}.\quad C = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Diagonalizing Matrices 1

(1) Find the *eigenvalues* $\{\lambda_i\}$ (n counting multiplicities);
(2) Find a *basis* for each *eigenspace* E_{λ_i} (need $\dim E_{\lambda_i} = \text{mult}\,\lambda_i$).
(3) Construct $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ from (1),
and $P = [\vec{v}_1\ \vec{v}_2\ \ldots\ \vec{v}_n]$ from eigenvectors in (2)
(4) *CHECK* that $AP = PD$.

$$A = \begin{bmatrix} 8 & -6 & 6 \\ 12 & -10 & 12 \\ 3 & -3 & 5 \end{bmatrix} \implies \chi(\lambda) \begin{aligned} &= -\lambda^3 + 3\lambda^2 - 4 \\ &= -(\lambda+1)\cdot(\lambda-2)^2. \end{aligned}$$

$$A + I = \begin{bmatrix} 9 & -6 & 6 \\ 12 & -9 & 12 \\ 3 & -3 & 6 \end{bmatrix} \overset{\mathcal{R}}{\sim} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E_1 = \text{Span}\left\{ \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \right\}.$$

$$A - 2I = \begin{bmatrix} 6 & -6 & 6 \\ 12 & -12 & 12 \\ 3 & -3 & 3 \end{bmatrix} \overset{\mathcal{R}}{\sim} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E_2 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

$$AP = \begin{bmatrix} 8 & -6 & 6 \\ 12 & -10 & 12 \\ 3 & -3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Diagonalizing Matrices 2

(1) Find the *eigenvalues* $\{\lambda_i\}$ (n counting multiplicities);
(2) Find a *basis* for each *eigenspace* E_{λ_i} (need $\dim E_{\lambda_i} = \operatorname{mult} \lambda_i$).
(3) Construct $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ from (1),
and $P = [\vec{v}_1\ \vec{v}_2\ \ldots\ \vec{v}_n]$ from eigenvectors in (2)
(4) *CHECK* that $AP = PD$.

$$B = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 1 & 2 \end{bmatrix} \implies \lambda_1 = -1,\ \lambda_2 = \lambda_3 = 2.$$

$$B - 2I = \begin{bmatrix} -3 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 1 & 0 \end{bmatrix} \overset{\mathcal{R}}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E_2 = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$