Video Lecture B3: The nullspace of a matrix

Tom Roby

Outline & Objectives

- Define the *nullspace* $\operatorname{Nul} A$ of a matrix A, show that it is a *subspace*, and connect it with solving homogeneous systems of equations.

The nullspace of a matrix

Definition ($\operatorname{Nul} A$)

For a matrix $A \in \mathbb{R}^{m\times n}$, define the nullspace of A by

$$\operatorname{Nul} A := \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}.$$

Proposition

For any matrix $A \in \mathbb{R}^{m\times n}$, $\operatorname{Nul} A$ is a subspace of $\mathbb{R}^n$. Thus, so is the set of homogenous solutions to an $m \times n$ system of equations.

Proof: *(1)* Is $\vec{0} \in \operatorname{Nul} A$? *(2)* Is $\operatorname{Nul} A$ closed under vector addition? *(3)* Is $\operatorname{Nul} A$ closed under scalar multiplication?

EG: If $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $\vec{x} \in \operatorname{Nul} A \iff \begin{cases} x_1 = 2x_2 \\ x_3 = 0 \end{cases}$

So $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, a line in $\mathbb{R}^3$.

Another example

$$\begin{cases} x_1 \phantom{{}-x_2} + x_3 + x_4 = 0 \\ 2x_1 - x_2 \phantom{{}+3x_3} + x_4 = 0 \\ x_1 + x_2 + 3x_3 + 2x_4 = 0 \end{cases} \mapsto \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & -1 & 0 & 1 \\ 1 & 1 & 3 & 2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mapsto \begin{cases} x_1 \phantom{{}x_2} = -x_3 - x_4 \\ x_2 = -2x_3 - x_4 \end{cases},$$

$$\vec{x} = \begin{bmatrix} -x_3 - x_4 \\ -2x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = \cdots = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = k \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} + \ell \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \text{ where } k, \ell \in \mathbb{R}.$$