Video Lecture B12: Changes of Basis

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Outline & Objectives

- Construct the *change of basis* matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ between any two bases $\mathcal{B}$ and $\mathcal{C}$ of a (finite-dim) vector space V.
- Analyze how $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is related to $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$.

Change of Basis

Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$, and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ be bases for $\mathbb{R}^2$. We can express $\vec{x} \in \mathbb{R}^2$ in terms of either $\mathcal{B}$ or $\mathcal{C}$. How to go between the two?

Suppose we know that $\vec{b}_1 = 3\vec{c}_1 - 2\vec{c}_2$ and $\vec{b}_2 = 2\vec{c}_1 - \vec{c}_2$. So $[\vec{b}_1]_{\mathcal{C}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}_{\mathcal{C}}$ and $[\vec{b}_2]_{\mathcal{C}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}_{\mathcal{C}}$. Now if $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}_{\mathcal{B}}$, then $\vec{x} = 4\vec{b}_1 - 5\vec{b}_2 \implies [\vec{x}]_{\mathcal{C}} = [4\vec{b}_1 - 5\vec{b}_2]_{\mathcal{C}} \overset{!}{=} 4[\vec{b}_1]_{\mathcal{C}} - 5[\vec{b}_2]_{\mathcal{C}} =$ $4\begin{bmatrix} 3 \\ -2 \end{bmatrix} - 5\begin{bmatrix} 2 \\ -1 \end{bmatrix} \overset{!}{=} \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}_{\mathcal{C}}$. So $[\vec{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\vec{x}]_{\mathcal{B}}$, where $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\vec{b}_1]_{\mathcal{C}} & [\vec{b}_2]_{\mathcal{C}} \end{bmatrix}$.

What if $\mathcal{C}$ were the standard basis $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$?

Theorem

Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$, and $\mathcal{C} = \{\vec{c}_1, \ldots, \vec{c}_n\}$ be bases for V. Then the (unique) matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\vec{b}_1]_{\mathcal{C}} & [\vec{b}_2]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}} \end{bmatrix}$ is the change of basis matrix such that $[\vec{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\vec{x}]_{\mathcal{B}}$.

EG: $\mathcal{B} = \{\vec{b}_1 = 1 - t,\ \vec{b}_2 = 1 + 4t + t^2,\ \vec{b}_3 = 4 - 2t^2\}$, and
$\mathcal{C} = \{\vec{c}_1 = 1,\ \vec{c}_2 = 1 + t,\ \vec{c}_3 = 1 + t + t^2\}$
are bases of $V = \mathbb{P}_2$. We can write:
$\vec{b}_1 = 1 - t = 2\vec{c}_1 - \vec{c}_2$; $\vec{b}_2 = 1 + 4t + t^2 = -3\vec{c}_1 + 3\vec{c}_2 + \vec{c}_3$;
$\vec{b}_3 = 4 - 2t^2 = 4\vec{c}_1 + 2\vec{c}_2 - 2\vec{c}_3$; hence,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 2 & -3 & 4 \\ -1 & 3 & 2 \\ 0 & 1 & -2 \end{bmatrix}.$$

Corollary

(1) $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1}$. *(2)* $\left(\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}\right)\left(\underset{\mathcal{C}\leftarrow\mathcal{D}}{P}\right) = \underset{\mathcal{B}\leftarrow\mathcal{D}}{P}$.