# Video Lecture E8: Matrix Equations & Spanning Sets

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## Outline & Objectives

Justify the equivalence of the following four statements for an $m \times n$ matrix $A$

1. For every $\vec{b} \in \mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.
2. Every $\vec{b} \in \mathbb{R}^m$ is a linear combination of the columns of $A$.
3. The span of the columns of $A$ is (all of) $\mathbb{R}^m$.
4. $A$ has a pivot position in every row.

and use this to answer questions about sets of vectors.

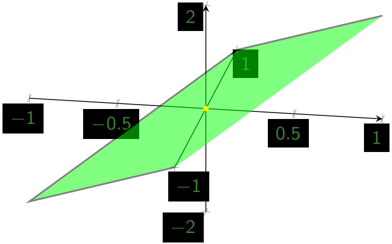
# Recall: Three ways to view a linear system

**Theorem**

*Let $A = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n]$ be an $m \times n$ matrix, $\vec{b} \in \mathbb{R}^m$. TFAE*
*(1) the solutions $\vec{x} \in \mathbb{R}^n$ to $A\vec{x} = \vec{b}$ (The Following Are Equal)*
*(2) the solutions to $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$*
*(3) the solutions to augmented matrix $\left[\begin{array}{cccc|c} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n & \vec{b} \end{array}\right]$*

When is there a solution?
Provided RREF avoids $\left[\begin{array}{ccc|c} 0 & \cdots & 0 & (\neq 0) \end{array}\right]$
When is there a solution for *every* $b \in \mathbb{R}^m$?
Provided $A$ has a pivot in every row!

$$\left[\begin{array}{ccc|c} -1 & 5 & 2 & b_1 \\ 3 & -14 & 5 & b_2 \\ -2 & 10 & 4 & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -5 & -2 & -b_1 \\ 0 & 1 & -1 & 3b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 + b_3 \end{array}\right]$$




When is this consistent?

### Theorem

*Let* $A = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n]$ *be an* $m \times n$ *matrix. TFAEquivalent:*
*(1) For every* $\vec{b} \in \mathbb{R}^n$*, the matrix eqn.* $A\vec{x} = \vec{b}$ *has a solution.*
*(2) Every* $\vec{b} \in \mathbb{R}^n$ *is a linear combination of the columns of* $A$*.*
*(3) The columns of* $A$ *span* $\mathbb{R}^m$*.*
*(4)* $A$ *has a pivot in every row.*

### Proof.

$(1) \iff (2) \iff (3)$ just from definitions.
If (4) holds, then by row reduction $[A \mid \vec{b}] \sim [U \mid \vec{c}]$, where $U$ has a pivot in each row. Use back-substitution to solve for each non-free variable in terms of the $c_i$'s and the free variables. If (4) is false, then bottom row of $U$ consists of all zeroes, so system will be inconsistent when $c_m = 1$. Reversing all row ops to $[A \mid \vec{b}]$, gives a $\vec{b}$ with no solution. ■

## Theorem

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*(3) The columns of* $A$ *span* $\mathbb{R}^m$*.*
*(4)* $A$ *has a pivot in every row.*

*Q:* Do the columns of the following matrix span $\mathbb{R}^3$?

$$\begin{bmatrix} -1 & 5 & 2 \\ 3 & -14 & 5 \\ -2 & 10 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & -3 & 5 & -2 \\ -3 & 1 & 2 & -2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 1.3 \\ 0 & 1 & 0 & 1.7 \\ 0 & 0 & 1 & 0.1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 4 & 5 \\ 2 & 6 & 3 & 5 \\ -1 & -3 & 1 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$