Strategy to transform continuous random variables.

- Consider a random variable $\boldsymbol{X}$. Make sure you understand its p.d.f. $f_X(x)$ and its c.d.f. $F_X(x)$. These relations are important:

$$F'_X(x) = f_X(x)$$
$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \mathbb{P}(X < x)$$
$$F_X(b) - F_X(a) = \int_{a}^{b} f_X(t)dt = \mathbb{P}(a < X < b)$$

- Consider a function $g(x)$ and the transformed random variable $Y = g(X)$. Denote $y = g(x)$, which ***we can solve for*** x, to obtain $x = g^{-1}(y)$.
- For now, assume that g is a ***monotone increasing function***, and we understand the range of Y.

$$\min(Y) = g(\min(X)) \leqslant Y \leqslant \max(Y) = g(\max(X))$$

(1) The c.d.f. of Y can be computed as

$$F_Y(y) = F_X(x) = F_X(g^{-1}(y))$$

(2) The p.d.f. of Y can be computed as

$$f_Y(y) = F'_Y(y) = \frac{d}{dy} F_X(g^{-1}(y))$$

(3) The expected value $\mathbb{E}Y$ can be computed in two different ways as

$$\mathbb{E}Y = \int\limits_{min(Y)}^{max(Y)} y f_Y(y)dy = \int\limits_{min(X)}^{max(X)} g(x) f_X(x)dx$$

(4) To compute $\mathbf{Var}\,(Y)$, use

$$\mathbb{E}Y^2 = \int\limits_{min(Y)}^{max(Y)} y^2 f_Y(y)dy = \int\limits_{min(X)}^{max(X)} (g(x))^2 f_X(x)dx$$

and then

$$\mathbf{Var}\,(Y) = \mathbb{E}Y^2 - (\mathbb{E}Y)^2$$

- ***If g is a monotone decreasing function***, then some things are reversed

$$\min(Y) = g(\max(X)) \leqslant Y \leqslant \max(Y) = g(\min(X))$$
$$F_Y(y) = 1 - F_X(x) = 1 - F_X(g^{-1}(y))$$

- *If g is neither monotone decreasing nor monotone increasing, then we have to consider cases.*