# AN INTRODUCTION TO ANALYSIS ON FRACTALS

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ANALYSIS ON GRAPHS AND ITS APPLICATIONS
ISAAC NEWTON INSTITUTE FOR MATHEMATICAL SCIENCES
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## Major general references (books)


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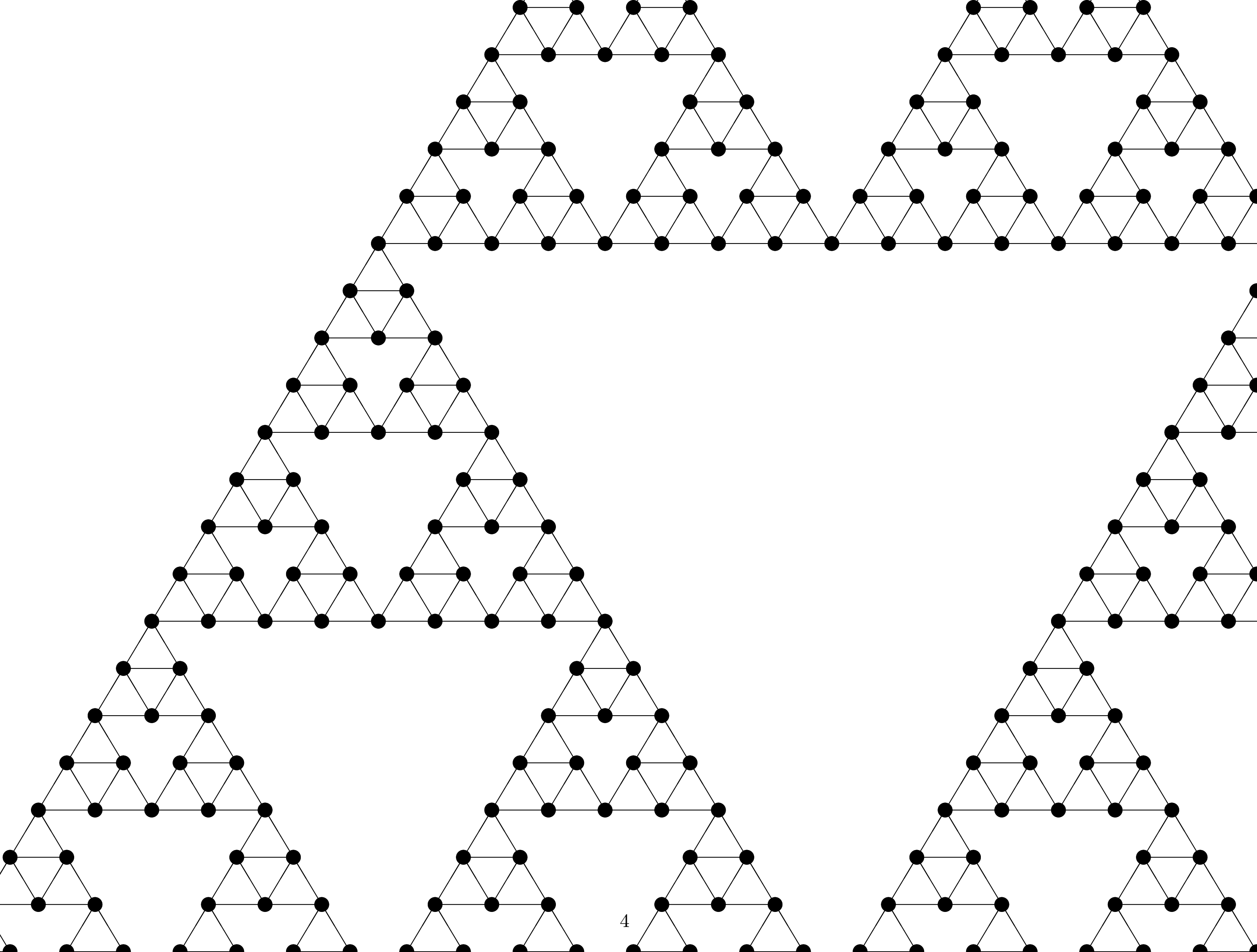
CONTENTS

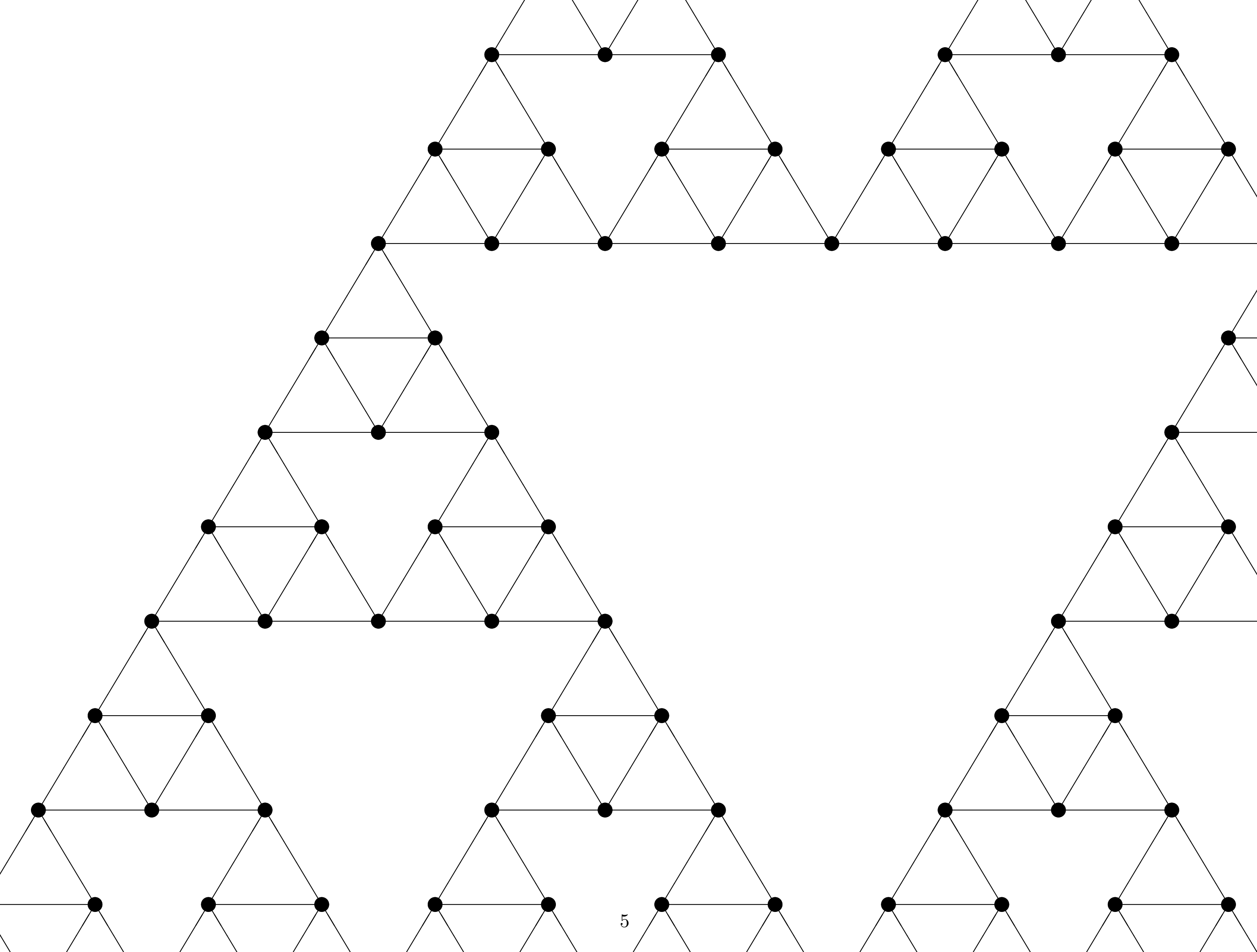
Major general references (books) 2

Lecture 1
Laplacians on self-similar graphs and relation to self-similar groups 6
Abbreviated list of references 21

Lecture 2
Laplacians on self-similar fractals and spectral zeta functions 27
Abbreviated list of references 45

Lecture 3
Kigami's resistance forms on fractals and relation to quantum graphs 55
Abbreviated list of references 78





# Lecture 1
# Laplacians on self-similar graphs and relation to self-similar groups


R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters.* J. Physique Letters **44** (1983), L13–L22.

