Show all work: either write at least a sentence explaining your reasoning, or annotate your math work with brief explanations. Correct answer with no solution will give only a partial credit. There is NO need to simplify, and NO calculators are needed.

• You may leave your answer in terms of sums, products, factorials or binomial coefficients, and fractions. Use the notation \( \Phi(x) \) for the \( \mathcal{N}(0, 1) \) distribution function, that is \( \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy = \mathbb{P}(Z < x) \) where \( Z \) is the standard normal random variable. You do not need a table of values of \( \Phi \).

(1) Consider a standard deck of 52 cards. What is the probability of a four of a kind? (This occurs when the cards have denominations \( a, a, a, a, b \).)

(2) Consider a roulette wheel consisting of 50 numbers 1 through 50, 0, and 00. If Phan always bets that the outcome will be one of the numbers 1 through 20, what is the probability that

(a) Phan will lose his first 7 bets,
(b) if his first win will occur on his \( X \)th bet, what is the distribution of \( X \)? What is \( \mathbb{P}(X = 9) \)?

(3) The monthly worldwide average number of airplane crashes of commercial airlines is 3.5. What is the probability that at most 1 accident will occur in next 2 months?

(4) The r.v. \( X \) has a mgf given by

\[ m_X(t) = \frac{1}{1-t}, \quad t < 1. \]

If \( u \) is some unknown number greater than 0, what is \( \mathbb{P}(X > 1 + u \mid X > u) \)?

(5) A manufacturing company sources widgets from three different suppliers (A, B, and C). Based on the company’s quality control data, it appears that 3 percent of widgets coming from A are faulty, as are 5 percent of the widgets coming from B, and 2 percent coming from C. Based on recent purchasing records, suppliers A, B, and C supply 30 percent, 20 percent, and 50 percent of the company’s stock of widgets, respectively.

(a) What is the probability that a random widget from the company’s stock is faulty?
(b) Given that a widget is faulty, what is the probability that it came from supplier C?
(c) Using the definition of independence of events, determine whether the events \( F = \{ \text{widget is faulty} \} \) and \( C = \{ \text{widget came from supplier C} \} \) are independent or not.

(6) Suppose the joint density function of the random variables \( X \) and \( Y \) is

\[ f(x, y) = \begin{cases} c(x + y) & 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases} \]

(a) Find the value of \( c \).
(b) Compute \( \mathbb{P}(X^2 + Y^2 \leq 1) \)
(c) Compute \( \mathbb{E}[X^2 Y] \).

(7) Suppose \( X \) is a normal r.v. with mean 1 and variance 1 and let \( Y \) be an independent Poisson r.v. with parameter 2. What is \( \text{Var}(2X - Y) \)? What are \( \mathbb{E}(2X - Y) \) and \( \mathbb{E}(2X - Y)^2 \)?

(8) Let \( X \) be a uniform random variable over \((1, 6)\). Find the moment generating function of \( X \). Show all steps.

(9) Suppose \( X \) has the following moment generating function

\[ m_X(t) = \frac{e^t}{1 - t^2}. \]

Find \( \mathbb{E}[X] \). (This distribution is known as the Laplace distribution)

(10) A person has 100 light bulbs whose lifetimes are independent exponentials with mean 5 hours. If the bulbs are used one at a time, with a failed bulb being replaced immediately by a new one, approximate the probability that there is still a working light bulb after 525 hours.
answer key:

(1) \[ \frac{13 \cdot 12 \cdot 4}{\binom{52}{5}} \]

(2) (a) \( \left( \frac{32}{52} \right)^7 \)

(b) Geometric, \( \left( \frac{32}{52} \right)^8 \left( \frac{20}{52} \right) \)

(3) \( 8e^{-7} \)

(4) \( e^{-1} \)

(5) (a) \( .01 + .01 + .009 = \frac{29}{1000} \)

(b) \[ \frac{.01}{.029} = \frac{10}{29} \]

(c) no.

(6) (a) \( c = 1 \)

(b) \( \frac{2}{3} \)

(c) \[ \frac{1}{8} + \frac{1}{9} = \frac{17}{72} \]

(7) 6, 0 and 6

(8) \[ \frac{e^{6t} - e^t}{5t} \]

(9) 1

(10) \[ 1 - \Phi\left( \frac{1}{2} \right) \]