Video Lecture E11: Linear Independence

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Outline & Objectives

- Memorize the fundamental definitions of *linear dependence* and *linear independence* of sets of vectors in $\mathbb{R}^n$.
- Compute whether a given set of vectors is linearly dependent or independent, and in the former case, compute a *linear dependence relation*.
- Use theorems to quickly analyze whether a given set of vectors is linearly dependent or independent.

Linear dependence & independence

A homogeneous matrix equation is equivalent to a vector eqn:

$$\begin{bmatrix} 1 & 5 & 3 \\ 3 & 7 & 1 \\ 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff x_1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 7 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Definition

A set of vectors $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ is called linearly independent if the vector equation $x_1\vec{v}_1 + \cdots + x_p\vec{v}_p = \vec{0}$ has only the trivial soln. (all $x_i = 0$). The set is linearly dependent if there exists a nontrivial solution, c_i not all zero, such that $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$. [$\rightsquigarrow$ linear dependence relation]

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies \begin{cases} x_1 & = & 2x_3 \\ x_2 & = & -x_3 \\ x_3 & \text{is} & \text{free} \end{cases} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

Special Cases

1. *One nonzero vector:* Let $S = \{\vec{v}_1\}$ with $\vec{v}_1 \neq 0$.
2. *A set containing $\vec{0}$:*
3. *A set with two vectors:*
   Let $\vec{u}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -3 \\ 6 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ -6 \end{bmatrix},$.
4. *A set containing too many vectors:*
   $$\left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Theorems (Labor-saving devices)

Theorem

If a set $S = \{\vec{v}_1, \ldots, \vec{v}_p\} \subseteq \mathbb{R}^n$ *contains the zero vector, then it is linearly dependent.*

Theorem

If $p > n$, *then the set* $S = \{\vec{v}_1, \ldots, \vec{v}_p\} \subseteq \mathbb{R}^n$ *is linearly dependent.*

(a) $\left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix} \right\}$, (b) $\left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$,

(c) $\left\{ \begin{bmatrix} 4 \\ -6 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ -9 \\ 3 \end{bmatrix} \right\}$, (d) $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}$.