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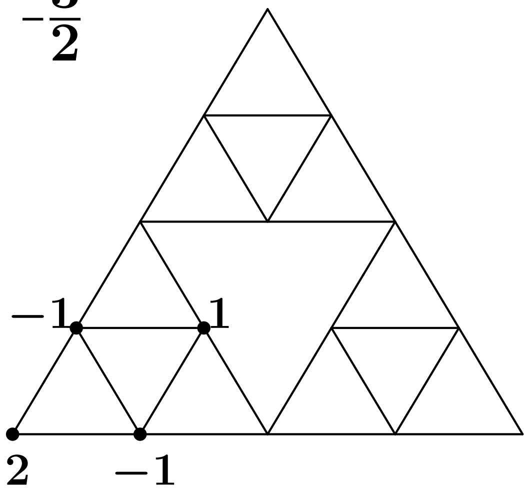
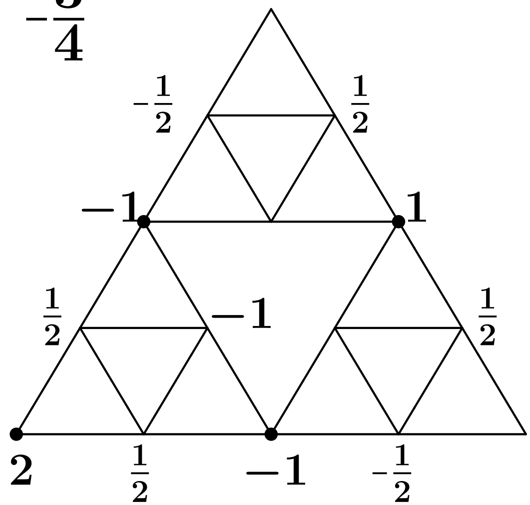
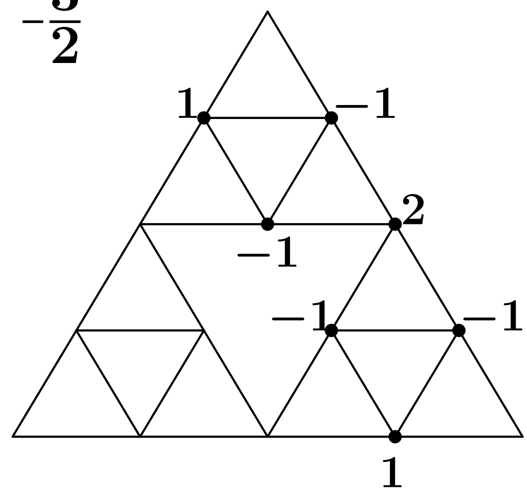
$z = -\frac{3}{2}$



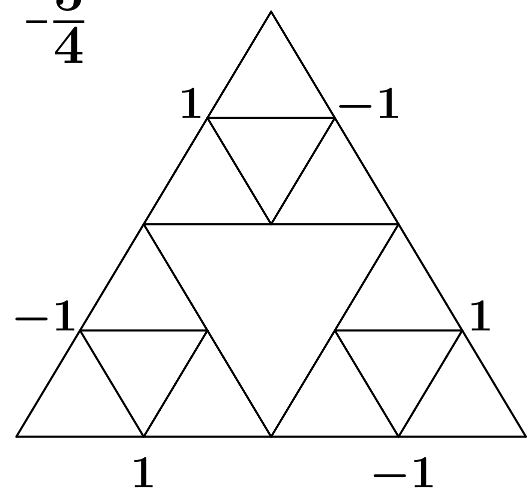

$z = -\frac{3}{4}$




$z = -\frac{3}{2}$




$z = -\frac{5}{4}$



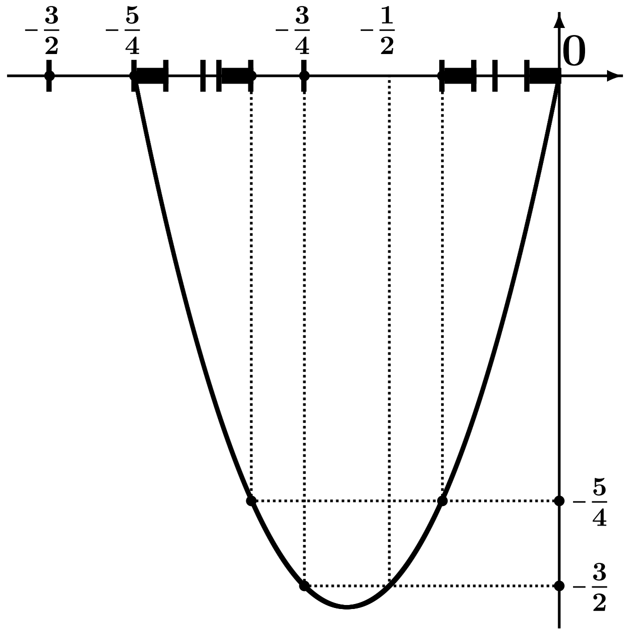
Let $\boldsymbol{\Delta}$ be the probabilistic Laplacian (generator of a simple random walk) on the **Sierpiński lattice**. If $z \neq -\frac{3}{2}, -\frac{5}{4}, -\frac{1}{2}$, and $R(z) = z(4z+5)$, then

$$R(z) \in \sigma(\Delta) \iff z \in \sigma(\Delta)$$

$$\sigma(\Delta) = \mathcal{J}_R \bigcup \mathcal{D}$$

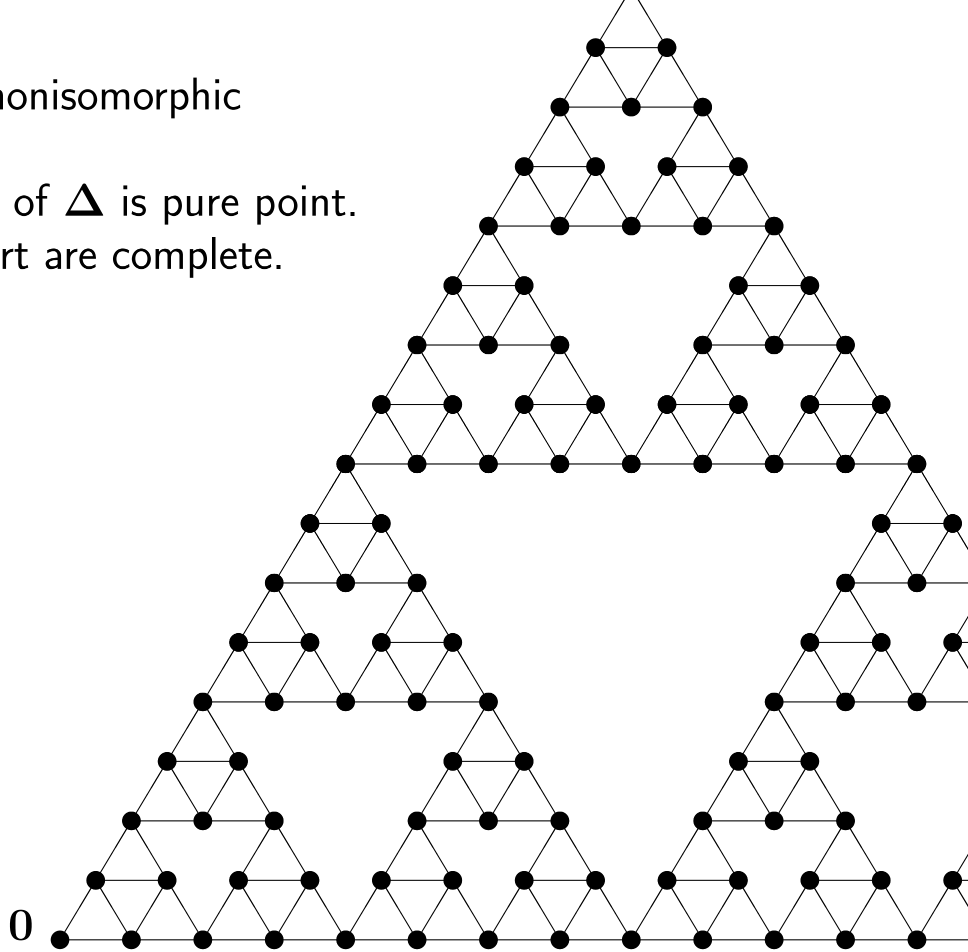
where $\mathcal{D} \stackrel{\text{def}}{=} \{-\frac{3}{2}\} \bigcup \left( \bigcup_{m=0}^{\infty} R^{-m}\{-\frac{3}{4}\} \right)$

