Resistance of random Sierpiński gaskets

Daniel Fontaine, Thomas Smith, and Alexander Teplyaev

Abstract. We analyze resistance (Dirichlet) forms on the random Sierpiński gaskets introduced in an earlier work of Meyers, Strichartz and the last author. Such resistance networks are continuous limits of quantum graphs.

Contents

1. Introduction

In this paper we analyze random resistance networks or, equivalently, energy (Dirichlet) forms, on random Sierpiński gaskets. Such resistance networks can be described as certain limits of quantum graphs, as demonstrated in [**49**]. More precisely, we analyze the approximating sequences of quantum graphs which satisfy appropriate compatibility conditions. We use a specific algorithm, introduced in [**39**], that constructs all compatible Dirichlet forms based on a random choice of parameters. Defining an energy form on the Sierpiński gasket allows one to introduce such objects as Laplacian, diffusion processes etc.

It was proved in [**39**] that the harmonic functions are continuous with probability one. We prove that with probability one the random Sierpiński gasket has a homeomorphic harmonic embedding into the two dimensional Euclidean space, i.e. one can use a pair of harmonic functions as coordinates. Then it is possible to write the energy of a smooth function as the integral, with respect to a reference

2000 *Mathematics Subject Classification.* Primary: 28A80; Secondary: 34B45, 60D05, 60J60, 78A48, 94C99.

Key words and phrases. Self-similar Dirichlet form, resistance network, random fractal, Sierpinski gasket, quantum graphs.

Supported in part by NSF grant DMS-0505622.

energy measure, of the norm squared of the gradient. Then one can represent the energy measure Laplacian as a second derivative in a sense. In addition, we show that, in harmonic coordinates, the angles of approximating triangles tend to zero with probability one at every junction point.

A computer program was written that executes the randomization algorithm and calculates resistances which define energy (Dirichlet) forms on the Sierpiński gasket. We attempted to answer certain questions that arose naturally in our examination: Do the calculated resistances form a log normal distribution? How does the data change when we alter the domain of the parameters? The analysis of the data is done in greater detail in Section 4. We include pictures of the random Sierpiński gaskets in harmonic coordinates as well as histograms pertaining to our data. We demonstrate that individual resistance tend to zero if the disorder is small, but for larger disorders "most" of the resistances tend to zero, but there are some resistances that approach infinity. We conjecture that for small disorder with probability one the effective resistance topology coincides with the standard topology on the Sierpiński gasket. However, as the disorder gets larger, with probability one there are points in the Sierpiński gasket which are at infinite effective resistance distance from the boundary. Equivalently, for small disorder the diffusion process is point recurrent on the Sierpiński gasket, and for larger disorder the diffusion process is point recurrent on a proper subset of the Sierpiński gasket.

The analysis on self-similar fractals was first developed in the physics and engineering literature, see [**2, 14, 23, 42, 43, 45**] and references therein. There are three mathematical books, [**3, 28, 47**], that provide background to the analysis on self-similar fractals. The following papers deal with the analysis on the Sierpiński gasket and other fractals which is relevant to our work: [**1, 8, 21, 22, 24, 25, 26, 27, 29, 37, 39, 40, 44, 46, 48, 49**]. Also, there are many probabilistic works on the diffusions and random walks on self-similar fractals and graphs, see for instance [**7, 12, 15, 35, 36**] and references therein. It was recently demonstrated by Kigami, using the new theory of heat kernel estimates on metric measure spaces (see [**4, 5, 6, 18, 30, 31, 34**] and references therein), that the energy measure diffusion on the standard Sierpiński gasket has Gaussian asymptotics in harmonic coordinates. Random fractals, primarily various modifications of random Sierpiński gaskets, were considered in [**19, 16, 17**], although the randomization procedure was significantly different from our work.

Acknowledgment. The last author thanks Ofer Zeitouni for helpful discussions.

2. Energy forms and effective resistance

In this section we recall from [**28, 29**] some basic facts about limits of resistance networks. Although we state the results of this section for the Sierpiński gasket, they can be applied for general p.c.f. fractals with only minor changes.

If V is a finite set then an energy form $\mathcal{E}$ on V can be defined by

$$\mathcal{E}(f,f) = \sum_{x,y\in V} c(x,y)(f(x)-f(y))^2$$

where $c(x,y) \geqslant 0$ is called the conductance between x and y. Then the set V with the energy form $\mathcal{E}$ is often called a finite resistance network where the resistance between x and y is defined as $1/c(x,y)$ if $c(x,y) > 0$ and infinity otherwise. It is assumed that $c(x,x) = 0$ for all $x \in V$, and that the network is connected in the sense that any pair of points can be connected by a sequence of positive conductances.

Suppose $V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq ...$ is an increasing sequence of finite sets, and an energy form $\mathcal{E}_n$ is defined on each V_n. Then this sequence of resistance networks is called compatible if for any function f_n on V_n there is a function f_{n+1} on V_{n+1} such that

$$\mathcal{E}_n(f_n, f_n) = \mathcal{E}_{n+1}(f_{n+1}, f_{n+1})$$

and f_n is the restriction of f_{n+1} to V_n. Then it is easy to see that such f_{n+1} is unique, and that

$$\mathcal{E}_n(f\big|_{V_n}, f\big|_{V_n}) \leqslant \mathcal{E}_{n+1}(f, f)$$

for any function f on V_{n+1}. In this case each one can see that $\mathcal{E}_n$ is equal to the so called trace on V_n of the energy form $\mathcal{E}_{n+1}$. Moreover, $\mathcal{E}_n$ is equal to the so called trace on V_n of the energy form $\mathcal{E}_{n+k}$ for any $k \geqslant 0$.

The limiting energy (Dirichlet) form $\mathcal{E}$ on $V_* = \bigcup_{n=0}^{\infty}$ is defined by

$$\mathcal{E}(f, f) = \lim_{n\to\infty} \mathcal{E}_n(f\big|_{V_n}, f\big|_{V_n}).$$

By definition the domain of $\mathcal{E}$, denoted by $\operatorname{Dom}\mathcal{E}$, consists of all function for which this increasing limit is finite. It is not hard to see that $\mathcal{E}_n$ is the trace on V_n of $\mathcal{E}$ (see [**38, 20**] and references therein). It is also easy to see that every point of $V_* = \bigcup_{n\geq 0} V_n$ has positive capacity.

