Video Lecture E5: Vectors, Linear Combinations, and Span

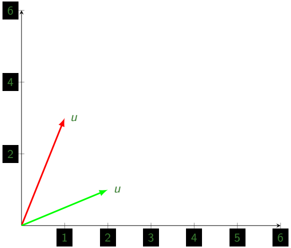
Tom Roby

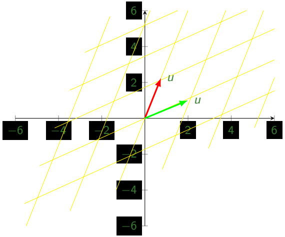
Outline & Objectives

- Compute sums, scalar multiples, and linear combinations of given vectors. Visualize examples in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Analyze whether a certain vector is in the *span* of a given set of vectors.

Definition

A vector in $\mathbb{R}^n$ is an n-tuple of numbers, which can be represented as $(u_1, u_2, \ldots, u_n)$ or (usually) as a one-column matrix $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$. *We can add two vectors coordinate-wise, and multiply a vector by any $k \in \mathbb{R}$ (called a scalar).*





Algebraic Properties of $\mathbb{R}^n$

Proposition (Vector Space Axioms)

For all vectors $\vec{u}, \vec{v}, \vec{w}$ *and all scalars* c *and* d*:*

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$
3. $\vec{u} + \vec{0} = 0 + \vec{u} = \vec{u}$
4. $\vec{u} + (-\vec{u}) = -\vec{u} + \vec{u} = 0$
5. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
6. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
7. $c(d\vec{u}) = (cd)\vec{u}$
8. $1\vec{u} = \vec{u}$

Linear Combinations of Vectors

Definition

Given scalars $c_1, c_2, \ldots, c_p$, *we call*

$$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$$

a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$.

Linear Combinations Inverse problem

Given scalars $c_1, c_2, \ldots, c_p$, and vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$, it's easy to write down the linear combination:
$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$

Question

Given $\vec{y}$ and $\{\vec{v}_i\}$, can we write $\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$ for some c_i?

Example

Write $\vec{y} = \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$ *as lin. comb. of* $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}.$

The span of a set of vectors

Definition

The span of a set of vectors $\{\vec{v}_i\}$ is the set of all lin.combs:
$\mathrm{Span}\{\vec{v}_i\} = \{c_1\vec{v}_1 + \cdots + c_p\vec{v}_p : c_i \in \mathbb{R}\}$

1. $\mathrm{Span}\left\{\vec{0}\right\} =$
2. $\mathrm{Span}\left\{\begin{bmatrix}1\\3\end{bmatrix}\right\} =$
3. $\mathrm{Span}\left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\} =$
4. $\mathrm{Span}\left\{\begin{bmatrix}1\\3\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\right\} =$
5. $\mathrm{Span}\left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\6\end{bmatrix}\right\} =$
6. $$\mathrm{Span}\left\{\begin{bmatrix}1\\-2\\4\end{bmatrix}, \begin{bmatrix}2\\1\\-3\end{bmatrix}, \begin{bmatrix}3\\-1\\1\end{bmatrix}\right\} =$$