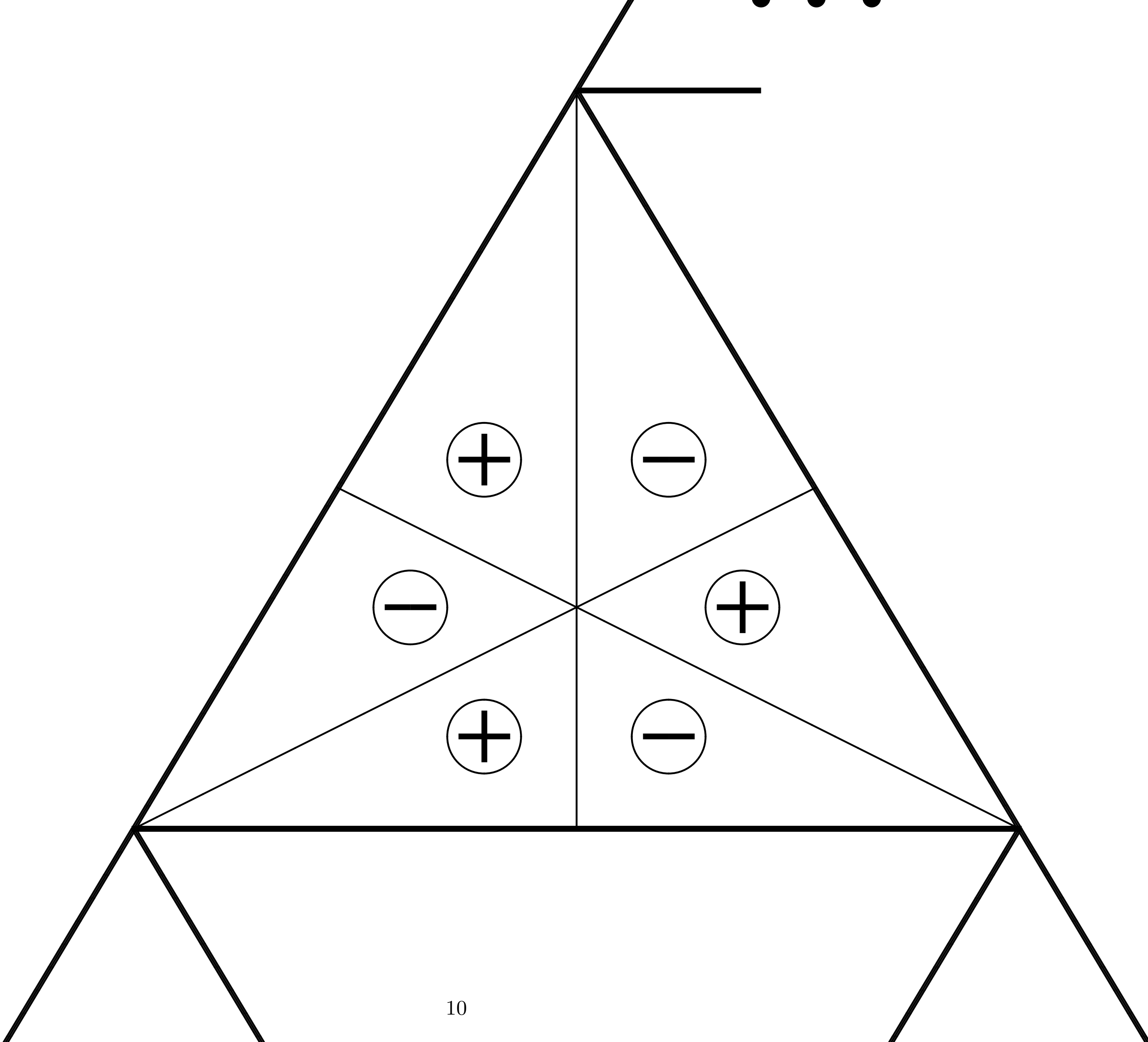
and $\mathcal{J}_R$ is the Julia set of $R(z)$.



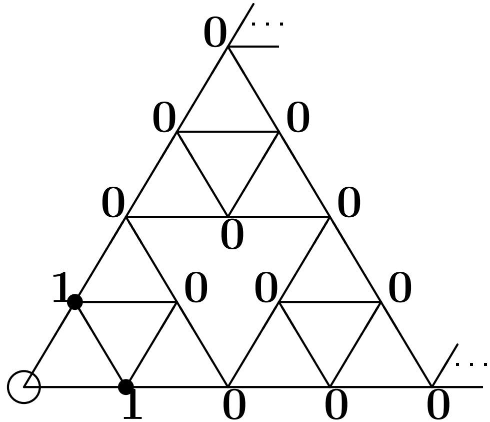
There are uncountably many nonisomorphic Sierpiński lattices.

**Theorem (T).** The spectrum of $\Delta$ is pure point. Eigenfunctions with finite support are complete.





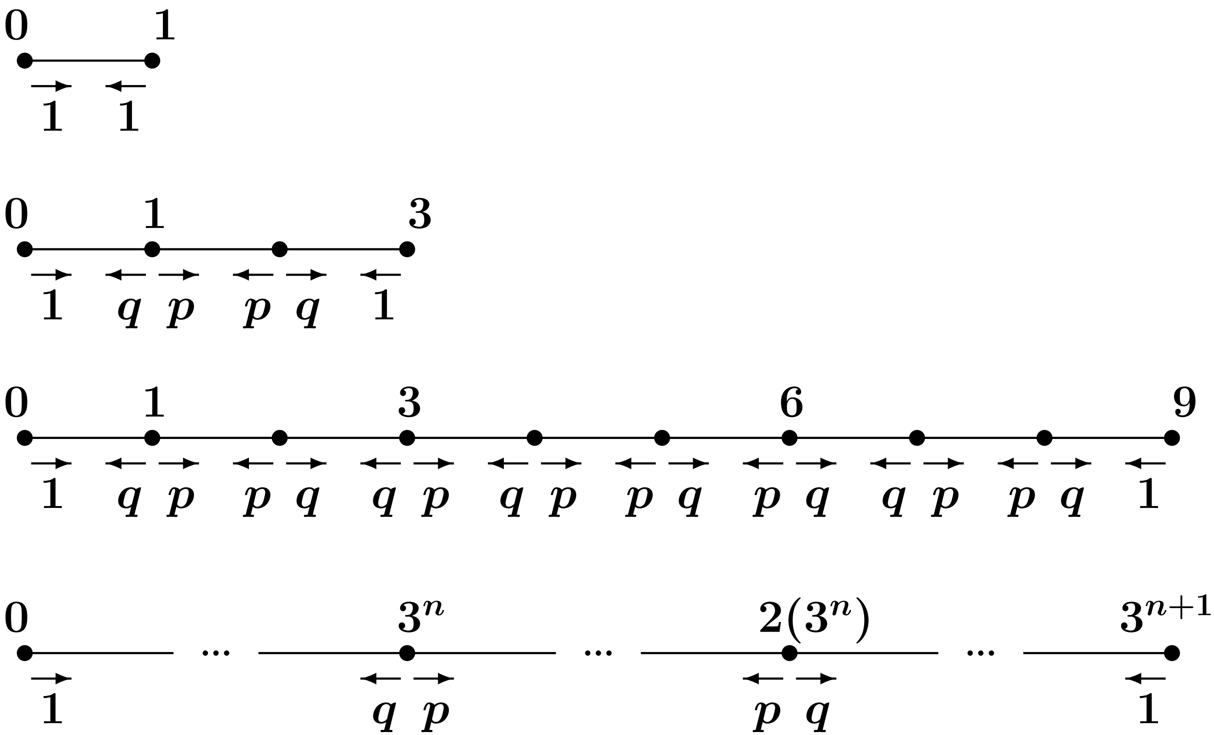
Let $\mathbf{\Delta^{(0)}}$ be the Laplacian with zero (Dirichlet) boundary condition at $\partial L$. Then the compactly supported eigenfunctions of $\mathbf{\Delta^{(0)}}$ are **not** complete (eigenvalues in $\mathcal{E}$ is not the whole spectrum).