Many questions related to such sequences of Dirichlet forms $\mathcal{E}_n$ are studied in [**28, 29**] in detail. An important tool in this study is the so called effective resistance R, which is defined for any $x, y \in V_*$ by

$$R(x, y) = \Big(\min_u \{\mathcal{E}(u, u) | u(x) = 1, u(y) = 0\}\Big)^{-1}.$$

Here minimum is taken over all functions on V_*. Note that $x, y \in V_n$ for large enough n and $R(x, y)$ does not change if $\mathcal{E}$ is replaced by $\mathcal{E}_n$ because of the compatibility condition. By [**28**, Theorem 2.1.14], $R(x, y)$ is a metric on V_*. In what follows we will write R-continuity, R-closure etc. for continuity, closure etc. with respect to the effective resistance metric R. If $\mathcal{E}(u, u) < \infty$ then u is R-continuous by [**28**, Theorem 2.2.6(1)]. The main ingredient of the proof of this fact is the following inequality, which follows directly from the definition of R,

$$|u(x) - u(y)|^2 \leq R(x, y)\mathcal{E}(u, u).$$

It implies, in particular, that any function of finite energy is R–Hölder continuous with respect to the effective resistance metric.

If Ω is the R–completion of V_*, then any function in $\operatorname{Dom}\mathcal{E}$ is a restriction of an R–continuous function on Ω. In other words, if u is a function on V_* such that $\mathcal{E}(u, u) < \infty$ then u has a unique continuation to Ω that is R–continuous. We will denote this continuation by the same symbol u and the set of such functions by $\operatorname{Dom}\mathcal{E}$.

A function h defined on a finite resistance network V is said to be harmonic if

$$\sum_{y\in V} (h(x) - h(y))c(x, y) = 0$$

for every $x \in V\backslash\partial V$, where ∂V is a subset of V which is called the boundary. It is well known fact that on a connected resistance network a harmonic function is uniquely determined by its boundary values on ∂V. In the case of the Sierpiński gasket it is the most natural and convienient to choose the boundary consisting of the three corners of the largest triangle.

An important question is whether Ω is equal to the Sierpiński gasket S. The answer is positive if all the conductances tend to infinity. This happens, for example, in the case of a so called regular self-similar harmonic structure (see [**25, 28**]). Thus it is natural to say that a harmonic structure is regular if $\Omega = S$ and nonregular otherwise. It is easy to see that a harmonic structure is regular if all the conductances tend to infinity, but the converse is not true. It is proved in [**28**, Proposition 3.3.2] that if harmonic functions are continuous then there is a continuous injective map $\theta : \Omega \to S$ which is the identity on V_*. Therefore in this case we can (and will) consider Ω as a subset of S. Then Ω is the R-closure of V_*. In a sense, Ω is the set where the energy form $\mathcal{E}$ "lives". If Ω is not just an abstract completion then the name "energy form on the Sierpiński gasket S" is more justified. Strictly speaking [**28**, Proposition 3.3.2] is formulated for self-similar harmonic structures, but self-similarity is not used in the proof.

It is proved in [**28**, Theorem 3.5.6] that if $x \in \Omega$ then $\{x\}$ has positive capacity. The converse of this statement is proved in [**28**] for any self-similar harmonic structure.

To define the Green's function we use the construction invented by Kigami for the self-similar harmonic structures. Let Green's function on $V_n \backslash \partial V$ be defined as $G_n = (X_n)^{-1}$ where X_n is the matrix of the energy form $\mathcal{E}_n$, and the inverse defined only for functions with zero boundary conditions. Then the compatibility condition implies that the restriction of Green's function on V_{n+1} to V_n is Green's function on V_n. Naturally, this allows to define Green's function on V_*, which is denoted by $g(\cdot, \cdot)$.

Theorem 1 ([**28, 29**]).

1. *$g(x, y) = g(y, x)$ for all $x, y \in V_*$;*
2. *$g(x, y) > 0$ for all $x, y \in V_* \backslash \partial V$;*
3. *$g(x, y) = 0$ for all $x \in \partial V$;*
4. *$g(x, \cdot) \in \operatorname{Dom} \mathcal{E}$, in particular it is R–continuous;*
5. *$\mathcal{E}(g(x, \cdot), h) = 0$ for any harmonic function h;*
6. *$\mathcal{E}(g(x, \cdot), f) = u(x)$ for any $f \in \operatorname{Dom} \mathcal{E}$ which vanishes on the boundary;*
7. *$g(\cdot, \cdot)$ has a continuation from $V_* \times V_*$ to $\Omega \times \Omega$, in particular if $x \in \Omega$ then $g(x, x) < \infty$.*
8. *For any probability measure μ on Ω, which is non zero on any nonempty open set, the energy form $\mathcal{E}$ is a local regular Dirichlet form on L^2_μ.*

Note, again, that if harmonic functions are continuous then the this theorem holds for Ω which is the R-closure of V_* in S. For discussion related to the last item of this theorem see also [**49**].

3. Parametrization of energy forms on the Sierpiński gasket

In this section we provide brief background information about the construction of non self-similar Dirichlet forms on the Sierpiński gasket. More detailed description is given in [**39**].

Suppose we start with a triangle with vertices v_0, v_1, and v_2 and some initial resistance values between these points. In our notation resistance r_j connects points v_{j-1} and v_{j+1}, where $j = 0, 1, 2$, and we consider the indices $mod(3)$, that is v_3 is

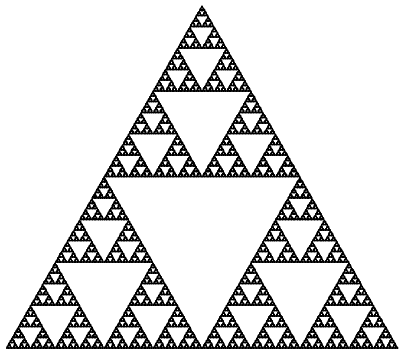

FIGURE 1. The usual self-similar representation of the Sierpiński gasket.

the same as v_0 etc. Then this network is transformed by the $\Delta - Y$ transformation into the upside down Y shape, which is our initial resistance network Γ_0. The formulas for $\Delta - Y$ and $Y - \Delta$ transformations are

$$R_j = \frac{r_{j-1}r_{j+1}}{r_0 + r_1 + r_2}$$

and

$$r_j = \frac{R_0R_1 + R_0R_2 + R_1R_2}{R_j} \tag{3.1}$$

respectively.

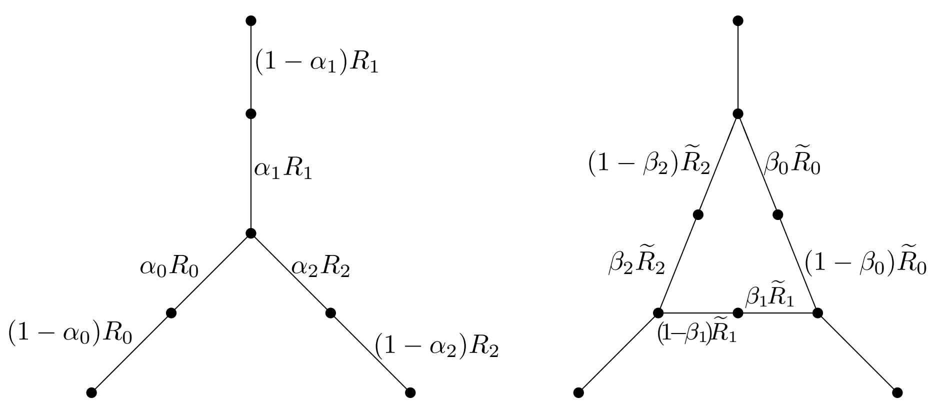

FIGURE 2. Networks Γ_0 and Γ_1.

