Video Lecture D1: Introduction to Determinants

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Outline & Objectives

- Memorize the inductive definition of the *determinant* of a square matrix and compute examples.
- Generalize this to the *Laplace expansion*, which computes $\det A$ by *cofactors*, then analyze how to simplify computations via the choice of which row or column to expand over.
- Leverage this to demonstrate the simple formula for the determinant of a *triangular* matrix.

Inductive definition of determinants

Definition (Determinant of a square matrix)

For $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, *let* $A_{k\ell} \in \mathbb{R}^{n-1\times n-1}$ *be* A *with the* k*th row and* ℓ*th column of* A *deleted. Set* $\det[a] = a$ *and for* $k \geq 2$

$|A| = \det A = a_{11}\det A_{11} - a_{12}\det A_{12} + \ldots + (-1)^{1+n} a_{1n}\det A_{1n}$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ -1 & 2 & 3 \end{bmatrix}$$

Definition (Cofactor)

The (i,j)*-cofactor of* A *is* $C_{ij} := (-1)^{i+j}\det A_{ij}$.

Theorem (Laplace Expansion)

$\forall\, k, \ell,\ \det A = a_{k1}C_{k1} + \cdots + a_{kn}C_{kn} = a_{1\ell}C_{1\ell} + \cdots + a_{n\ell}C_{n\ell}$.

Can compute $\det A$ by expanding along *any* row or column!

Triangular matrices

How should we expand $\begin{bmatrix} 2 & 4 & 1 & 3 \\ 0 & 3 & -1 & -2 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$?

Definition

Call a matrix $A = [a_{ij}]$ *upper triangular if* $a_{ij} = 0$ *for* $i > j$ *and lower triangular if* $a_{ij} = 0$ *for* $i < j$*. Call* A *triangular either way.*

Theorem

For a triangular matrix A, $\det A = a_{11} a_{22} \dots a_{nn}$ *(product of its diagonal entries).*