Let $\partial L^{(0)}$ be the set of two points adjacent to $\partial L$ and $\omega_{\Delta}^{(0)}$ be the spectral measure of $\mathbf{\Delta^{(0)}}$ associated with $\mathbb{1}_{\partial L^{(0)}}$. Then $\mathrm{supp}(\omega_{\Delta}^{(0)}) = \mathfrak{J}_R$ has Lebesgue measure zero and

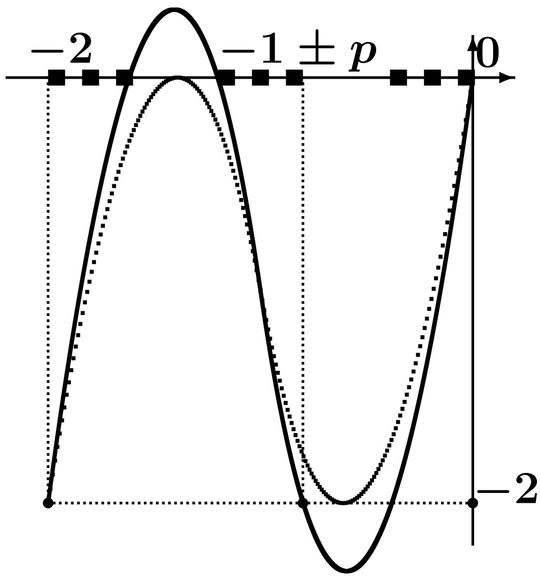
$$\frac{d(\omega_{\Delta}^{(0)} \circ R_{1,2})}{d\omega_{\Delta}^{(0)}}(z) = \frac{(8z+5)(2z+3)}{(2z+1)(4z+5)}$$

Fix $p, q > 0$, $p + q = 1$, and define probabilistic Laplacians $\Delta_n$ on the segments $[0, 3^n]$ of $\mathbb{Z}_+$ inductively as a generator of the random walks:




and let $\Delta = \lim_{n \to \infty} \Delta_n$ be the corresponding probabilistic Laplacian on $\mathbb{Z}_+$.

If $z \neq -1 \pm p$ and $R(z) = z(z^2 + 3z + 2 + pq)/pq$, then $R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$




**Theorem(T).** $\sigma(\Delta) = \mathcal{J}_R$, the Julia set of $R(z)$.

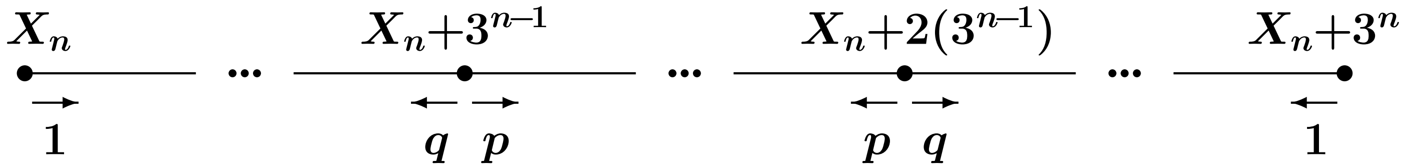
If $p = q$, then $\sigma(\Delta) = [-2, 0]$, spectrum is a.c.

If $p \neq q$, then $\sigma(\Delta)$ is a Cantor set of Lebesgue measure zero, spectrum is singularly continuous.

There are uncountably many "random" self-similar Laplacians $\Delta$ on $\mathbb{Z}$:
For a sequence $\mathcal{K} = \{k_j\}_{j=1}^{\infty}$, $k_j \in \{0, 1, 2\}$, let

$$X_n = -\sum_{j=1}^{n} k_j 3^j$$

and $\Delta_n$ is a probabilistic Laplacian on $[X_n, X_n + 3^n]$:

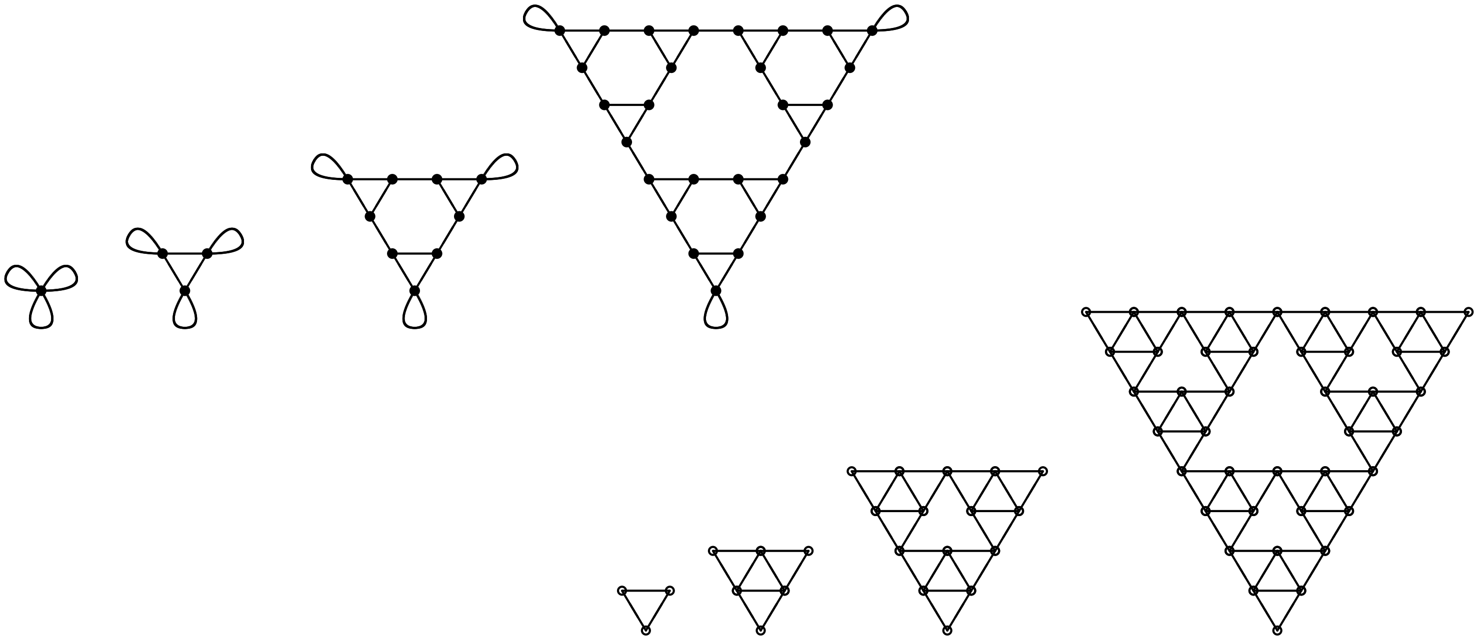


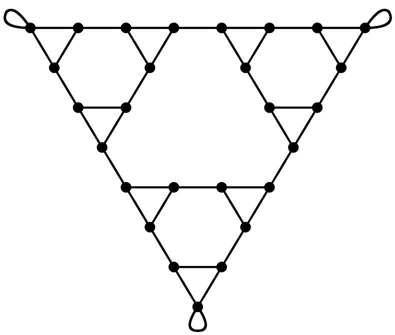

In the previous example $k_j = 0$ for all $j$.

