Video Lecture E10: Solving general systems

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Outline & Objectives

- Compute all solutions to a general system $A\vec{x} = \vec{b}$, and connect them with the corresponding homogenous system $A\vec{x} = \vec{0}$.
- Visualize the geometry of the solution sets.

Nonhomogenous systems

Definition

A system of linear equations whose matrix form is $A\vec{x} = \vec{b}$ *where* $\vec{b} \neq 0$ *is called* ***nonhomogenous*** *(aka* ***inhomogenous****).*

We've seen that some nonhomogenous systems have no solns.

$$\left[\begin{array}{ccc|c} -1 & 5 & 2 & b_1 \\ 3 & -14 & -7 & b_2 \\ -2 & 10 & 4 & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -7 & 14b_1 + 5b_2 \\ 0 & 1 & -1 & 3b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 + b_3 \end{array}\right]$$

But if it *does* have a soln, how to find them *all*?

$$\left[\begin{array}{ccc|c} -1 & 5 & 2 & 1 \\ 3 & -14 & -7 & -5 \\ -2 & 10 & 4 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -7 & -11 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 7x_3 - 11$

$x_2 = x_3 - 2$

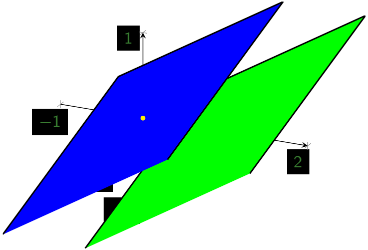
x_3 is free

So any vector of the form $\begin{bmatrix} -11 \\ -2 \\ 0 \end{bmatrix} + k \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}$ is a solution (line in $\mathbb{R}^3$).

Another example

$$\begin{cases} x_1 \phantom{{}-x_2} + x_3 + x_4 = 2 \\ 2x_1 - x_2 \phantom{{}+3x_3} + x_4 = 3 \\ x_1 + x_2 + 3x_3 + 2x_4 = 3 \end{cases} \mapsto \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 2 & -1 & 0 & 1 & 3 \\ 1 & 1 & 3 & 2 & 3 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \mapsto \begin{cases} x_1 = 2 - x_3 - x_4 \\ x_2 = 1 - 2x_3 - x_4 \end{cases}$$

$$\vec{x} = \begin{bmatrix} 2 - x_3 - x_4 \\ 1 - 2x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + k \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} + \ell \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$



Theory and Practice of solving SLE

Theorem

Say the equation $A\vec{x} = \vec{b}$ is consistent for given A, $\vec{b}$, and let $\vec{p}$ be any solution (so $A\vec{p} = \vec{b}$). Then the solution set is all vectors of form $\vec{u} = \vec{p} + \vec{v}_h$, where $\vec{v}_h$ is a solution to the homogeneous equation $A\vec{x} = \vec{0}$.

Practice

1. Row reduce augmented $[A \mid \vec{b}]$ to RREF;
2. Express each basic variable in terms of constants and free variables;
3. Write a typical soln. $\vec{x}$ using the above.
4. Decompose it into a linear combination of *numeric* vectors (with free variables as coefficients/parameters).