We choose, at random, parameters $(\alpha_0, \alpha_1, \alpha_2) \in (0, 1)$ to split the resistances R_j and to create, using the $Y-\Delta$ transform (3.1), an inner triangle with new resistance values $\widetilde{R}_0$, $\widetilde{R}_1$, and $\widetilde{R}_2$. Then we choose, again at random, parameters $(\beta_0, \beta_1, \beta_2) \in (0, 1)$ to split these new resistances $\widetilde{R}_0$, $\widetilde{R}_1$, and $\widetilde{R}_2$. After that we have three Y-shaped networks, and we apply these steps to each of them independently. The effective resistances between vertices v_0, v_1, and v_2 remain the same under all these transformations. Moreover, the compatibility conditions of Section 2 hold in this case.

Note that it is convenient to label the 3^k triangles of depth k by words of length k of the alphabet $0, 1, 2$. For example, the whole Sierpiński gasket has an empty label; it is subdivided into three triangles with labels 0, 1 and 2. Each of these triangle, say labeled j, is subdivided into triangles $j0$, $j1$ and $j2$, and so on. The rule is that a depth k triangle labeled by word w is subdivided into three depth $k+1$ triangles labeled $w0$, $w1$ and $w2$ in the natural order.

To obtain a parametric description of all compatible Dirichlet forms on Γ_k we just use the algorithm iteratively on each of the triangles on each depth up to k. To transform our networks, for each word w we chose a random 6-dimensional vector $\xi_w = (\alpha_0^w, \alpha_1^w, \alpha_2^w, \beta_0^w, \beta_1^w, \beta_2^w) \in (0,1)^6$; the choices of the 6 parameters can be made independently for each w.

Lemma 3.1 ([**39**]). *The space of all Dirichlet forms on Γ_1 compatible with a fixed Dirichlet form on Γ_0 is a manifold of dimension 6. More generally, the space of all Dirichlet forms on Γ_k compatible with a fixed Dirichlet form on Γ_0 is a manifold of dimension $6(1 + 3 + ... + 3^{k-1}) = 3(3^k - 1)$.*

Theorem 2 ([**39**]). *All the local regular resistance Dirichlet forms, in the sense of* [**29**]*, on the Sierpiński gasket are in one to one correspondence, via the algorithm defined above, with the set of vectors $\xi_w = (\alpha_0^w, \alpha_1^w, \alpha_2^w, \beta_0^w, \beta_1^w, \beta_2^w) \in (0,1)^6$ where $w \in \bigcup_{k=0}^{\infty}\{0,1,2\}^k$.*

Example 3.2. It is an easy exercise to show that if the resistances in the initial network Γ_0 are equal to one, and for all i and w we set $\alpha_i^w = \frac{2}{5}, \beta_i^w = \frac{1}{2}$, then in the network Γ_k all resistances are equal to $\left(\frac{3}{5}\right)^k$. This corresponds to the so called standard energy form on the Sierpiński gasket. According to the results of [**39, 28, 29**] this is the only, up to a constant multiple, local regular Dirichlet form on the Sierpiński gasket which satisfies the following two assumptions: points have positive capacity and the effective resistance topology coincides with the standard topology; locally the Dirichlet form is preserved by the local symmetries of the Sierpiński gasket. The standard Dirichlet form on the Sierpiński gasket is self-similar with weights equal to $\frac{3}{5}$.

4. Distribution of energy and resistances

A computer program was written that executes the randomization algorithm of Section 3, and calculates resistances that define Dirichlet forms on the Sierpiński gasket. In particular, we are interested in how the data changes when we alter the domain of the parameters. Recall that in Example 3.2 the standard non random Dirichlet form on the Sierpiński gasket corresponds to the non random parameters values $\alpha = \frac{2}{5}$, $\beta = \frac{1}{2}$. Note however that in this case the sub triangles have various energies, and so one can consider the distribution of energies even in this non random case. The histogram of this distribution is shown on Figure 3. This distribution is very interesting and has not been studied theoretically although papers [**8, 41**] contain some related results.

In our study we ran the program several times, and within each run the random parameters $(\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2)$ were chosen from different intervals. We started with the nonrandom values of Example 3.2 and then increased the domains of random parameters as follows:

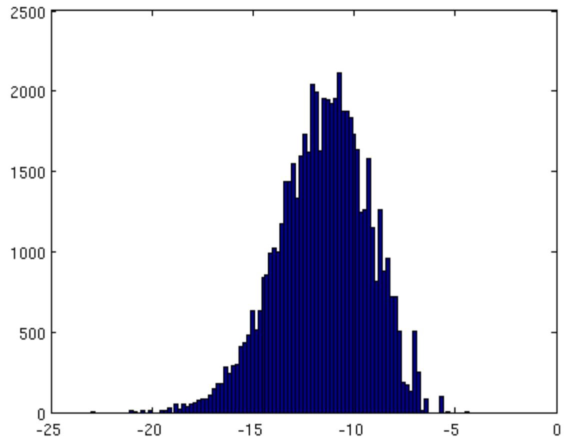

FIGURE 3. The distribution of energy corresponding to the non-random standard Dirichlet form on the Sierpiński gasket at depth $n = 9$.