**Theorem (T).**

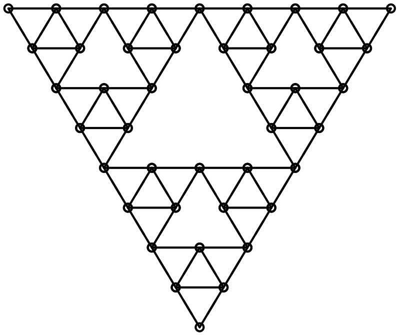
For any sequence $\mathcal{K}$ we have $\sigma(\Delta) = \mathcal{J}_R$. The same is true for the Dirichlet Laplacian on $\mathbb{Z}_+$ (when $k_j \equiv 0$).

R. Grigorchuk and Z. Sunik, *Asymptotic aspects of Schreier graphs and Hanoi Towers groups,* preprint.





Sierpiński 3-graph
(Hanoi Towers-3 group)



Sierpiński 4-graph
(standard)

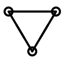
These three polynomials are conjugate:

Sierpiński 3-graph (Hanoi Towers-3 group): $\boldsymbol{f(x) = x^2 - x - 3}$
$\boldsymbol{f(3) = 3}$, $\boldsymbol{f'(3) = 5}$

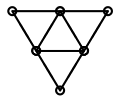
Sierpiński 4-graph, "adjacency matrix" Laplacian: $\boldsymbol{P(\lambda) = 5\lambda - \lambda^2}$
$\boldsymbol{P(0) = 0}$, $\boldsymbol{P'(0) = 5}$

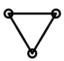
Sierpiński 4-graph, probabilistic Laplacian: $\boldsymbol{R(z) = 4z^2 + 5z}$
$\boldsymbol{R(0) = 0}$, $\boldsymbol{R'(0) = 5}$

**Theorem.** Eigenvalues and eigenfunctions on the Sierpiński 3-graphs and Sierpiński 4-graphs are in one-to-one correspondence, with the exception of the eigenvalue $z = -\frac{3}{2}$ for the 4-graphs.

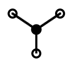


$\underset{4z^2 + 5z}{\longleftarrow}$

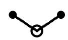


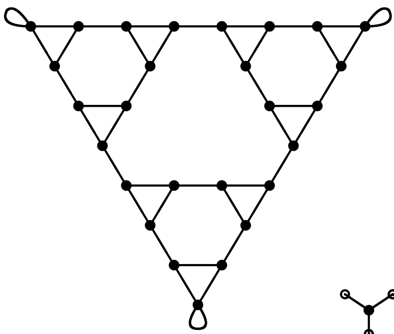


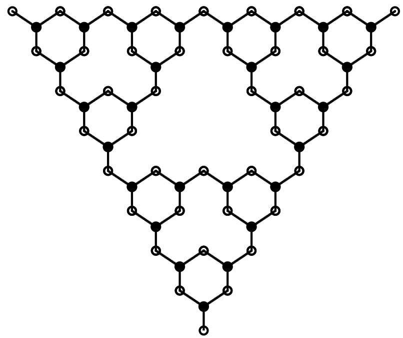
$\underset{\frac{4}{3}z^2 + \frac{8}{3}z}{\longleftarrow}$

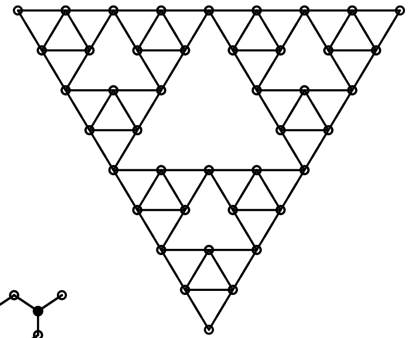


$\underset{2z^2 + 4z}{\longleftarrow}$



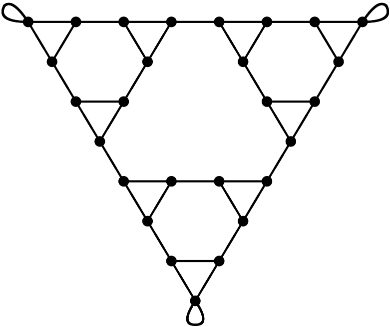


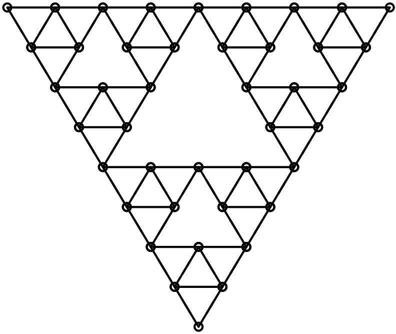


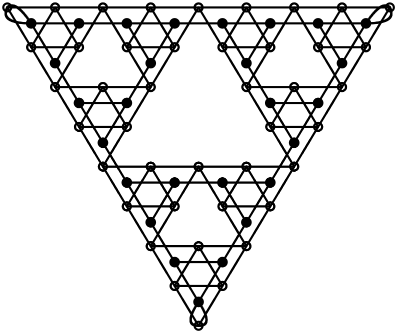


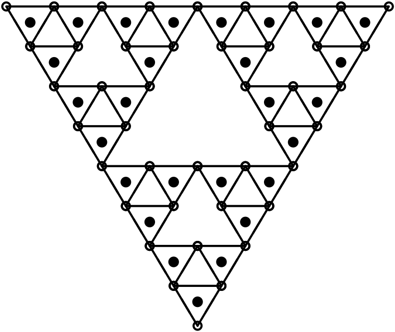
Sierpiński 3-graph
(Hanoi Towers-3 group)
$\boldsymbol{R(z) = 2z^2 + 4z}$

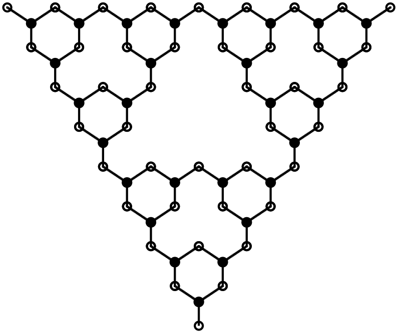
Sierpiński 4-graph
(standard)
$\boldsymbol{R(z) = \frac{4}{3}z^2 + \frac{8}{3}z}$

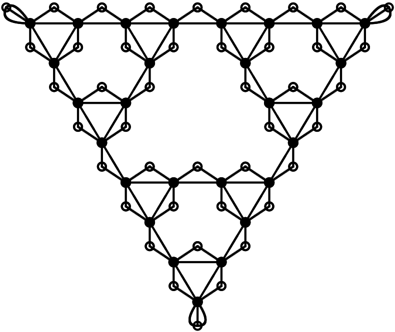


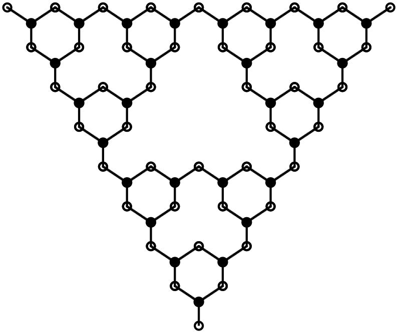


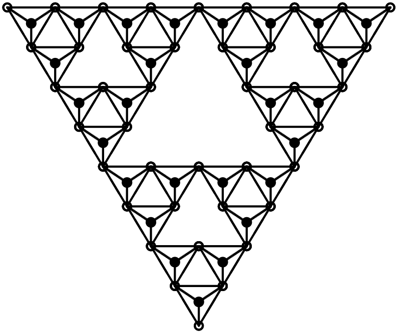












Let $\mathcal{H}$ and $\mathcal{H}_0$ be Hilbert spaces, and $U : \mathcal{H}_0 \longrightarrow \mathcal{H}$ be an isometry.

