MATH 3160 - Probability - Fall 2017
Test 2, Wednesday November 15

(1) Two balls are withdrawn randomly without replacement from a bowl containing $\boldsymbol{3}$ white and $\boldsymbol{3}$ black balls. Let $\boldsymbol{X}$ be the number of white balls among the withdrawn balls. What are the probability mass function of $\boldsymbol{X}$, $\mathbb{E}\boldsymbol{X}$ and $\mathbf{Var}\,(\boldsymbol{X})$?

Solution: $\mathbb{P}(\boldsymbol{X}=\mathbf{0})=\binom{3}{2}\Big/\binom{6}{2}=\mathbf{3/15=1/5}$

$\mathbb{P}(\boldsymbol{X}=\mathbf{1})=\binom{3}{1}\cdot\binom{3}{1}\Big/\binom{6}{2}=\mathbf{9/15=3/5}$, $\mathbb{P}(\boldsymbol{X}=\mathbf{2})=\binom{3}{2}\Big/\binom{6}{2}=\mathbf{3/15=1/5}$

Answer:

p.m.f.: $\boldsymbol{p}(\mathbf{0})=\boldsymbol{p}(\mathbf{2})=\mathbf{1/5}$, $\boldsymbol{p}(\mathbf{1})=\mathbf{3/5}$

$\mathbb{E}\boldsymbol{X}=\mathbf{0+2/5+3/5=1}$

$\mathbf{Var}\,(\boldsymbol{X})=\mathbb{E}(\boldsymbol{X}-\boldsymbol{EX})=\mathbf{1/5+0+1/5=2/5}$

(2) Suppose that earthquakes occur on the West coast of the U.S. on average at a rate of 3 per week (including very mild ones) and follow Poisson probability distribution. What is the probability that there will be 2 earthquakes next week, if we suppose that at least one will happen? *(Hint: use conditional probability).*

Solution: $\boldsymbol{P}(\boldsymbol{X}=\mathbf{2})=\mathbf{3^2}\boldsymbol{e}^{-\mathbf{3}}/\mathbf{2}$, $\boldsymbol{P}(\boldsymbol{X}\geqslant\mathbf{1})=\mathbf{1}-\boldsymbol{e}^{-\mathbf{3}}$

Answer:

$\boldsymbol{P}(\boldsymbol{X}=2|\boldsymbol{X}\geqslant 1)=\dfrac{\mathbf{3^2}\boldsymbol{e}^{-\mathbf{3}}}{\mathbf{2(1}-\boldsymbol{e}^{-\mathbf{3}})}$

(3) Suppose $\boldsymbol{X}$ is exponentially distributed with the mean $\mathbb{E}\boldsymbol{X}=\mathbf{2}$. What is the probability $\mathbf{3}<\boldsymbol{X}<\mathbf{5}$ if we know that $\boldsymbol{X}>\mathbf{2}$? *(Hint: use conditional probability and the basic properties of the exponentially distribution).*

Solution: Exponentials are memory-less: $\mathbb{P}(\boldsymbol{X}>\boldsymbol{s}+\boldsymbol{t}\mid\boldsymbol{X}>\boldsymbol{t})=\mathbb{P}(\boldsymbol{X}>\boldsymbol{s})$.
Hence $\boldsymbol{P}(3<\boldsymbol{X}<5|\boldsymbol{X}>2)=\boldsymbol{P}(3<\boldsymbol{X}|\boldsymbol{X}>2)-1+\boldsymbol{P}(\boldsymbol{X}>5|\boldsymbol{X}>2)=\boldsymbol{P}(1<\boldsymbol{X}<3)$

Here $\boldsymbol{\lambda}=\frac{\mathbf{1}}{\mathbf{2}}$

Answer:

$\boldsymbol{P}(3<\boldsymbol{X}<5|\boldsymbol{X}>2)=\boldsymbol{e}^{-\mathbf{1/2}}-\boldsymbol{e}^{-\mathbf{3/2}}$

(4) Suppose $X = \mathcal{N}(\mu, \sigma^2)$, $P(X < 0) = 0.15866 = \Phi(-1)$ and $P(X < 5) = 0.97725 = \Phi(2)$. Find μ and σ.

Solution: $\mu - \sigma = 0$, $\mu + 2\sigma = 5$. This implies

Answer:

$\mu = \sigma = 5/3$

(5) Suppose we toss a fair coin 16 times. Find the formula for the best possible normal approximation of the probability that there are at least 9 heads. You do not have to evaluate the numeral value but your answer should include $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-y^2/2} dy = \mathbb{P}(Z < x)$, where Z is the standard normal random variable.

Solution: $\mu = 16/2 = 8$, $\sigma = \sqrt{16/4} = 2$. This implies $\mathbb{P}(X \geqslant 9) \approx \mathbb{P}(8 + 2Z > 8.5) = \mathbb{P}(Z > 0.25)$

Answer:

$\mathbb{P}(X \geqslant 9) \approx 1 - \Phi(0.25)$

This is approximately equal to 0.40129 but this was not part of the test.

A different but also correct solution is

$\mathbb{P}(X \geqslant 9) = \mathbb{P}(16 \geqslant X \geqslant 9) \approx \mathbb{P}(16.5 > 8 + 2Z > 8.5) = \mathbb{P}(4.25 > Z > 0.25) = \Phi(4.25) - \Phi(0.25)$, which numerically is about 0.40128

The exact probability, using the binomial distribution, is $\frac{26333}{65536} \approx 0.40180969$

(6) Suppose the random variable X is uniformly distributed in the interval $[0, 2]$ and $Y = X^3$. Find the c.d.f. $F_Y(y)$ and $\mathbb{E}Y$.

Solution: Let $y = x^3$, $x = \sqrt[3]{y}$. We have $0 < Y < 8$. Then $F_Y(y) = \mathbb{P}(Y < y) = \mathbb{P}(X < x) = F_X(\sqrt[3]{y}) = \frac{1}{2}\sqrt[3]{y}$ when $0 < y < 8$.

$$\mathbb{E}Y = \frac{1}{2}\int_0^2 x^3 dx = \frac{1}{2}\frac{1}{4}x^4\Big|_0^2 = 2$$

Answer:

$F_Y(y) = 0$ when $y \leq 0$, $F_Y(y) = \frac{1}{2}\sqrt[3]{y}$ when $0 < y < 8$, $F_Y(y) = 1$ when $y \geq 8$

$\mathbb{E}Y = 2$