Video Lecture D2: Determinants and Row operations

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Outline & Objectives

- Explore how row operations affect the determinant of a matrix. Use this to compute examples.
- Demonstrate another equivalent condition in the Invertible Matrix Theorem: $\det A \neq 0$.

Determinants and row reduction

Theorem (How row operations change the determinant)

Let $A \in \mathbb{R}^{n\times n}$.

1. *Rescaling a row of A by k rescales $\det A$ by k.*
2. *Interchanging two rows of A changes the sign of $\det A$.*
3. *Adding a multiple of one row to another leaves $\det A$ unchanged.*

$$\begin{vmatrix} 3 & 6 & 0 & 9 \\ 0 & 0 & -4 & 2 \\ 2 & 5 & 1 & 6 \\ -1 & 0 & 3 & 1 \end{vmatrix} = 3\cdot \begin{vmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & -4 & 2 \\ 2 & 5 & 1 & 6 \\ -1 & 0 & 3 & 1 \end{vmatrix} = 3(-1)\begin{vmatrix} 1 & 2 & 0 & 3 \\ 2 & 5 & 1 & 6 \\ 0 & 0 & -4 & 2 \\ -1 & 0 & 3 & 1 \end{vmatrix} = -3\begin{vmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -4 & 2 \\ 0 & 2 & 3 & 4 \end{vmatrix} =$$

$$-3\begin{vmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & 1 & 4 \end{vmatrix} = -3\begin{vmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & 4.5 \end{vmatrix} = (-3)(1)(1)(-4)(4.5) = 54.$$

Computing Determinants via Row Reduction

Theorem (Computing $\det A$ via reduction to triangular mx)

Suppose $A \sim U$ via row replacements and r row interchanges, where U is upper-triangular. Then $\det A = (-1)^r u_{11} u_{22} \dots u_{nn}$.

Corollary (Determinant condition for invertible matrix)

A square matrix $A \in \mathbb{R}^{n \times n}$ is invertible $\iff$ $\det A \neq 0$.

Proof: Use that A is invertible $\iff$ A has n (nonzero) pivots $\iff$ every $u_{ii} \neq 0$ $\iff$ $u_{11} \cdots u_{nn} \neq 0$. ■

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 9 & -2 & 4 \\ 6 & 0 & 4 \end{bmatrix}$$