**Definition.** We call an operator $H$ **spectrally similar** to an operator $H_0$ with functions $\varphi_0$ and $\varphi_1$ if

$$U^*(H-z)^{-1}U = (\varphi_0(z)H_0 - \varphi_1(z))^{-1}$$

In particular, if $\varphi_0(z) \neq 0$ and $R(z) = \varphi_1(z)/\varphi_0(z)$, then

$$U^*(H-z)^{-1}U = \frac{1}{\varphi_0(z)}(H - R(z))^{-1}.$$

If $H = \begin{pmatrix} S & \bar{X} \\ X & Q \end{pmatrix}$ then

$$S - zI_0 - \bar{X}(Q - zI_1)^{-1}X = \varphi_0(z)H_0 - \varphi_1(z)I_0$$

**Theorem (Malozemov, Teplyaev).** If $\Delta$ is the graph Laplacian on a self-similar symmetric infinite graph, then

$$\mathcal{J}_R \subseteq \sigma(\Delta_\infty) \subseteq \mathcal{J}_R \cup \mathcal{D}_\infty$$

where $\mathcal{D}_\infty$ is a discrete set and $\mathcal{J}_R$ is the Julia set of the rational function $R$.

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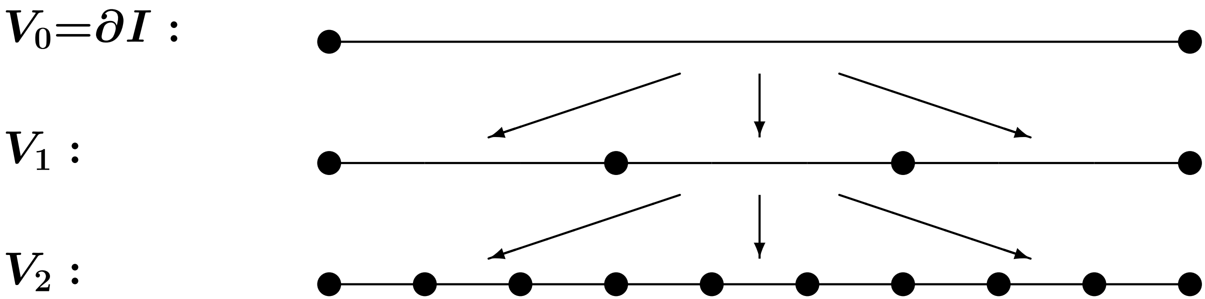
# Lecture 2
# Laplacians on self-similar fractals and spectral zeta functions

Three **contractions** $F_1, F_2, F_3 : \mathbb{R}^1 \longrightarrow \mathbb{R}^1$, $F_j(x) = \frac{1}{3}(x+p_j)$, with fixed points $p_j = 0, \frac{1}{2}, 1$. The interval $I=[0,1]$ is a unique compact set such that

$$I = \bigcup_{j=1,2,3} F_j(I)$$

The *boundary* of $I$ is $\partial I = V_0 = \{0, 1\}$ and the *discrete approximations* to $I$ are $V_n = \bigcup_{j=1,2,3} F_j(V_{n-1}) = \left\{\frac{k}{3^n}\right\}_{k=0}^{3^n}$



**Definition.** The *discrete Dirichlet (energy) form* on $V_n$ is

$$\mathcal{E}_n(f) = \sum_{\substack{x,y \in V_n \\ y \sim x}} (f(y) - f(x))^2$$

and the *Dirichlet (energy) form* on $I$ is $\mathcal{E}(f) = \lim_{n\to\infty} 3^n \mathcal{E}_n(f) = \int_0^1 |f'(x)|^2 dx$

**Definition.** A function $h$ is *harmonic* if it minimizes the energy given the boundary values.

**Proposition.** $3\mathcal{E}_{n+1}(f) \geqslant \mathcal{E}_n(f)$ and $3\mathcal{E}_{n+1}(h) = \mathcal{E}_n(h) = 3^{-n}\mathcal{E}(h)$ for a harmonic $h$.

**Proposition.** The Dirichlet (energy) form on $I$ is *self-similar* in the sense that

$$\mathcal{E}(f) = 3 \sum_{j=1,2,3} \mathcal{E}(f \circ F_j)$$

**Definition.** The $\boldsymbol{discrete\ Laplacians}$ on $V_n$ are

$$\Delta_n f(x) = \frac{1}{2}\sum_{\substack{y\in V_n\\ y\sim x}} f(y)-f(x), \quad x\in V_n\setminus V_0$$

and the Laplacian on $I$ is $\Delta f(x) = \lim_{n\to\infty} 9^n \Delta_n f(x) = f''(x)$

**Gauss–Green (integration by parts) formula:**

$$\mathcal{E}(f) = -\int_0^1 f\Delta f dx + ff'\Big|_0^1$$

**Spectral asymptotics:** Let $\rho(\lambda)$ be the $\boldsymbol{eigenvalue\ counting\ function}$ of the Dirichlet or Neumann Laplacian $\Delta$:

$$\rho(\lambda) = \#\{j : \lambda_j < \lambda\}.$$

Then

$$\lim_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} = \frac{1}{\pi}$$

where $d_s = 1$ is the spectral dimension.

**Definition.** The ***spectral zeta function*** is $\zeta_\Delta(s) = \sum_{\lambda_j \neq 0} (-\lambda_j)^{-s/2}$
Its poles are the ***complex spectral dimensions.***

Let $R(z)$ be a polynomial of degree $N$ such that its Julia set $\mathfrak{J}_R \subset (-\infty, 0]$, $R(0) = 0$ and $c = R'(0) > 1$.

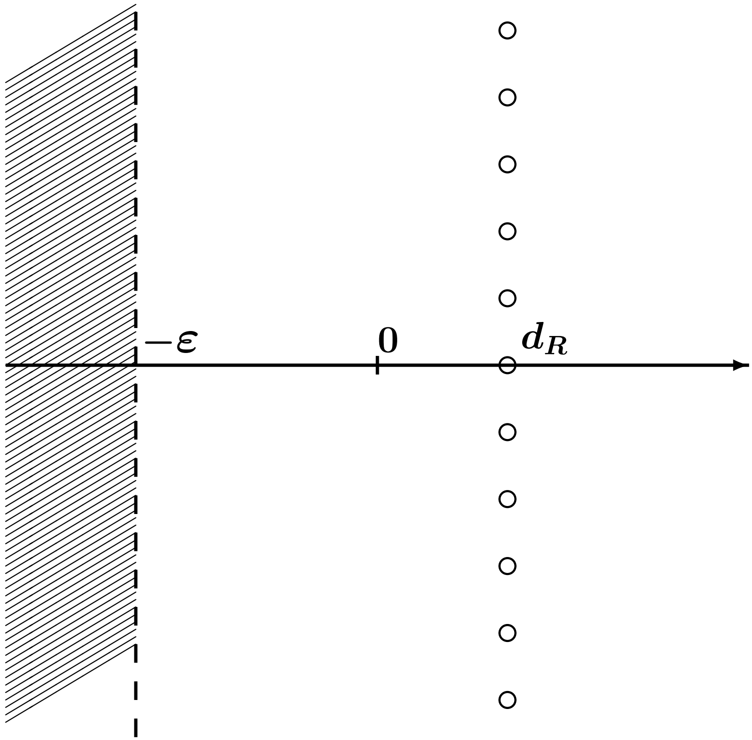
**Definition.** The ***zeta function of*** $R(z)$ for $\mathrm{Re}(s) > d_R = \frac{2 \log N}{\log c}$ is

$$\zeta_R^{z_0}(s) = \lim_{n \to \infty} \sum_{z \in R^{-n}\{z_0\}} (-c^n z)^{-s/2} = \sum \lambda_j^{-s/2}$$

**Theorem.** $\zeta_R^{z_0}(s) = \dfrac{f_1(s)}{1 - N c^{-s/2}} + f_2^{z_0}(s)$, where $f_1(s)$ and $f_2^{z_0}(s)$ are analytic for $\mathrm{Re}(s) > 0$. If $\mathfrak{J}_R$ is totally disconnected, then this meromorphic continuation extends to $\mathrm{Re}(s) > -\varepsilon$, where $\varepsilon > 0$.

