Video Lecture B4: The column space of a matrix

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Outline & Objectives

- Define the *column space* $\mathrm{Col}\,A$ of a matrix A, show that it is a *subspace*, and connect it with the *Span* of the columns of A.
- Compare and contrast the *column space* with the *null space* of a matrix A.
- Define the notions of *linear transformation*, *kernel*, and *range* for general vector spaces and connect these with *nullspace* and *column space* of the corresponding matrix.

The column space of a matrix

Definition ($\mathrm{Col}\, A$)

For a matrix $A \in \mathbb{R}^{m \times n}$, define the column space of A by

$$\mathrm{Col}\, A := \{\vec{y} \in \mathbb{R}^m : \vec{y} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^n\}.$$

Easy to see: $\vec{y} \in \mathrm{Col}\, A \iff \vec{y} = A\vec{x} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n \iff \vec{y} \in \mathrm{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$, explaining the term *column* space.

Proposition

For any matrix $A \in \mathbb{R}^{m \times n}$, $\mathrm{Col}\, A$ is a subspace of $\mathbb{R}^m$.

Proof: *(1)* Is $\vec{0} \in \mathrm{Col}\, A$? *(2)* Is $\mathrm{Col}\, A$ closed under vector addition? *(3)* Is $\mathrm{Col}\, A$ closed under scalar multiplication? *Or* just use proposition from VL#B2 that $\mathrm{Span}\{\vec{a}_i\}$ is a subspace.

EG: If $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $\vec{b} \in \mathrm{Col}\, A \iff$

$$b \in \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \iff b \in \mathbb{R}^2.$$

Comparing $\operatorname{Nul} A$ with $\operatorname{Col} A$

Let $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 3 & -1 & 2 & 6 \\ 2 & 2 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\operatorname{Nul} A$	$\operatorname{Col} A$
is a subset of $\mathbb{R}^n$ (domain)	is a subset of $\mathbb{R}^m$ (target space)
is defined implicitly	is given explicitly as a Span.
hard to find vectors therein	easy to find vectors therein
$\vec{v} \in \operatorname{Nul} A \iff A\vec{v} = \vec{0}$	$\vec{v} \in \operatorname{Col} A \iff A\vec{x} = \vec{v}$ is consistent
Given $\vec{v}$, easy to check whether $\vec{v} \in \operatorname{Nul} A$.	Given $\vec{v}$, hard to check whether $\vec{v} \in \operatorname{Col} A$.
$\operatorname{Nul} A = \{\vec{0}\} \iff A\vec{x} = 0$ has only trivial soln.	$\operatorname{Col} A = \mathbb{R}^m \iff A\vec{x} = \vec{b}$ is consistent $\forall \vec{b} \in \mathbb{R}^m$.
$\operatorname{Nul} A = \{\vec{0}\} \iff$ the lin transf $\vec{x} \mapsto A\vec{x}$ is 1-1	$\operatorname{Col} A = \mathbb{R}^m \iff$ the lin transf $\vec{x} \mapsto A\vec{x}$ is onto

Linear Transformations for general vector spaces

Definition

A function $T : V \to W$ *between two vector spaces is a linear transformation if it satisfies:*
(a) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})\ \forall\, \vec{u}, \vec{v} \in V$.
(b) $T(c\vec{v}) = cT(\vec{v})\ \forall\, \vec{v} \in V$ *and* $\forall$ *scalar* c.
Define the kernel of T *by* $\ker T = \{\vec{x} \in V : T(\vec{x}) = \vec{0}\}$.
Define the range of T *by* $\mathrm{range}\, T \langle \{T(\vec{x}) : \vec{x} \in V\} \subseteq W$.

So if $T(\vec{x}) = A\vec{x}$, then $\ker T = \mathrm{Nul}\, A$ and $\mathrm{range}\, T = \mathrm{Col}\, A$.