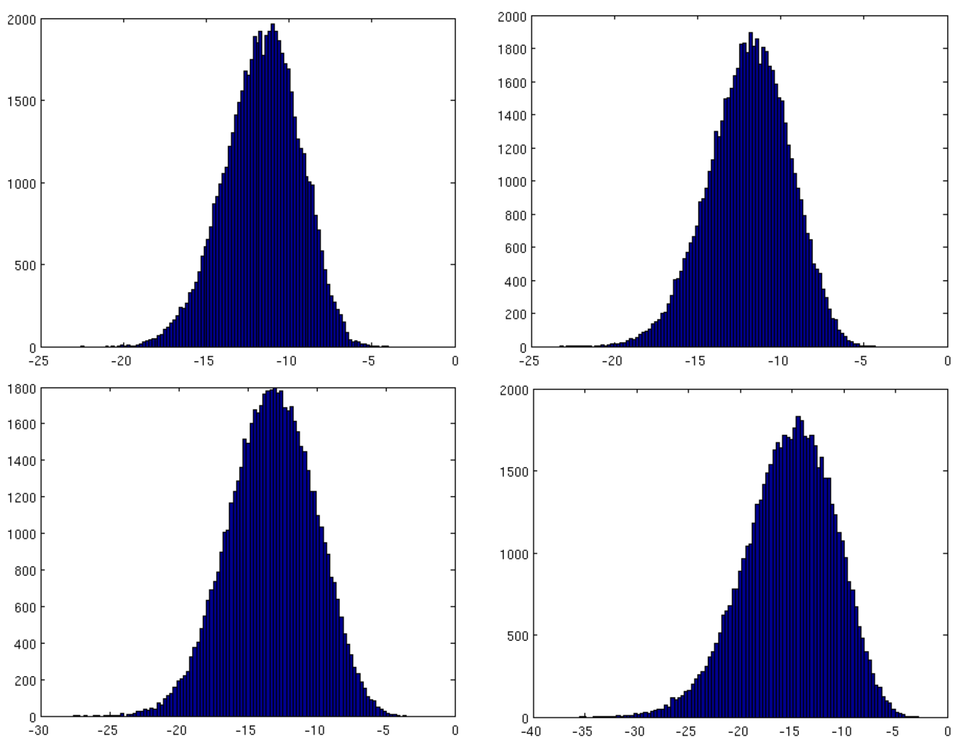

FIGURE 4. Samples of the distribution of energy in the Sierpiński gasket at depth $n = 9$ corresponding to the increasing domains of random parameters $(*)$.

$$\begin{array}{ll} \alpha \in (.3, .5), & \beta \in (.4, .6) \\ \alpha \in (.2, .7), & \beta \in (.3, .7) \\ \alpha \in (.1, .9), & \beta \in (.1, .9) \\ \alpha \in (0, 1), & \beta \in (0, 1) \end{array} \qquad (*)$$

As we changed the domains of random parameters, the resulting resistance networks and data changed significantly.

Each run of the program was done to a depth of nine, meaning one run consisted of executing the above three steps 3^9 times. Thus, the program produced data for the resistance networks on Γ_1, ..., Γ_9. We computed the mean, standard deviation, fourth moment, and 5th and 95th percentiles for each depth which are shown on Figures 6 and 7.

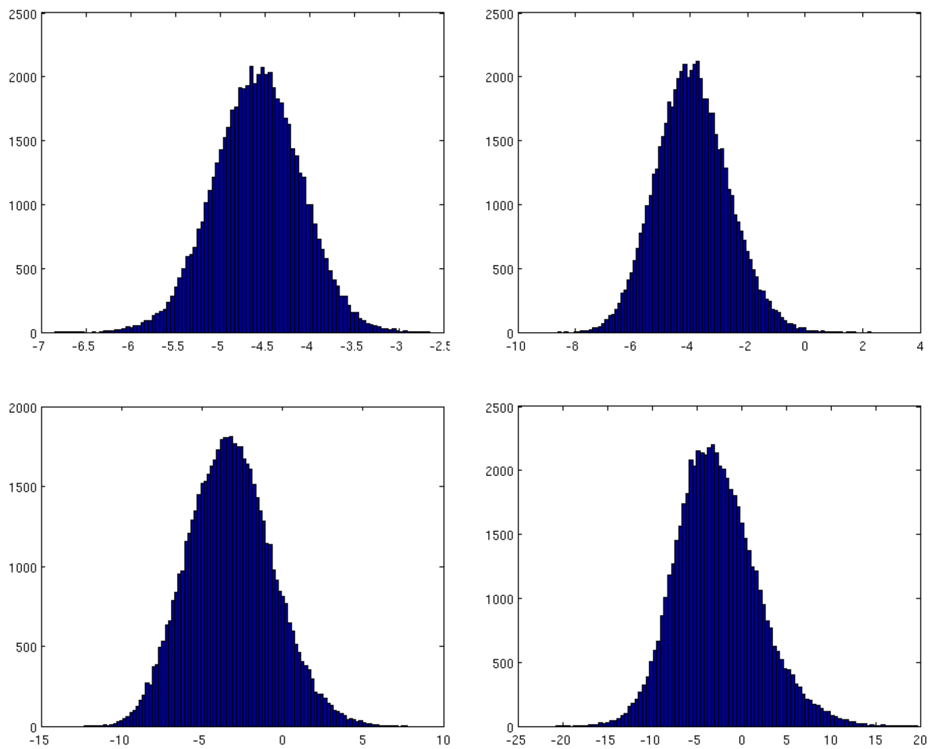

FIGURE 5. Samples of the distribution of resistances in the Sierpiński gasket at depth $n = 9$ corresponding to the increasing domains of random parameters $(*)$.

The histograms of the distribution of resistances on Figure 5 change as we alter the domain of α and β. In Figure 5 one can see that individual resistance tend to zero if the disorder is small, but for larger disorders “most” of the resistances tend to zero, there are some resistances that approach infinity. The statistical distribution of resistances resembles log normal for small domains of random parameters, but for larger domains our data does not allow to decide conclusively if the limiting distribution is log normal . The statistical distribution of the energies seems to be not log normal for various choices of parameters (see Figure 4), but again it does not allow to make conclusion about the limiting distribution.