In the case of polynomials this theorem has been improved by Grabner et al.

$d_R \in$ the poles of $\zeta_R^{z_0} \subseteq \left\{ \frac{2 \log N + 4in\pi}{\log c} : n \in \mathbb{Z} \right\}$



**Theorem.** $\zeta_\Delta(s) = \zeta^0_R(s)$ where $R(z) = z(4z^2{+}12z{+}9)$.

The Riemann zeta function $\zeta(s)$ satisfies $\zeta(s) = \pi^s \zeta^0_R(s)$ The only complex spectral dimension is the pole at $s = 1$.

A sketch of the proof: If $z \neq -\frac{1}{2}, -\frac{3}{2}$, then

$$R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$$

and so $\zeta_\Delta(s) = \zeta^0_R(s)$ since the eigenvalues $\lambda_j$ of $\Delta$ are limits of the eigenvalues of $9^n \Delta_n$.

Also $\lambda_j {=} {-}\pi^2 j^2$ and so

$$\zeta_\Delta(s) = \sum_{j=1}^{\infty} \left(\pi^2 j^2\right)^{-s/2} = \pi^{-s}\zeta(s)$$

where $\zeta(s)$ is the Riemann zeta function. *Q.E.D.*

$$\zeta(s) = \pi^s \lim_{n\to\infty} \sum_{\substack{z\in R^{-n}\{0\} \\ z\neq 0}} \left(-9^n z\right)^{-s/2}$$

**Definition.** $\Delta_\mu$ is $\mu$*–Laplacian* if

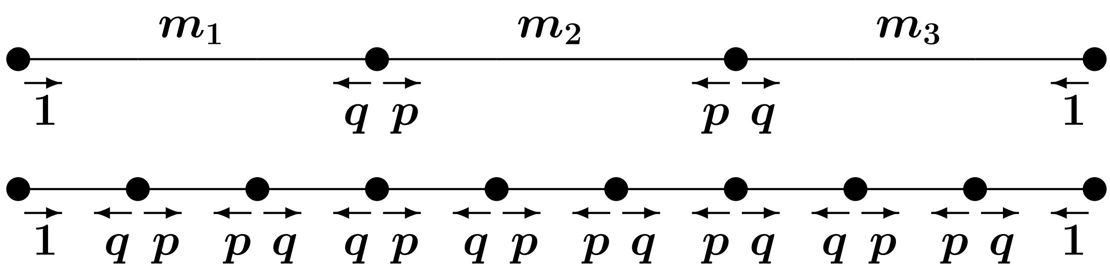
$$\mathcal{E}(f) = \int_0^1 |f'(x)|^2 dx = -\int_0^1 f\Delta_\mu f d\mu + ff'\Big|_0^1.$$

**Definition.** A probability measure $\mu$ is ***self-similar*** with weights $m_1, m_2, m_3$ if $\mu = \sum_{j=1,2,3} m_j \mu \circ F_j$.

**Proposition.** $\Delta_\mu f(x) = \frac{f''}{\mu} = \lim_{n\to\infty} \left(1+\frac{2}{pq}\right)^n \Delta_n f(x)$.

$$\Delta_n f\left(\tfrac{k}{3^n}\right) = \begin{cases} pf\left(\frac{k-1}{3^n}\right) + qf\left(\frac{k+1}{3^n}\right) - f\left(\frac{k}{3^n}\right) \\ qf\left(\frac{k-1}{3^n}\right) + pf\left(\frac{k+1}{3^n}\right) - f\left(\frac{k}{3^n}\right) \end{cases}$$

where $m_1 = m_3$, $p = \frac{m_2}{m_1+m_2}$, $q = \frac{m_1}{m_1+m_2}$, and



**Spectral asymptotics:** If $\rho(\lambda)$ is the eigenvalue counting function of the Dirichlet or Neumann Laplacian $\Delta_\mu$, then

$$0 < \liminf_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} \leqslant \limsup_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} < \infty$$

where the spectral dimension is

$$d_s = \frac{\log 9}{\log(1+\frac{2}{pq})} \leqslant 1.$$

All the inequalities are strict if and only if $p \neq q$.

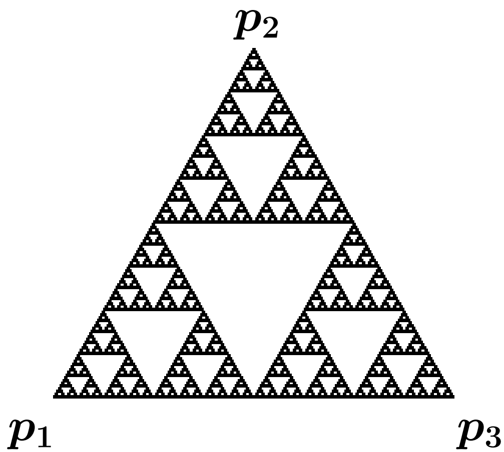
**Proposition.** $R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$

where $z \neq -1 \pm p$ and $R(z) = z(z^2+3z+2+pq)/pq$.

Note that $R'(0) = 1 + \frac{2}{pq}$, and $d_s = d_R$.

**Theorem.** $\zeta_{\Delta_\mu}(s) = \zeta_R^0(s)$

Three **contractions** $F_1, F_2, F_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$,
$F_j(x) = \frac{1}{2}(x+p_j)$, with fixed points $p_1, p_2, p_3$.




The **Sierpiński gasket** is a unique compact set $S$ such that

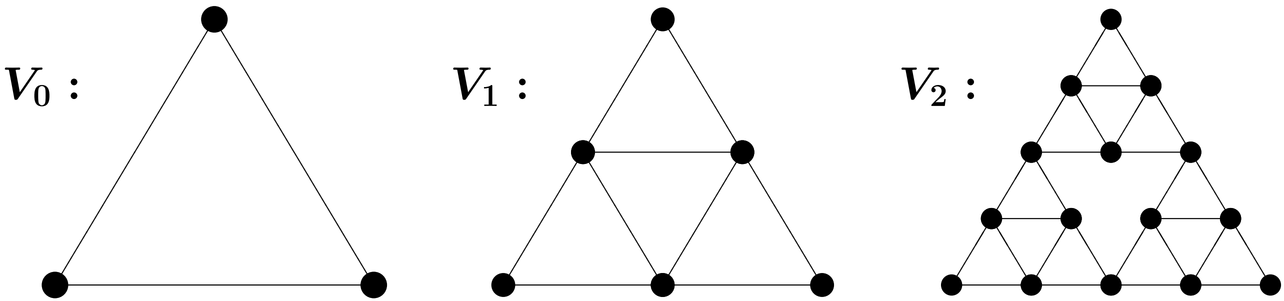
$$S = \bigcup_{j=1,2,3} F_j(S)$$

**Definition.** The $\boldsymbol{boundary}$ of $\boldsymbol{S}$ is

$$\partial S = V_0 = \{p_1, p_2, p_3\}$$

and $\boldsymbol{discrete\ approximations}$ to $\boldsymbol{S}$ are

$$V_n = \bigcup_{j=1,2,3} F_j(V_{n-1})$$



**Definition.** The ***discrete Dirichlet (energy) form*** on $V_n$ is

$$\mathcal{E}_n(f) = \sum_{\substack{x,y\in V_n \\ y\sim x}} (f(y)-f(x))^2$$

and the ***Dirichlet (energy) form*** on $S$ is

$$\mathcal{E}(f) = \lim_{n\to\infty} \left(\tfrac{5}{3}\right)^n \mathcal{E}_n(f)$$

**Definition.** A function $h$ is ***harmonic*** if it minimizes the energy given the boundary values.

**Proposition.** $\frac{5}{3}\mathcal{E}_{n+1}(f) \geqslant \mathcal{E}_n(f)$
$\frac{5}{3}\mathcal{E}_{n+1}(h)=\mathcal{E}_n(h)=\left(\frac{5}{3}\right)^{-n}\mathcal{E}(h)$ for a harmonic $h$.

