MATH 3160 - Probability - Fall 2017
Quiz 13-14, Wednesday December 6

(1) Find the moment generating function $m_X(t)$ for a random variable X with the p.d.f.

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{if } 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

Solution: $m_X(t) = \int_1^3 \frac{1}{2} e^{tx} dx = \frac{1}{2t} e^{tx} \Big|_{x=1}^{x=3}$

Answer:

$$m_X(t) = \frac{e^{3t} - e^t}{2t}$$

(2) For the moment generating function $m_X(t)$ in problem (1), find $m'_X(0)$ and $m''_X(0)$ by expressing them as moments of X, and computing moments as integrals.

Solution: $\mathbb{E}X = \int_1^3 \frac{1}{2} x \, dx = \frac{1}{4} x^2 \Big|_{x=1}^{x=3} = (9-1)/4 = 2$

$\mathbb{E}X^2 = \int_1^3 \frac{1}{2} x^2 \, dx = \frac{1}{6} x^3 \Big|_{x=1}^{x=3} = (3^3 - 1)/6$

Answer:

$m'_X(0) = 2$ $\quad$ $m''_X(0) = (3^3 - 1)/6 = \frac{13}{3}$ $\quad$ simplification was not required

(3) Find $\mathbb{E}X$ and $\mathbb{E}X^2$ if the random variable X has the moment generating function $m_X(t) = \dfrac{1}{(1-2t)^3}$

Solution: $m'_X(t) = \dfrac{2 \cdot 3}{(1-2t)^4}$ $\quad$ $m''_X(t) = \dfrac{2 \cdot 3 \cdot 2 \cdot 4}{(1-2t)^5}$

Answer:

$\mathbb{E}X = 6$ $\quad$ $\mathbb{E}X^2 = 2 \cdot 3 \cdot 2 \cdot 4 = 48$ $\quad$ simplification was not required

(4) Let $X_1, X_2, \dots, X_{16}$ be independent exponential random variables with parameter $\lambda = 2$. Use the Central Limit Theorem to approximate $\mathbb{P}\left(\sum_{i=1}^{16} X_i > 12\right)$. Use the notation $\Phi(x)$ for the $\mathcal{N}(0,1)$ distribution function. Hint: recall that λ is not the same as $\mu = \mathbb{E}X$ or $\sigma^2 = \mathrm{Var}(X)$.

Solution: $\mu_X = \frac{1}{2}$, $\sigma_X^2 = \frac{1}{4}$, $\mathbb{P}\left(\sum_{i=1}^{16} X_i > 12\right) = \mathbb{P}\left(Z > \dfrac{12 - n\mu}{\sigma\sqrt{n}}\right) = \mathbb{P}\left(Z > \dfrac{12 - \frac{16}{2}}{\frac{1}{2}\sqrt{16}}\right)$

Answer:

$\mathbb{P}\left(\sum_{i=1}^{16} X_i > 12\right) = \mathbb{P}(Z > 2) = 1 - \Phi(2) \approx 0.02275$ $\quad$ simplification was not required