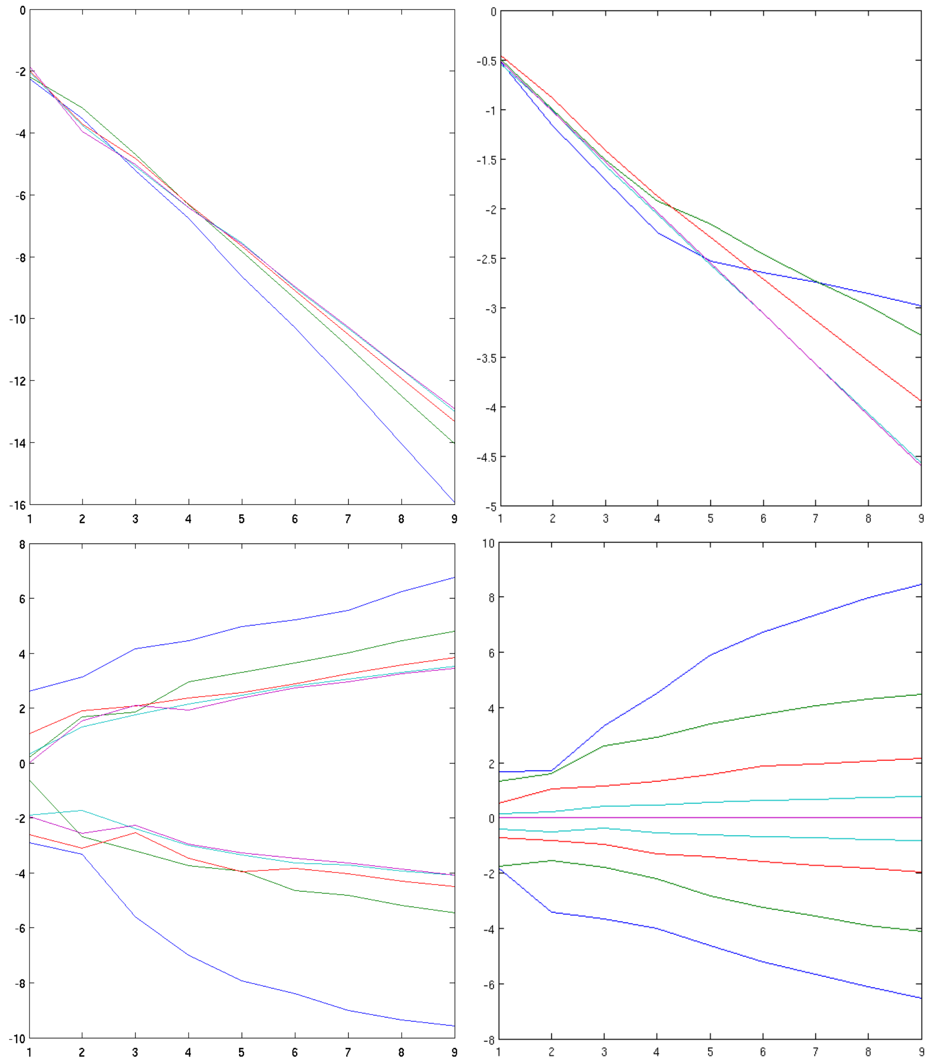

FIGURE 6. The top two graphs are medians of the energies (left) and resistances (right) at each depth $n = 1, ..., 9$ corresponding to the increasing domains of random parameters $(*)$. As the randomness increases, the median of the resistance values increases, while the median of the energy values decreases. The bottom two graphs are the $5th$ and $95th$ percentiles from which the median is subtracted.

In our examination of percentiles, we subtracted the mean from each value. Thus, for the case when $\alpha = .4$ and $\beta = .5$, the percentiles are equal to each other and are both equal to the mean (see example 3.2). That is why the difference between the percentile and the mean is zero, as shown. The graph in Figure 6 demonstrates that for most intervals, the top and bottom percentiles were linear, except for the case when $\alpha,\ \beta \in (0, 1)$. It should also be noted that the top

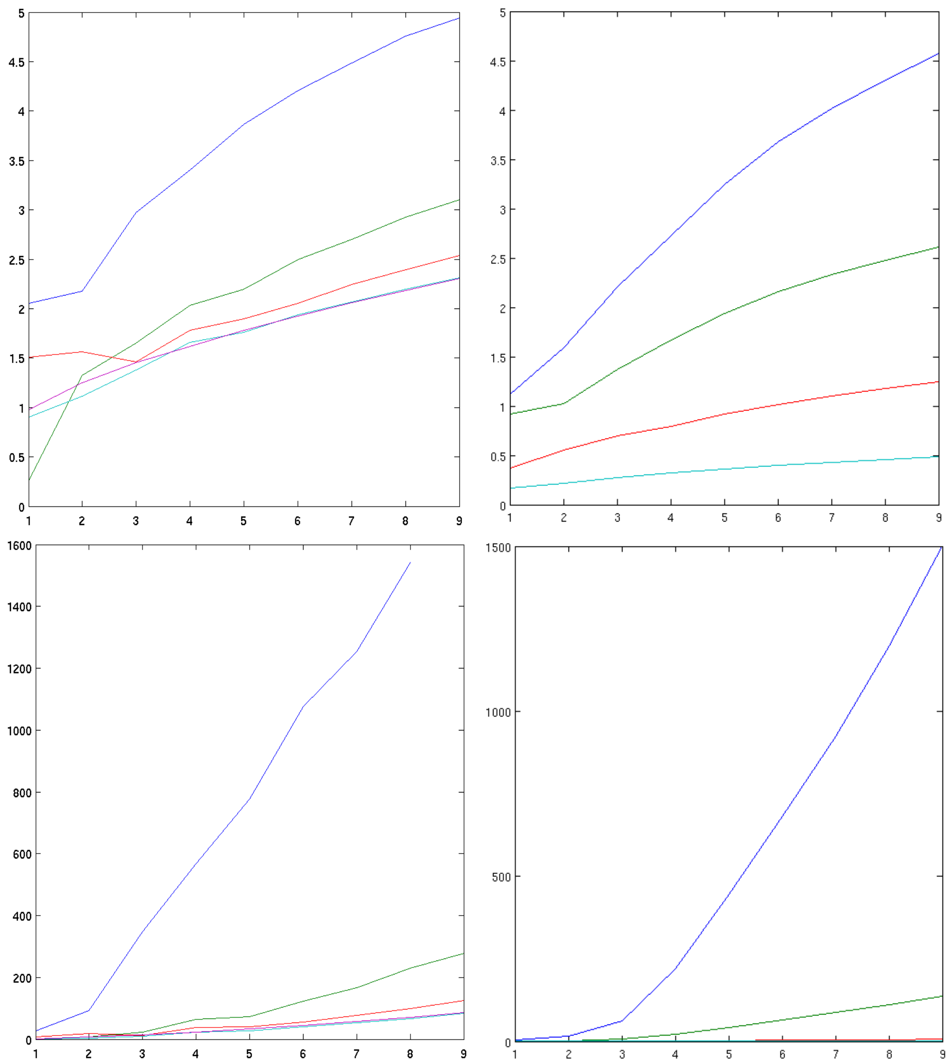

FIGURE 7. Standard deviation (top) and the fourth moment (bottom) of the energies (left) and resistances (right) at each depth $n = 1, ..., 9$ corresponding to the increasing domains of random parameters $(*)$. The standard deviations and the fourth moments increase significantly as the randomness increases.

percentile approximately reflected the bottom percentile over the horizontal graph of $\alpha = .4$ and $\beta = .5$, and that the top percentile increases as the domain of random parameters increases.

The graph in Figure 7 indicates that the fourth moment for $\alpha, \beta \in (0, 1)$ increased dramatically while the fourth moment for $\alpha \in (.1, .9), \beta \in (.1, .9)$ increased slightly. For all other intervals, the fourth moment was relatively flat.

5. Harmonic functions on random Sierpiński gaskets

Harmonic functions play an important role in the analysis on fractals. First, recall that a function h defined on a finite set V is said to be harmonic if

$$\sum_{y\in V}(h(x)-h(y))c(x,y)=0$$

for every $x \in V\backslash\partial V$, where $c(x,y)$ is the conductance between vertices x and y. Conductance is the reciprocal of resistance, so for any resistance R, $c=\frac{1}{R}$. If $c(x,y)>0$, then x and y are said to be connected. On a connected resistance network a harmonic function is uniquely determined by it boundary values. In the case of the Sierpiński gasket it is the most natural and convienient to choose the boundary $\partial V=\{v_0,v_1,v_2\}$ of the three corners of the largest triangle.

