Video Lecture F1: Eigenvalues & Eigenvectors

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Outline & Objectives

- Memorize the fundamental definition of an *eigenvector* and its corresponding *eigenvalue* for any square matrix $A \in \mathbb{R}^{n \times n}$, and visualize it geometrically.
- Analyze how row reduction can be used to find the eigenvectors corresponding to a given eigenvalue.
- Explore basic properties that follow from definitions.

Eigenvectors and Eigenvalue

Motivation: Can we find a better basis for viewing the (*repeated*) action on $\mathbb{R}^n$ of $\vec{x} \mapsto A\vec{x}$, i.e., vectors $\vec{x}$ s.t. $A\vec{x}$ is "simple"?

Definition

Let $A \in \mathbb{R}^{n\times n}$. If we can find nonzero $\vec{x} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ satisfying $A\vec{x} = \lambda\vec{x}$, then we call $\vec{x}$ an eigenvector of A, and λ the corresponding eigenvalue.

$$\begin{bmatrix} 2 & 0 & -3 \\ 2 & 1 & 4 \\ 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ -3 \end{bmatrix} = 3 \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \text{ but } A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 3 \end{bmatrix}.$$

Given eigenvalue λ, how to find a corr. eigenvector $\vec{x}$?

$$A\vec{x} = \lambda\vec{x} \iff (A - \lambda I)\vec{x} = \vec{0} \iff \vec{x} \in \mathrm{Nul}(A - \lambda I).$$

$$A - 3I = \begin{bmatrix} -1 & 0 & -3 \\ 2 & -2 & 4 \\ 1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies \vec{x} \in \mathrm{Span}\left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

Q: What can you say about eigenvectors $\vec{x}$ for $\lambda = 0$?

Basic Properties of Eigenvalues & Eigenvectors

Q1: If $\vec{x}$ is an eigenvector for A corr. to λ, what about A^2?

Q2: If $\vec{x}$ is an eigenvector for A corr. to λ, what about A^{-1}?

Q3: What are the eigenvectors and eigenvalues of a diagonal matrix D?

Q4: How to visualize action of D geometrically?

Q5: Must every matrix have eigenvalues?

Q6: If $\vec{x}$ and $\vec{y}$ are eigenvectors for A, must $\vec{x} + \vec{y}$ also be?

Q7: If λ and μ are both eigenvalues for A, must $\lambda + \mu$ be?