MATH 3160 - Probability - Fall 2017
Quiz 10, Wednesday November 8

Use the notation $\Phi(x)$ for the $\mathcal{N}(0,1)$ distribution function, that is $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-y^2/2} dy = \mathbb{P}(Z < x)$ where Z is the standard normal random variable.

. . . .

(1) Let X be a uniform random variable on the interval $[0,4]$, and $Y = \sqrt{X}$. Find the c.d.f. $F_Y(y)$ of Y. *Hint: it may be useful if first you find the range of Y, and use cases to define the c.d.f.*

Solution: we know from the text that $F_X(x) = \frac{x}{4}$ if $x \in [0,4]$. Let $y = \sqrt{x}$, and so $x = y^2$. Hence $F_Y(y) = \mathbb{P}(Y < y) = \mathbb{P}(\sqrt{X} < \sqrt{x}) = \mathbb{P}(X < x) = F_X(x) = F_X(y^2) = \frac{y^2}{4}$

Answer: $\boxed{F_Y(y) = \dfrac{y^2}{4}}$ when $0 < y < 2$; $F_Y(y) = 0$ when $y \leqslant 0$; $F_Y(y) = 1$ when $2 \leqslant y$,

(2) Find the p.d.f. $f_Y(y)$ of Y.

Solution: since $f = F'$, we have that $f_Y(y) = \frac{d}{dy}\frac{y^2}{4} = \frac{y}{2}$ when $0 < y < 2$.

Answer: $\boxed{f_Y(y) = \dfrac{y}{2}}$ when $0 < y < 2$ and zero otherwise.

(3) Find $\mathbb{E}Y$

Solution: $\mathbb{E}Y = \int y f(y) dy = \int_0^2 y \frac{y}{2} dy = \int_0^2 \frac{y^2}{2} dy = \frac{y^3}{6}\Big|_0^2 = \frac{8}{6} = \frac{4}{3}$

Answer: $\boxed{\mathbb{E}Y = \dfrac{4}{3}}$

(4) Find $\mathbb{E}Y^2$

Solution: $\mathbb{E}Y^2 = \int y f(y) dy = \int_0^2 y^2 \frac{y}{2} dy = \int_0^2 \frac{y^3}{2} dy = \frac{y^4}{8}\Big|_0^2 = \frac{16}{8} = 2$

Answer: $\boxed{\mathbb{E}Y^2 = 2}$

(5) If Z is the standard normal $\mathcal{N}(0,1)$ random variable, find the c.d.f. and the p.d.f. of $|Z|$. *Hint: you can use function $\Phi(x)$ and cases.*

Solution: $\mathbb{P}(|Z| < x) = \mathbb{P}(-x < Z < x) = \Phi(x) - \Phi(-x)$

Answer: $\boxed{\Phi(x) - \Phi(-x) = 2\Phi(x) - 1}$ if $x \geqslant 0$, and zero otherwise.

Extra credit question: what is $\mathbb{E}|Z|$ and $\mathbf{Var}\,|Z|$? Answer: $\mathbb{E}|Z| = \frac{2}{\sqrt{2\pi}} \int\limits_0^{\infty} x e^{-x^2/2} dx = \sqrt{\frac{2}{\pi}}$ by the substitution $u = x^2/2,\ du = x\,dx$, $\mathbb{E}|Z|^2 = 1$, and so $\mathbf{Var}\,|Z| = \mathbb{E}|Z|^2 - (\mathbb{E}|Z|)^2 = 1 - \frac{2}{\pi}$.