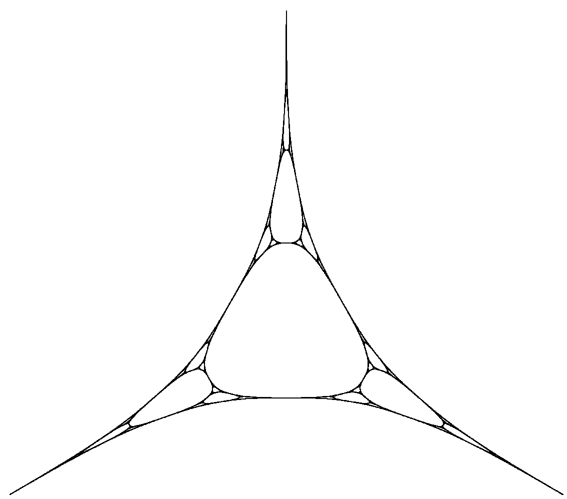

FIGURE 8. Sierpiński gasket in the standard non random harmonic coordinates.

The next proposition is a consequence of general results on traces of resistance forms, see for instance [**10, 39, 28, 29**] and Section 2.

PROPOSITION 5.1. *Suppose x is a vertex of a triangle of depth k and h is a harmonic function in the network Γ_m. Then $h(x)$ is uniquely determined by the boundary values of h on $\partial V=\{v_0,v_1,v_2\}$, and is independent of m provided $m\geqslant k$.*

THEOREM 3 ([**39**]). *(1) Suppose that $\xi_w=(\alpha_0^w,\alpha_1^w,\alpha_2^w,\beta_0^w,\beta_1^w,\beta_2^w)\in(0,1)^6$ are independent identically distributed random 6-dimensional vectors indexed by the words w of finite length. Then with probability one harmonic functions are continuous.*
(2) Suppose that there is $\varepsilon>0$ such that $\alpha_j^w,\beta_j^w\in[\varepsilon,1-\varepsilon]$ for all w,j. Then harmonic functions are Hölder continuous with Hölder exponent $1-\varepsilon^2$.

To define harmonic coordinates on the Sierpiński gasket S, we fix two harmonic functions h_1 and h_2, which are linearly independent over constants. Then for each point $x\in S$ we define

$$\psi: S\to\mathbb{R}^2,$$
$$\psi(x)=(h_1(x),h_2(x)).$$

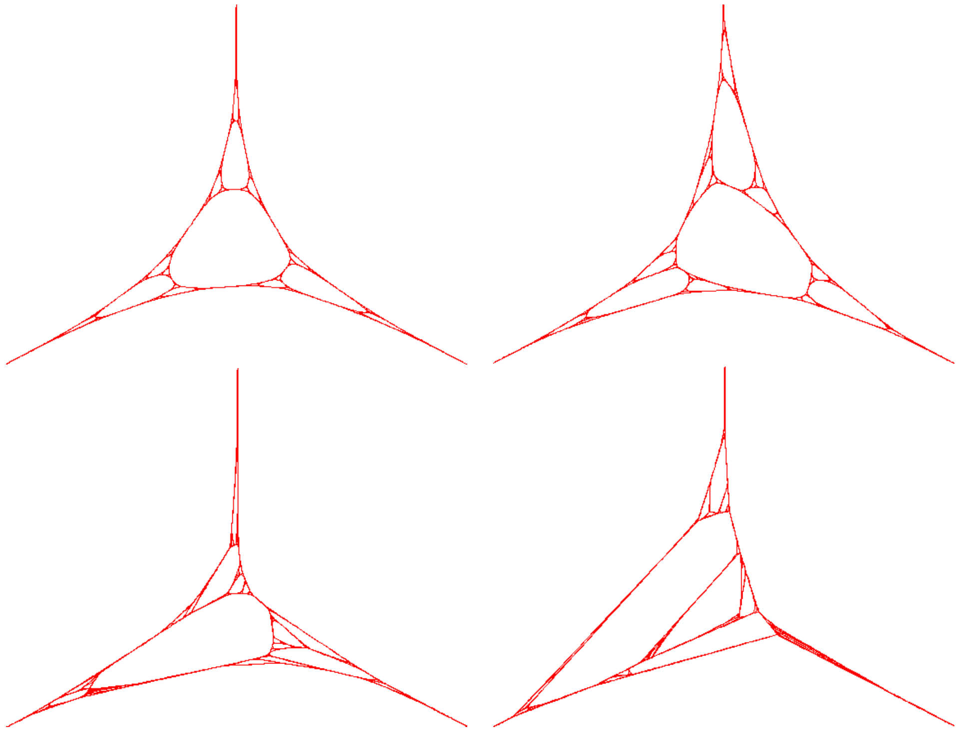

FIGURE 9. Samples of the Sierpiński gasket in random harmonic coordinates corresponding to the increasing domains of random parameters $(*)$.

For convenience we choose the harmonic functions h_1 and h_2 in such way that the boundary points of the Sierpiński gasket in harmonic coordinates are vertices of an isosceles triangle, that is $h_1(v_0) = -1$, $h_2(v_0) = 0$, $h_1(v_1) = 0$, $h_2(v_1) = \sqrt{3}$, $h_1(v_2) = 1$, $h_2(v_2) = 0$.

In [**26**] ψ is proved to be a homeomorphism in the case of the standard Dirichlet form on the Sierpiński gasket (see Example 3.2). This Kigami's result can be generalized as follows

THEOREM 4. *Suppose that* $\xi_w = (\alpha_0^w, \alpha_1^w, \alpha_2^w, \beta_0^w, \beta_1^w, \beta_2^w) \in (0,1)^6$ *are independent identically distributed random 6-dimensional vectors indexed by the words* w *of finite length. Then with probability one the coordinate map* $\psi : S \to S_\psi := \psi(S)$ *is a homeomorphism.*

PROOF. By Theorem 3 harmonic functions are continuous with probability one, and they separate points by [**39**, Lemma 5.3 and Proposition 6.1]. □

The next theorem says, essentially, that the angles of the curvilinear triangles that made the Sierpiński gasket in harmonic coordinates are zero.

THEOREM 5. *Suppose that* $\xi_w = (\alpha_0^w, \alpha_1^w, \alpha_2^w, \beta_0^w, \beta_1^w, \beta_2^w) \in (0,1)^6$ *are independent identically distributed random 6-dimensional vectors indexed by the words* w *of finite length. Then with probability one the following is true. Let* $\{T_k\}_{k=0}^{\infty}$ *be* *a sequence of triangles such that $T_{k+1} \subsetneq T_k$ for all k. Then the sum of the angles of the triangle T_k tend to zero as k tend to infinity.*

PROOF. The result follows from [**39**, Lemma 4.1, Theorem 4.3 and Theorem 5.5]. □

We conjecture that some of the results of Kusuoka proved in [**35, 36, 8**] and recent results by Hino [**35, 36**] hold also with probability one for our random Sierpiński gaskets. In particular, we conjecture that the energy measures are singular with probability one with respect to all the product (Bernoulli) measures, and that the random Sierpiński gasket is one dimensional in any generic point. The latter means that the matrix Z (in Theorem 6 below) has rank one ν-almost everywhere.

