Video Lecture B11: The Rank of a Matrix

Tom Roby

Outline & Objectives

- Define the *row space*, $\operatorname{Row} A$, of a matrix A, and analyze how row operations affect it.
- Define the *rank*, $\operatorname{rank} A := \dim \operatorname{Col} A$, of a matrix A and prove that it is the same as $\dim \operatorname{Row} A$.
- Prove *The Rank Theorem* aka *The rank-nullity theorem*, which says that

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n$$

The Row Space of a Matrix

Definition (Row space of A)

For any $A \in \mathbb{R}^{m\times n}$, let $\operatorname{Row} A := \operatorname{Span}\{\textit{all rows of } A\} \subseteq \mathbb{R}^n$.
Equivalently, $\operatorname{Row} A = \operatorname{Col} A^T$.

$$A = \begin{bmatrix} 1 & 2 & -3 & -2 & 3 \\ 2 & 5 & -8 & 1 & 2 \\ 1 & 1 & -1 & -7 & 7 \\ 2 & 4 & -6 & -1 & 3 \end{bmatrix} \sim B = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem

Suppose $A \overset{row}{\sim} B$. Then $\operatorname{Row} A = \operatorname{Row} B$. If B is in RREF, then the nonzero rows of B form a basis for $\operatorname{Row} B$.

What would be a basis for $\operatorname{Col} A$ above?

For $\operatorname{Row} A$, must use the rows of B, not of A (cf. for $\operatorname{Col} A$).

Definition (Rank of a matrix)

Define the rank of $A \in \mathbb{R}^{m\times n}$ *to be* $\dim \operatorname{Col} A$.

Theorem (The Rank Theorem)

For $A \in \mathbb{R}^{m\times n}$, $\dim \operatorname{Row} A = \dim \operatorname{Col} A = \operatorname{rank} A = \#$ *pivots in* A.
Also, $\operatorname{rank} A + \dim \operatorname{Nul} A = n =$ *#columns of* A.

Pf: Each column of A either contains a pivot (corresponding to a basis vector of $\operatorname{Col} A$) or corresponds to a free variable (corresponding to a basis vector of $\operatorname{Nul} A$). ■

What can you say about $\dim \operatorname{Nul} A$ for $A \in \mathbb{R}^{5\times 11}$?

An engineer has found five solutions to a homogenous system of 50 equations in 54 variables. If none of her solutions is a linear combination of the other four, can she be sure that she can solve any inhomogenous system with the same coefficients?

Additions to the Invertible Matrix Theorem

Theorem

For any square matrix $A \in \mathbb{R}^{n\times n}$ TFAE (The Following Are Equiv.):

1. A is invertible.
2. A is row-equivalent to I_n.
3. A has n pivot positions.
4. $A\vec{x} = \vec{0}$ has only the trivial soln. $\vec{x} = \vec{0}$.
5. Columns of A are linearly independent.
6. $A \mapsto A\vec{x}$ is one-to-one.
7. $\forall\, b \in \mathbb{R}^n$, $A\vec{x} = b$ has ≤ 1 soln.
8. $\forall\, b \in \mathbb{R}^n$, $A\vec{x} = b$ has ≥ 1 soln.
9. $A \mapsto A\vec{x}$ is onto.
10. Columns of A span $\mathbb{R}^n$.
11. $\exists\, C \in \mathbb{R}^{n\times n}$ such that $CA = I_n$.
12. $\exists\, D \in \mathbb{R}^{n\times n}$ such that $AD = I_n$.
13. A^T is invertible.
14. Cols (or rows) of A are basis for $\mathbb{R}^n$.
15. $\operatorname{Col} A = \mathbb{R}^n$.
16. $\dim \operatorname{Col} A = n$.
17. $\operatorname{rank} A = n$.
18. $\operatorname{Nul} A = \{\vec{0}\}$.
19. $\dim \operatorname{Nul} = 0$.