(1) Suppose a random variable $X$ has the p.d.f. $f(x)$, which is defined as $f(x) = ax(1-x)$ when $0 < x < 1$, and $f(x) = 0$ when $x$ is negative or larger than 1. Find $a$ and $E X$.

Solution: $\int_0^1 x(1-x) \, dx = \frac{x^2}{2} - \frac{x^3}{3} \bigg|_{x=1}^{x=0} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$. Hence $a = 6$.

$6 \int_0^1 x^2(1-x) \, dx = 2x^3 - 3x^4 \bigg|_{x=1}^{x=0} = 2 - \frac{3}{2} = \frac{1}{2}$

Answer: $a = 6$, $EX = \frac{1}{2}$ Note that $EX = \frac{1}{2}$ can be obtained by just looking at the picture because this is the center of the parabola.

(2) Suppose a different random variable $X$ has the p.d.f. $f(x)$, which is defined as $f(x) = ax(1-x)$ when $0 < x < \frac{1}{2}$, and $f(x) = 0$ when $x$ is negative or larger than $\frac{1}{2}$. Find $a$ and $EX$.

Solution: $\int_{\frac{1}{2}}^1 x(1-x) \, dx = \frac{x^2}{2} - \frac{x^3}{3} \bigg|_{x=1}^{x=\frac{1}{2}} = \frac{1}{8} - \frac{1}{24} = \frac{1}{12}$. Hence $a = 12$.

$12 \int_{\frac{1}{2}}^1 x^2(1-x) \, dx = 4x^3 - 3x^4 \bigg|_{x=\frac{1}{2}}^{x=0} = \frac{1}{2} - \frac{3}{16} = \frac{5}{16}$

Answer: $a = 12$, $EX = \frac{5}{16}$ Note that $a = 12$ can be obtained by just looking at the picture because $\frac{1}{2}$ this is the center of the parabola and so the answer for $a$ should be $2 \cdot 6$, where 6 comes from the previous question.

(3) Suppose another random variable $X$ has the p.d.f. $f(x)$, which is defined as $f(x) = \frac{1}{x}$ when $1 < x < b$, and $f(x) = 0$ when $x$ is smaller than 1 or larger than $b$. Here $b$ is a certain number, which can appear in your answer. Find formulas for $EX$ and Var$(X)$. Do not simplify.

Solution: $EX = \int_1^b \frac{x}{x} \, dx = b - 1$. $EX^2 = \int_1^b x \, dx = (b^2 - 1)/2$.

Answer: $EX = b - 1$, $\text{Var}(X) = EX^2 - (EX)^2 = (b^2 - 1)/2 - (b - 1)^2$

Bonus question 1: What is $b$ in the previous question? Solution: $\int_1^b \frac{1}{x} \, dx = \log(b) - \log(1) = \log(b) = 1$. We know $\log(e) = 1$ because $e^1 = e$. Answer: $b = e$

Bonus question 2: Can you find a number $c$ such that $f(x) = \frac{c}{x}$ is a p.d.f. on the half-infinite interval $[1, \infty)$? Solution: $\int_1^\infty \frac{1}{x} \, dx = \log(x) \bigg|_1^\infty = +\infty$. Answer: not possible, d.n.e.

Bonus question 3: Suppose $c$ and $p$ are numbers such that $f(x) = \frac{c}{x^p}$ is a p.d.f. on the half-infinite interval $[1, \infty)$. Can you find a relation between $p$ and $c$? Solution: $\int_1^\infty \frac{1}{x^p} \, dx = \frac{1}{1-p}x^{1-p} \bigg|_1^\infty = p-1$ if $p > 1$, and is not defined if $p \leq 1$. Answer: $c = 1 - p$ if $p > 1$, and is not defined if $p \leq 1$