6. Approximation by quantum graphs

In this section we assume that a random Sierpiński gasket is homeomorphically embedded into $\mathbb{R}^2$ using harmonic coordinates, which is possible by Theorem 5. In our exposition we follow [**49**]. For the background on quantum graphs see [**32, 33**] and references therein.

We start with defining a different sequence of approximating energy forms. In various situations these forms are associated with so called quantum graphs, photonic crystals and cable systems. If $f \in C^1(\mathbb{R}^2)$ then we define

$$\mathcal{E}_n^Q(f,g) = \sum_{x,y \in V_n} c_{n,x,y} \mathcal{E}_{x,y}^Q(f,f)$$

where

$$\mathcal{E}_{x,y}^Q(f,f) = \int_0^1 \left(\tfrac{d}{dt} f\big(x(1-t)+ty\big)\right)^2 dt$$

is the integral of the square of the derivative

$$\tfrac{d}{dt} f\big(x(1-t)+ty\big) = \langle \nabla f\big(x(1-t)+ty\big), y-x\rangle$$

of f along the straight line segment connecting x and y. Thus $\mathcal{E}_{x,y}^Q(f,f)$ is the usual one dimensional energy of a function on a straight line segment. If f is linear then $\mathcal{E}_{x,y}^Q(f,f) = \big(f(x)-f(y)\big)^2$. Therefore if f is piecewise harmonic then $\mathcal{E}_n^Q(f,f) = \mathcal{E}_n(f,f)$ for all large enough n. Also $\mathcal{E}_n^Q$ satisfies estimate

$$\mathcal{E}_n(f,f) \leqslant \|f\|^2_{C^1(\mathbb{R}^m)} \nu(F). \tag{6.1}$$

Therefore for any $C^1(R^2)$-function we have

$$\lim_{n\to\infty} \mathcal{E}_n^Q(f,f) = \mathcal{E}(f,f)$$

by [**49**, Theorem 5].

THEOREM 6 ([**49**]). *If f is the restriction to F of a $C^1(\mathbb{R}^m)$ function then $f \in \operatorname{Dom}\mathcal{E}$, and such functions are dense in $\operatorname{Dom}\mathcal{E}$. In particular we have the Kigami formula*

$$\mathcal{E}(f,f) = \int_F \langle \nabla f, Z \nabla f\rangle d\nu$$

for any $f \in C^1(\mathbb{R}^m)$, where Z is a positive trace one matrix defined ν-almost everywhere.

It is easy to see that if g is a $C^1(\mathbb{R}^2)$-function vanishing on V_0 and f is a $C^2(\mathbb{R}^2)$-function then

$$\mathcal{E}_n^Q(f,g) = \sum_{x,y\in V_n} c_{n,x,y}\int_0^1 g\big(x(1-t)+ty\big)\Big(\tfrac{d^2}{dt^2}f\big(x(1-t)+ty\big)\Big)dt$$

because after integration by parts all the boundary terms are canceled.

By [**29**] there is a densely defined operator Δ_ν, called the energy Laplacian (which is self-adjoint with Dirichlet or Neumann boundary conditions), such that for any function $g \in \operatorname{Dom}\mathcal{E}$, vanishing on the boundary V_0, and any function $f \in \operatorname{Dom}\Delta_\nu$, we have the analog of the Gauss-Green formula:

$$\mathcal{E}(f,g) = -\int_F g\Delta_\nu f d\nu,$$

see [**9, 13**].

Lemma 6.1 ([**49**]). *If f is the restriction to F of a $C^2(\mathbb{R}^2)$ function, and g is the restriction to F of a $C^1(\mathbb{R}^2)$ function, then*

$$|\mathcal{E}_n(f,g)| \leqslant const\|g\|_{C^1(\mathbb{R}^2)}\|f\|_{C^2(\mathbb{R}^2)}\nu(F).$$

Theorem 7 ([**49**]). *If f is the restriction to F of a $C^2(\mathbb{R}^2)$ function then $f \in \operatorname{Dom}\Delta_\nu$, and such functions are dense in $\operatorname{Dom}\Delta_\nu$. Moreover, ν-almost everywhere*

$$\Delta_\nu f = \operatorname{Tr}(ZD^2 f)$$

where $D^2 f$ is the matrix of the second derivatives of f.

Corollary 6.2. *If $f(x) = \|x\|^2$ then $\Delta_\nu f = 1$. Moreover, $\Delta_\nu f \in L^\infty(F)$ for any $f \in C^2(\mathbb{R}^2)$.*

One can also obtain Theorem 7 using the general theory of Dirichlet forms in [**9, 13**]. However there is a different constructive proof using the approximation by quantum graphs (see [**49**]).

References

[1] C. Avenancio-Leon and R. Strichartz, *Local behavior of harmonic functions on the Sierpiński gasket*, preprint.

[2] S. Alexander, *Some properties of the spectrum of the Sierpiński gasket in a magnetic field.* Phys. Rev. B **29** (1984), 5504–5508.

[3] M. T. Barlow, *Diffusions on fractals.* Lectures on Probability Theory and Statistics (Saint-Flour, 1995), 1–121, Lecture Notes in Math., **1690**, Springer, Berlin, 1998.

[4] M. T. Barlow, R. F. Bass and T. Kumagai, *Stability of parabolic Harnack inequalities on metric measure spaces.* J. Math. Soc. Japan, to appear.

[5] M.T. Barlow, T. Coulhon and A. Grigor'yan, *Manifolds and graphs with slow heat kernel decay.* Invent. Math. **144** (2001), 609–649.

[6] M. T. Barlow and B. M. Hambly, *Transition density estimates for Brownian motion on scale irregular Sierpiński gaskets.* Ann. Inst. H. Poincaré Probab. Statist., **33** (1997), 531–557.

[7] M.T. Barlow and E.A. Perkins, *Brownian motion on the Sierpiński gasket.* Probab. Theory Related Fields **79** (1988), 543–623.

[8] O. Ben-Bassat, R. S. Strichartz and A. Teplyaev, *What is not in the domain of the Laplacian on Sierpiński gasket type fractals.* J. Funct. Anal. **166** (1999), 197–217.

[9] N. Bouleau and F. Hirsch, *Dirichlet forms and analysis on Wiener space.* de Gruyter Studies in Math. **14**, 1991.

[10] B. Boyle, D. Ferrone, A. Polonsky, N. Rifkin, K. Savage and A. Teplyaev, *Electrical Resistance of N-gasket Fractal Networks,* preprint.

[11] D. Fontaine and A. Teplyaev, *Green's function and eigenvalues of random Sierpiński gaskets,* preprint.

