MATH 3160 - Probability - Fall 2017
sample final exam questions

Show all work: either write at least a sentence explaining your reasoning, or annotate your math work with brief explanations. Correct answer with no solution will give only a partial credit. There is NO need to simplify, and NO calculators are needed. • You may leave your answer in terms of sums, products, factorials or binomial coefficients, and fractions. Use the notation $\Phi(x)$ for the $\mathcal{N}(0,1)$ distribution function, that is $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy = \mathbb{P}(Z < x)$ where Z is the standard normal random variable. You do not need a table of values of Φ.

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(1) Find the moment generating function $m_X(t)$ for a random variable X with the p.d.f.

$$f_X(x) = \begin{cases} \frac{x}{2} & \text{if } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

For this moment generating function $m_X(t)$, find $m'_X(0)$ and $m''_X(0)$ by expressing them as moments of X, and computing moments as integrals.

(2) Find the moment generating function $m_X(t)$ for a random variable X with the p.d.f.

$$f_X(x) = \begin{cases} 2 - 2x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

For this moment generating function $m_X(t)$, find $m'_X(0)$ and $m''_X(0)$ by expressing them as moments of X, and computing moments as integrals.

(3) Find the moment generating function $m_X(t)$ for a random variable X with the p.d.f.

$$f_X(x) = \begin{cases} e^{1-x} & \text{if } x > 1 \\ 0 & \text{otherwise} \end{cases}$$

For this moment generating function $m_X(t)$, find $m'_X(0)$ and $m''_X(0)$.
There are three solutions to this question:

- computing $m'_X(0)$ and $m''_X(0)$ by differentiating;
- expressing $m'_X(0)$ and $m''_X(0)$ as moments of X, and computing moments as integrals (you'll need integration by parts).
- expressing $m'_X(0)$ and $m''_X(0)$ as moments of X, and connecting X to a random variable in the table (in this case, an exponential random variable).

(4) Find $\mathbb{E}X$ and $\mathbb{E}X^2$ if the random variable X has the moment generating function $m_X(t) = \dfrac{1}{(1-3t)^2}$

(5) Find $\mathbb{E}X$ and $\mathbb{E}X^2$ if the random variable X has the moment generating function $m_X(t) = \dfrac{1}{(1-5t)^4}$

(6) Find $\mathbb{E}X$ and $\mathbb{E}X^2$ if the random variable X has the moment generating function $m_X(t) = e^{t^2-3t}$

(7) Let $X_1, X_2, \ldots, X_{25}$ be independent exponential random variables with parameter $\lambda = 3$. Use the Central Limit Theorem to approximate $\mathbb{P}\left(\sum_{i=1}^{25} X_i > 12\right)$.

(8) Let $X_1, X_2, \ldots, X_{36}$ be independent uniform random variables on the interval $[1, 3]$. Use the Central Limit Theorem to approximate $\mathbb{P}\left(\sum_{i=1}^{36} X_i > 80\right)$.

(9) Let $X_1, X_2, \ldots, X_{49}$ be independent random variables from questions (1)–(6) above. In each case, use the Central Limit Theorem to approximate $\mathbb{P}\left(\sum_{i=1}^{49} X_i > 99\right)$.

$$\boxed{\mathbb{E}X = \mu \text{ and } \mathbf{Var}\,(X) = \sigma^2}$$

$$\boxed{S_n = X_1 + \ldots + X_n = \sum_{i=1}^{n} X_i}$$

$$\boxed{\mathbb{E}S_n = n\mu \text{ and } \mathbf{Var}\,(S_n) = n\sigma^2}$$

$$\boxed{\mathrm{SD}(S_n) = \sigma\sqrt{n}}$$

L.L.N..:

$$\boxed{\frac{S_n}{n} \to \mu}$$

as $n \to \infty$

C.L.T.:

$$\boxed{S_n \sim n\mu + \sigma\sqrt{n}Z}$$

$$\boxed{\frac{S_n - n\mu}{\sigma\sqrt{n}} \sim Z}$$

as $n \to \infty$

$$\boxed{\mathbb{P}(S_n > A) = \mathbb{P}\left(Z > \frac{A - n\mu}{\sigma\sqrt{n}}\right) = 1 - \Phi\left(\frac{A - n\mu}{\sigma\sqrt{n}}\right)}$$

1. $\boldsymbol{m(t) = \frac{1}{2}(e^{2t}(2t-1)+1)/t^2}$, $\boldsymbol{\mathbb{E}X = \frac{4}{3}}$, $\boldsymbol{\mathbb{E}X^2 = 2}$

2. $\boldsymbol{m(t) = 2(e^t - 1 - t)/t^2}$, $\boldsymbol{\mathbb{E}X = \frac{1}{3}}$, $\boldsymbol{\mathbb{E}X^2 = \frac{1}{6}}$

3. $\boldsymbol{m(t) = e^t/(1-t)}$, $\boldsymbol{\mathbb{E}X = 2}$, $\boldsymbol{\mathbb{E}X^2 = 5}$

4. $\boldsymbol{m(t) = \frac{1}{(1-3t)^2}}$, $\boldsymbol{\mathbb{E}X = 6}$, $\boldsymbol{\mathbb{E}X^2 = 54}$

5. $\boldsymbol{m(t) = \frac{1}{(1-5t)^4}}$, $\boldsymbol{\mathbb{E}X = 20}$, $\boldsymbol{\mathbb{E}X^2 = 500}$

6. $\boldsymbol{m(t) = e^{t^2-3t}}$, $\boldsymbol{\mathbb{E}X = -3}$, $\boldsymbol{\mathbb{E}X^2 = 11}$, $\boldsymbol{\mathrm{Var}\,(X) = 2}$

7. $\boldsymbol{\mathbb{E}X = \frac{1}{3}}$, $\boldsymbol{\mathbb{E}X^2 = \frac{2}{9}}$, $\boldsymbol{\mathrm{Var}\,(X) = \frac{1}{9}}$

$$\mathbb{P}\left(\sum_{i=1}^{25} \boldsymbol{X_i > 12}\right) \boldsymbol{\approx 1 - \Phi\left(\frac{11}{5}\right)}.$$

8. $\boldsymbol{\mathbb{E}X = 2}$, $\boldsymbol{\mathbb{E}X^2 = 13/3}$, $\boldsymbol{\mathrm{Var}\,(X) = \frac{1}{3}}$

$$\mathbb{P}\left(\sum_{i=1}^{36} \boldsymbol{X_i > 80}\right) \boldsymbol{\approx 1 - \Phi\left(\frac{4\sqrt{3}}{3}\right)}.$$

9. $$\mathbb{P}\left(\sum_{i=1}^{49} \boldsymbol{X_i > 99}\right) \boldsymbol{\approx 1 - \Phi\left(\frac{99 - 49\mathbb{E}X}{7\sqrt{\mathbb{E}X^2 - (\mathbb{E}X)^2}}\right)}.$$

...

9(6). Note that in this case the variables are Gaussian, and $\mathbb{P}\left(\sum_{i=1}^{49} \boldsymbol{X_i > 99}\right) \boldsymbol{= 1 - \Phi\left(\frac{99 + 49 \cdot 3}{7\sqrt{2}}\right)}$.