MATH 3160 - Probability - Fall 2017
Quiz 7, Wednesday October 25

(1) Suppose a random variable X has the p.d.f. $f(x)$, which is defined as $f(x) = ax(1-x)$ when $0 < x < 1$, and $f(x) = 0$ when x is negative or larger than 1. Find a and $\mathbb{E}X$.

Solution: $\int_0^1 x(1-x)\,dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_{x=0}^{x=1} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$. Hence $a = 6$.
$6\int_0^1 x^2(1-x)\,dx = 2x^3 - 3\frac{x^4}{2}\Big|_{x=0}^{x=1} = 2 - \frac{3}{2} = \frac{1}{2}$

Answer: $\boxed{a = 6, \quad \mathbb{E}X = \frac{1}{2}}$ Note that $\mathbb{E}X = \frac{1}{2}$ can be obtained by just looking at the picture because this is the center of the parabola.

(2) Suppose a different random variable X has the p.d.f. $f(x)$, which is defined as $f(x) = ax(1-x)$ when $0 < x < \frac{1}{2}$, and $f(x) = 0$ when x is negative or larger than $\frac{1}{2}$. Find a and $\mathbb{E}X$.

Solution: $\int_0^{\frac{1}{2}} x(1-x)\,dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_{x=0}^{x=\frac{1}{2}} = \frac{1}{8} - \frac{1}{24} = \frac{1}{12}$. Hence $a = 12$.
$12\int_0^{\frac{1}{2}} x^2(1-x)\,dx = 4x^3 - 3x^4\Big|_{x=0}^{x=\frac{1}{2}} = \frac{1}{2} - \frac{3}{16} = \frac{5}{16}$

Answer: $\boxed{a = 12, \quad \mathbb{E}X = \frac{5}{16}}$ Note that $a = 12$ can be obtained by just looking at the picture because $\frac{1}{2}$ this is the center of the parabola and so the answer for a should be $2 \cdot 6$, where 6 comes from the previous question.

(3) Suppose another random variable X has the p.d.f. $f(x)$, which is defined as $f(x) = \dfrac{1}{x}$ when $1 < x < b$, and $f(x) = 0$ when x is smaller than 1 or larger than b. Here b is a certain number, which can appear in your answer. Find formulas for $\mathbb{E}X$ and $\mathrm{Var}(X)$. Do not simplify.

Solution: $\mathbb{E}X = \int_1^b \frac{x}{x}\,dx = b - 1$. $\mathbb{E}X^2 = \int_1^b x\,dx = (b^2-1)/2$.

Answer: $\boxed{\mathbb{E}X = b - 1, \quad \mathrm{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = (b^2-1)/2 - (b-1)^2}$

Bonus question 1: What is b in the previous question? Solution: $\int_1^b \frac{1}{x}\,dx = \log(b) - \log(1) = \log(b) = 1$. We know $\log(e) = 1$ because $e^1 = e$. Answer: $\boxed{b = e}$

Bonus question 2: Can you find a number c such that $f(x) = \dfrac{c}{x}$ is a p.d.f. on the half-infinite interval $[1, \infty)$? Solution: $\int_1^\infty \frac{1}{x}\,dx = \log(x)\Big|_1^\infty = +\infty$. Answer: $\boxed{\textit{not possible, d.n.e.}}$

Bonus question 3: Suppose c and p are numbers such that $f(x) = \dfrac{c}{x^p}$ is a p.d.f. on the half-infinite interval $[1, \infty)$. Can you find a relation between p and c? Solution: $\int_1^\infty \frac{1}{x^p}\,dx = \dfrac{1}{1-p}x^{1-p}\Big|_1^\infty = p-1$ if $p > 1$, and is not defined if $p \leqslant 1$. Answer: $\boxed{c = 1 - p \ \textit{ if } \ p > 1, \ \textit{ and is not defined if } \ p \leqslant 1}$