[12] M. Fukushima, *Dirichlet forms, diffusion processes and spectral dimensions for nested fractals.* Ideas and methods in Mathematical Analysis, Stochastics, and applications (Oslo, 1988), 151–161, Cambridge Univ. Press, Cambridge, 1992.

[13] M. Fukushima, Y. Oshima and M. Takada, *Dirichlet forms and symmetric Markov processes.* deGruyter Studies in Math. **19**, 1994.

[14] Y. Gefen, A. Aharony and B. B. Mandelbrot, *Phase transitions on fractals. I. Quasilinear lattices. II. Sierpiński gaskets. III. Infinitely ramified lattices.* J. Phys. A **16** (1983), 1267–1278; **17** (1984), 435–444 and 1277–1289.

[15] R. Grigorchuk and A. Żuk, *On the asymptotic spectrum of random walks on infinite families of graphs.* Random walks and discrete potential theory (Cortona, 1997), 188–204, Sympos. Math., XXXIX, Cambridge Univ. Press, Cambridge, 1999.

[16] B. M. Hambly, *Heat kernels and spectral asymptotics for some random Sierpiński gaskets.* Fractal geometry and stochastics, II (Greifswald/Koserow, 1998), 239–267, Progr. Probab., **46**, Birkhäuser, Basel, 2000.

[17] B. M. Hambly, *On the asymptotics of the eigenvalue counting function for random recursive Sierpiński gaskets.* Probab. Theory Related Fields, **117** (2000), 221–247.

[18] B. M. Hambly and T. Kumagai, *Heat kernel estimates for symmetric random walks on a class of fractal graphs and stability under rough isometries,* Fractal Geometry and applications: a jubilee of Benoit Mandelbrot, Proc. Symp. Pure Math. 72, Part 2, 233-259, 2004.

[19] B. M. Hambly and M.L. Lapidus, *Random fractal strings: their zeta functions, complex dimensions and spectral asymptotics,* to appear in Trans. Amer. Math. Soc.

[20] B. M. Hambly, V. Metz and A. Teplyaev, *Admissible refinements of energy on finitely ramified fractals*, preprint.

[21] M. Hino, *On singularity of energy measures on self-similar sets.* Probab. Theory Related Fields, **132** (2005), 265–290.

[22] M. Hino and K. Nakahara, *On singularity of energy measures on self-similar sets II,* preprint.

[23] R. G. Hohlfeld and N. Cohen, *Self–Similarity and the Geometric Requirements for Frequency Independence in Antennae.* Fractals **7** (1999), 79–84.

[24] J. Kigami, *A harmonic calculus on the Sierpiński spaces.* Japan J. Appl. Math. **6** (1989), 259–290.

[25] J. Kigami, *Harmonic calculus on p.c.f. self–similar sets.* Trans. Amer. Math. Soc. **335** (1993), 721–755.

[26] J. Kigami, *Harmonic metric and Dirichlet form on the Sierpiński gasket.* Asymptotic problems in probability theory: stochastic models and diffusions on fractals (Sanda/Kyoto, 1990), 201–218, Pitman Res. Notes Math. Ser., **283**, Longman Sci. Tech., Harlow, 1993.

[27] J. Kigami, *Effective resistances for harmonic structures on p.c.f. self-similar sets.* Math. Proc. Cambridge Philos. Soc. **115** (1994), 291–303.

[28] J. Kigami, *Analysis on fractals.* Cambridge Tracts in Mathematics **143**, Cambridge University Press, 2001.

[29] J. Kigami, *Harmonic analysis for resistance forms.* J. Functional Analysis **204** (2003), 399–444.

[30] J. Kigami, *Local Nash inequality and inhomogeneity of heat kernels.* Proc. London Math. Soc. (3) **89** (2004), 525–544.

[31] J. Kigami, *Volume doubling measures and heat kernel estimates on self-similar sets*, preprint.

[32] P. Kuchment, *Quantum graphs I. Some basic structures.* Waves in random media, **14** (2004), S107–S128.

[33] P. Kuchment, *Quantum graphs II. Some spectral properties of quantum and combinatorial graphs.* J. Phys. A. **38** (2005), 4887–4900.

[34] T. Kumagai, *Heat kernel estimates and parabolic Harnack inequalities on graphs and resistance forms.* Publ. Res. Inst. Math. Sci. **40** (2004), 793–818.

[35] S. Kusuoka, *Dirichlet forms on fractals and products of random matrices.* Publ. Res. Inst. Math. Sci. **25** (1989), 659–680.

[36] S. Kusuoka, *Lecture on diffusion process on nested fractals.* Lecture Notes in Math. **1567** 39–98, Springer-Verlag, Berlin, 1993.

[37] S. Kusuoka and X. Y. Zhou, *Waves on fractal-like manifolds and effective energy propagation.* Probab. Theory Related Fields **110** (1998), 473–495.

[38] V. Metz, *The cone of diffusions on finitely ramified fractals.* Nonlinear Anal. **55** (2003), 723–738.

[39] R. Meyers, R. Strichartz and A. Teplyaev, *Dirichlet forms on the Sierpinski gasket,* Pacific Journal of Mathematics **217** (2004), 149-174.

[40] J. Needleman, R. Strichartz, A. Teplyaev and P.-L. Yung, *Calculus on the Sierpiński gasket: polynomials exponentials and power series,* J. Funct. Anal. **215** (2004), 290–340.

[41] A. Öberg, R.S. Strichartz and A.Q. Yingst, *Level sets of harmonic functions on the Sierpiński gasket,* Ark. Mat. **40** (2002), 335–362.

[42] R. Rammal, *Spectrum of harmonic excitations on fractals.* J. Physique **45** (1984), 191–206.

[43] R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters.* J. Physique Letters **44** (1983), L13–L22.

[44] J. Stanley, R. Strichartz and A. Teplyaev, *Energy partition on fractals.* Indiana Univ. Math. J. **52** (2003), 133–156.

[45] R. B. Stinchcombe, *Fractals, phase transitions and criticality.* Fractals in the natural sciences. Proc. Roy. Soc. London Ser. A **423** (1989), 17–33.

[46] R. S. Strichartz, *Analysis on fractals.* Notices AMS, **46** (1999), 1199-1208.

[47] R. S. Strichartz, *Differential equations on fractals: a tutorial.* a book in preparation.

[48] A. Teplyaev, *Energy and Laplacian on the Sierpiński gasket.* Fractal Geometry and Applications: A Jubilee of Benoit Mandelbrot. Proc. Sympos. Pure Math. **72**, Part 1, AMS, December 2004.

[49] A. Teplyaev, *Harmonic coordinates on fractals with finitely connected cell structure,* preprint.

Department of Mathematics, University of Connecticut, Storrs CT 06269 USA
E-mail address: `Daniel.Fontaine@uconn.edu`
E-mail address: `smith@math.uconn.edu`
E-mail address: `teplyaev@math.uconn.edu`
URL: `http://www.math.uconn.edu/~teplyaev/fractals/`