# Video Lecture E12: Linear Independence in Theory

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# Outline & Objectives

- Understand *linear dependence* informally as characterizing whether certain vectors are "redundant" in terms of creating linear combinations.
- Analyze the reasoning behind the definition of linear (in)dependence and one vector being a linear combination of others in the set.

# Linear dependence & redundancy

**Definition**

*A set of vectors $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ is called linearly independent if the vector equation $x_1\vec{v}_1 + \cdots + x_p\vec{v}_p = \vec{0}$ has only the trivial soln. (all $x_i = 0$). The set is linearly dependent if there exists a nontrivial solution, $c_i$ not all zero, such that $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$. [$\rightsquigarrow$ linear dependence relation]*

**Theorem**

*An indexed set $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ of $p \geq 2$ vectors is linearly dependent $\iff$ at least one vector is a linear combination of the others. In fact, if $\vec{v}_1 \neq 0$, then there is $j \geq 2$ such that $v_j = d_1\vec{v}_1 + \cdots + d_{j-1}\vec{v}_{j-1}$, some $d_i \in \mathbb{R}$.*

*Proof:* $(\Rightarrow)$ Suppose $\vec{v}_1 \neq 0$, and let $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$ be the linear dependence relation. Let $j$ be the *largest* index such that $c_j \neq 0$ in the LDR. Then moving all previous terms to other side, we get $c_j\vec{v}_j = -c_1\vec{v}_1 - \cdots - c_{j-1}\vec{v}_{j-1}$. Now divide through by $c_j$, which is nonzero: $\vec{v}_j = -\frac{c_1}{c_j}\vec{v}_1 - \cdots - \frac{c_{j-1}}{c_j}\vec{v}_{j-1}$.

## Rewriting Example

The columns $\vec{a}_1, \vec{a}_2, \vec{a}_3$ of the following matrix are lin. dependent:

$$A = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 7 & 1 \\ 2 & 5 & 1 \end{bmatrix} \rightsquigarrow 2\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 5 \\ 7 \\ 5 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = 0.$$

So we can rewrite $\vec{a}_3 = -2\vec{a}_1 + \vec{a}_2$.

This also lets us simplify linear combinations. Suppose we know that $(11, 25, 17) = 2\vec{a}_1 + 3\vec{a}_2 - 2\vec{a}_3$. Then we can replace $\vec{a}_3$ with $-2\vec{a}_1 + \vec{a}_2$ to get:
$(11, 25, 17) = 2\vec{a}_1 + 3\vec{a}_2 - 2\left(-2\vec{a}_1 + \vec{a}_2\right) = 6\vec{a}_1 + \vec{a}_2$. *(Check!)*

So $\left[\begin{array}{ccc|c} 1 & 5 & 3 & b_1 \\ 3 & 7 & 1 & b_2 \\ 2 & 5 & 1 & b_3 \end{array}\right]$ has a soln. $\iff$ $\left[\begin{array}{cc|c} 1 & 5 & b_1 \\ 3 & 7 & b_2 \\ 2 & 5 & b_3 \end{array}\right]$ does,

i.e., $\mathrm{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\} = \mathrm{Span}\{\vec{a}_1, \vec{a}_2\}$.

# True or False?

1. *T/F:* If $\vec{u}_2$ is a scalar multiple of $\vec{u}_1$, then $S = \{\vec{u}_1, \vec{u}_2\}$ is linearly dependent.
2. *T/F:* If $S = \{\vec{u}_1, \vec{u}_2\}$ is linearly dependent, then $\vec{u}_2$ is a multiple of $\vec{u}_1$.
3. *T/F:* If the equation $A\vec{x} = \vec{b}$ has a solution (other than $\vec{x} = \vec{0}$), then the columns of $A$ are linearly dependent.
4. *T/F:* If $\{w_1, w_2, w_3\}$ is linearly dependent, then so is $\{w_1, w_2, w_3, w_4\}$.