Video Lecture B8: Coordinate Systems & Isomorphic Vector Spaces

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Outline & Objectives

- Harness the notion of $\mathcal{B}$-*coordinates* for vectors in *arbitrary* vector spaces V to show *(linear) isomorphisms* between different vector spaces.
- In particular, if V has a basis $\mathcal{B}$ of cardinality n, then V is isomorphic to $\mathbb{R}^n$.

The Coordinate Mapping

Theorem

Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for V. Then the coordinate map φ given by $\vec{x} = c_1\vec{b}_1 + \cdots c_n\vec{b}_n \overset{\varphi}{\mapsto} [x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is a 1-1 and onto linear transformation from V to $\mathbb{R}^n$.

$\mathbb{P}_2$ has basis $\mathcal{P} : \{1, t, t^2\}$. Then

$\vec{g}(t) = c + bt + at^2 \overset{\varphi}{\mapsto} \begin{bmatrix} c \\ b \\ a \end{bmatrix} \in \mathbb{R}^3$. Suppose $\vec{h}(t) = f + et + dt^2$.

What are $\mathcal{P}$-coords of $\vec{g}(t) + \vec{h}(t)$? $\mathbb{P}_2$ also has basis $\mathcal{Q} = \{\vec{q}_1(t) = 1, \vec{q}_2(t) = 1 + t, \vec{q}_3(t) = 1 + t + t^2\}$. What are $\mathcal{Q}$-coordinates of $2 - 5t + 3t^2$?

Vector space isomorphisms

Definition

Let V and W be two vector spaces. If there exists a one-to-one and onto linear transformation $\varphi : V \to W$, then we call φ an isomorphism (Greek for "same form") and say that V is isomorphic to W, written $V \cong W$.

Above theorem showed that if V has a basis with n elements, then V is isomorphic to $\mathbb{R}^n$.

(1) $\mathbb{P}_2 \cong \mathbb{R}^3$. In general, $\mathbb{P}_d \cong \mathbb{R}^{d+1}$. (2) $\mathbb{R}^{2\times 2} \cong \mathbb{R}^4$. In general, $\mathbb{R}^{m\times n} \cong \mathbb{R}^{mn}$. (3) $\mathbb{P}_3 \cong \mathbb{R}^{2\times 2}$.

There can be many different isomorphisms between isomorphic vector spaces. $\mathbb{P}_2$ has bases $\mathcal{P} = \{1, t, t^2\}$ *and* $\mathcal{Q} : \{1, 1+t, 1+t+t^2\}$. Either way $\mathbb{P}_2 \cong \mathbb{R}^3$, but e.g.,

$$[1+t+t^2] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{P}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{Q}}.$$