Video Lecture F4: Similarity of Matrices

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Outline & Objectives

- Define the notion of *similar matrices* and show that similar matrices have the same eigenvalues.
- Define the *(algebraic) multiplicity* of an eigenvalue.

Similarity of Matrices

Definition (Similar matrices)

Call two square matrices $A, B \in \mathbb{R}^{n\times n}$ similar if $\exists$ an invertible $P \in \mathbb{R}^{n\times n}$ such that $A = PBP^{-1}$; then write $A \overset{\mathcal{S}}{\sim} B$.

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/7 & 1/7 \\ 5/7 & -2/7 \end{bmatrix}.$$

Proposition (Similarity is an equivalence relation)

$\forall\, A, B, C \in \mathbb{R}^{n\times n}$, we have (1) $A \overset{\mathcal{S}}{\sim} A$ (reflexive);
(2) $A \overset{\mathcal{S}}{\sim} B \implies B \overset{\mathcal{S}}{\sim} A$ (symmetric);
(3) $A \overset{\mathcal{S}}{\sim} B$ and $B \overset{\mathcal{S}}{\sim} C \implies A \overset{\mathcal{S}}{\sim} C$ (transitive);

Why is it useful that A is similar to a diagonal matrix? Say A is an monthly update matrix, and want to project out ten years.

$A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PD^2P^{-1}$.

Similarly $A^n = PD^nP^{-1}$, so $A^{120} = P \begin{bmatrix} 6^{120} & 0 \\ 0 & (-1)^{120} \end{bmatrix} P^{-1}$.

Similar matrices have the same eigenvalues

Theorem

If $A \overset{\mathcal{S}}{\sim} B$, then A and B have the same characteristic polynomial, hence the same eigenvalues (with the same multiplicities).

Proof: $\exists\, P$ invertible such that $A = PBP^{-1}$. Thus,
$|A - \lambda I| = |PBP^{-1} - \lambda PIP^{-1}| = |P(B - \lambda I)P^{-1}| =$
$|P| \cdot |B - \lambda I| \cdot |P^{-1}| = |P| \cdot \frac{1}{|P|} \cdot |B - \lambda I| = |B - \lambda I|$.

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} \overset{\mathcal{S}}{\sim} \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix} = B.$$

Definition (Multiplicity of an eigenvalue λ)

The (algebraic) multiplicity of an eigenvalue λ of A, is its multiplicity as a root of $\chi(\lambda)$.

Suppose an 11×11 matrix A has $\chi(\lambda) = -\lambda^{11} + \lambda^7 - \lambda^3$.