MATH 3160 - Probability - Fall 2017
Extra quiz problems December 14, 2017

(1) Suppose that, among $\boldsymbol{T}$ families, $\boldsymbol{D}$ of families own a dog, $\boldsymbol{C}$ of families own a cat, and $\boldsymbol{N}$ of the families own neither. How many families own both a cat and a dog? What is the probability to own both a cat and a dog, if a family is chosen at random?

Solution: There are $\boldsymbol{T-N}$ families who own $\boldsymbol{D+C}$ animals. Therefore $\boldsymbol{D+C+N-T}$ family owns a dog and a cat.

This is the contingency table:

	a dog	no dog	total
a cat	D+C+N-T	T-D-N	C
no cat	T-C-N	N	T-C
total	D	T-D	T

The probability to own two animals is $\mathbb{P}(\boldsymbol{cat\ and\ dog}) = \frac{\boldsymbol{N+D+C-T}}{\boldsymbol{T}}$

(2) A family is chosen at random among the same $\boldsymbol{T}$ families as in the previous problem, and found to have a dog. What is the probability they also own a cat?

$$\textbf{Solution: } \mathbb{P}(\boldsymbol{cat}|\boldsymbol{dog}) = \frac{\mathbb{P}(\boldsymbol{cat\ and\ dog})}{\mathbb{P}(\boldsymbol{dog})} = \frac{\boldsymbol{N+D+C-T}}{\boldsymbol{D}}$$

(3) In a multiple choice test, a student either knows the answer to a question or gives a random answer. Each question has $\boldsymbol{m}$ possible answers, and the student knows the answer to a question with probability $\boldsymbol{p}$. Find the probability that the student knows the answer to a question, given that the answer was correct.

Solution: If the student Knows the answer, this answer is Correct.

Hence we have $\boldsymbol{K \cap C = K}$, and therefore $\mathbb{P}(\boldsymbol{K}|\boldsymbol{C}) = \dfrac{\mathbb{P}(\boldsymbol{K \cap C})}{\mathbb{P}(\boldsymbol{C})} = \dfrac{\boldsymbol{p}}{\boldsymbol{p + (1-p)\cdot \frac{1}{m}}}$

Suppose that

- a flu test (correctly) indicates the presence of the flu $\boldsymbol{TPR}$ of the times when the patient actually has the flu *(this is called the true positive rate)*;
- the same test (incorrectly) indicates the presence of flu $\boldsymbol{FPR}$ of the times when flu is not actually present *(this is called the false positive rate)*;
- currently $\boldsymbol{r}$ is the rate of the flu in the population.

(4) For a random person, what is the probability that the flu test is positive?

Solution:

$$\mathbb{P}(\{\boldsymbol{test+}\}) = \mathbb{P}(\{\boldsymbol{flu}\} \cap \{\boldsymbol{test+}\}) + \mathbb{P}(\{\boldsymbol{no\ flu}\} \cap \{\boldsymbol{test+}\}) = \boldsymbol{r \cdot TPR + (1-r)\cdot FPR}$$

(5) Calculate the probability that a random person actually has the flu, given that the flu test is positive.

$$\textbf{Solution: } \mathbb{P}(\boldsymbol{flu}|\boldsymbol{test+}) = \frac{\mathbb{P}(\{\boldsymbol{flu}\} \cap \{\boldsymbol{test+}\})}{\mathbb{P}(\{\boldsymbol{test+}\})} = \frac{\boldsymbol{r \cdot TPR}}{\boldsymbol{r \cdot TPR + (1-r)\cdot FPR}}$$

(6) Car types $\boldsymbol{C1, C2, C3}$ are bought in numbers $\boldsymbol{N1, N2, N3}$ and have accident rates $\boldsymbol{R1, R2, R3}$ respectively. Given an accident, what is the probability that the car type $\boldsymbol{C1}$ is involved?

Solution:

$$\mathbb{P}(\boldsymbol{C1}|\boldsymbol{A}) = \frac{\mathbb{P}(\boldsymbol{C1 \cap A})}{\mathbb{P}(\boldsymbol{A})} = \frac{\boldsymbol{R1 \cdot \frac{N1}{T}}}{\boldsymbol{R1 \cdot \frac{N1}{T} + R2 \cdot \frac{N2}{T} + R3 \cdot \frac{N3}{T}}} = \frac{\boldsymbol{R1 \cdot N1}}{\boldsymbol{R1 \cdot N1 + R2 \cdot N2 + R3 \cdot N3}}$$

where the total number of cars is $\boldsymbol{T = N1 + N2 + N3}$.