# Video Lecture F3: The Characteristic Equation of a Matrix

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## Outline & Objectives

- Utilize the *characteristic equation* $\det(A - \lambda I)$ of a matrix $A$ to compute $A$'s eigenvalues.

# The Characteristic Equation of a matrix $A$

**Theorem (Characteristic Equation of $A$ has eigenvalue roots)**

$\lambda \in \mathbb{R}$ *is an eigenvalue of* $A \in \mathbb{R}^{n\times n} \iff \lambda$ *satisfies the characteristic equation of* $A$, *given by* $\chi(\lambda) := \det(A - \lambda I)$.

*Pf:* $A\vec{x} = \lambda\vec{x}$ has a nonzero soln $\iff \mathrm{Nul}(A-\lambda I)$ is nontrivial $\iff A - \lambda I$ is not invertible $\iff \chi(\lambda) = |A - \lambda I| = 0$ by IMT.

$$B = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} \text{ has } \chi(\lambda) = \begin{vmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda - 6 = (\lambda - 6)(\lambda + 1).$$

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 0 & -3 \\ 2 & 1 & 4 \\ 1 & -2 & 4 \end{bmatrix} \Rightarrow \chi(\lambda) = -\lambda^3 + 7\lambda^2 - 25\lambda + 39 = (3-\lambda)(\lambda^2 - 4\lambda + 13).$$

$$A - 3I = \begin{bmatrix} -1 & 0 & -3 \\ 2 & -2 & 4 \\ 1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies \vec{x} \in \mathrm{Span}\left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

# Eigenvalues of Triangular Matrices

$$C = \begin{bmatrix} 2 & 5 & 4 & -1 \\ 0 & 5 & -1 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix} \implies$$

$$|C - \lambda I| = \begin{vmatrix} 2-\lambda & 5 & 4 & -1 \\ 0 & 5-\lambda & -1 & 1 \\ 0 & 0 & -1-\lambda & 2 \\ 0 & 0 & 0 & 5-\lambda \end{vmatrix} \implies$$

$$\chi(\lambda) = (2-\lambda)(-1-\lambda)(5-\lambda)^2.$$

**Theorem**

*The eigenvalues of a triangular matrix are the diagonal entries.*

Can also prove this directly by analyzing when $A - \lambda I$ has a free variable for $A$ triangular.