**Theorem (Kigami).** $\mathcal{E}$ is a local regular Dirichlet form on $S$ which is self-similar in the sense that

$$\mathcal{E}(f) = \frac{5}{3} \sum_{j=1,2,3} \mathcal{E}(f\circ F_j)$$

**Definition.** The $\boldsymbol{discrete\ Laplacians}$ on $V_n$ are

$$\Delta_n f(x) = \frac{1}{4} \sum_{\substack{y \in V_n \\ y \sim x}} f(y) - f(x), \quad x \in V_n \setminus V_0$$

and the Laplacian on $S$ is

$$\Delta_\mu f(x) = \lim_{n \to \infty} 5^n \Delta_n f(x)$$

if this limit exists and $\Delta_\mu f$ is continuous.

**Gauss–Green (integration by parts) formula:**

$$\mathcal{E}(f) = -\int_S f \Delta_\mu f d\mu + \sum_{p \in \partial S} f(p) \partial_n f(p)$$

where $\mu$ is the normalized Hausdorff measure, which is self-similar with weights $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$:

$$\mu = \frac{1}{3} \sum_{j=1,2,3} \mu \circ F_j.$$

**Spectral asymptotics:** If $\rho(\lambda)$ is the eigenvalue counting function of the Dirichlet or Neumann Laplacian $\Delta_\mu$, then

$$0 < \liminf_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} < \limsup_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} < \infty$$

where the spectral dimension is

$$1 < d_s = \tfrac{\log 9}{\log 5} < 2.$$

**Proposition.** $R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$ where $z \neq -\frac{1}{2}, -\frac{3}{4}, -\frac{5}{4}$ and $R(z) = z(5 + 4z)$.

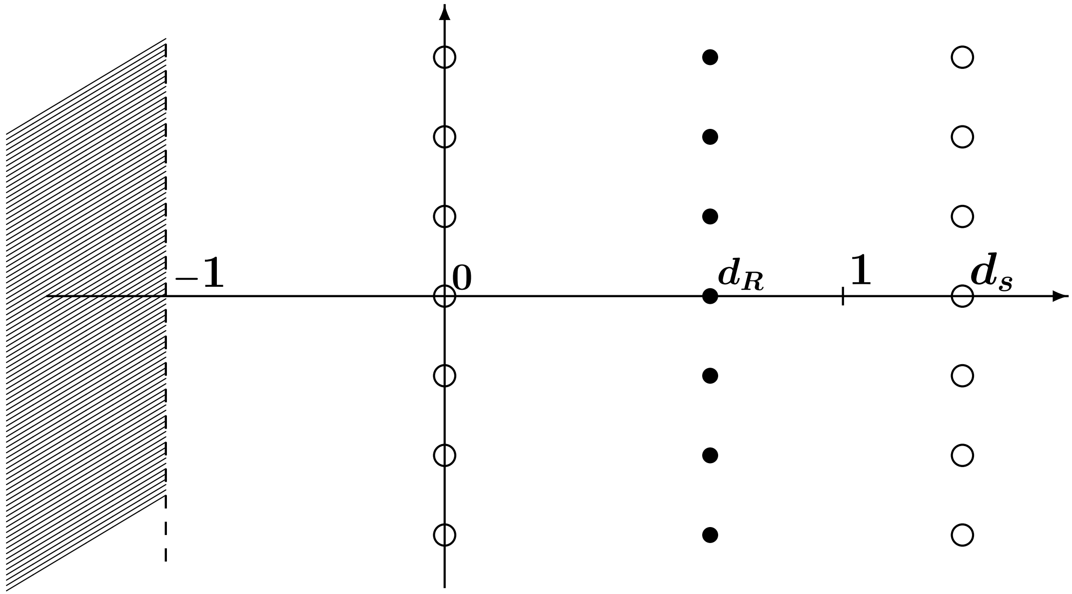
**Theorem (Fukushima, Shima).** Every eigenvalue of $\Delta_\mu$ has a form

$$\lambda = 5^m \lim_{n\to\infty} 5^n R^{-n}(z_0)$$

where $R^{-n}(z_0)$ is a preimage of $z_0 = -\frac{3}{4}, -\frac{5}{4}$ under the $n$-th iteration power of the polynomial $R(z)$. The multiplicity of such an eigenvalue is $C_1 3^m + C_2$.

**Theorem.** Zeta function of the Laplacian on the Sierpiński gasket is

$$\zeta_{\Delta_\mu}(s) = \frac{1}{2}\zeta_R^{-\frac{3}{4}}(s)\left(\frac{1}{5^{s/2}-3}+\frac{3}{5^{s/2}-1}\right) + \frac{1}{2}\zeta_R^{-\frac{5}{4}}(s)\left(\frac{3\cdot 5^{-s/2}}{5^{s/2}-3}-\frac{5^{-s/2}}{5^{s/2}-1}\right)$$




$$d_s = \frac{\log 9}{\log 5}$$

$$d_R = \frac{\log 4}{\log 5}$$

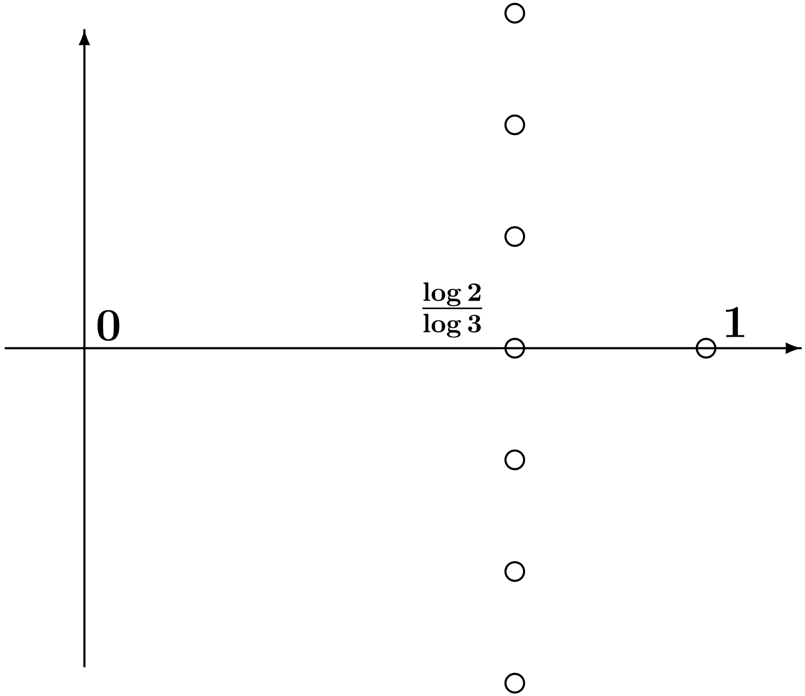
**Definition.** If $\mathfrak{L}$ is a fractal string, that is, a disjoint collection of intervals of lengths $l_j$, then its *geometric zeta function* is $\zeta_{\mathfrak{L}}(s) = \sum l_j^s$.

**Theorem (Lapidus).** If $A = -\frac{d^2}{dx^2}$ is a Neumann or Dirichlet Laplacian on $\mathfrak{L}$, then $\zeta_A(s) = \pi^{-s}\zeta(s)\zeta_{\mathfrak{L}}(s)$.

## Example: Cantor self-similar fractal string.

If $\mathfrak{L}$ is the complement of the middle third Cantor set in $[0, 1]$, then the complex spectral dimensions are $1$ and $\{\frac{\log 2 + 2in\pi}{\log 3} : n \in \mathbb{Z}\}$,

$$\zeta_{\mathfrak{L}}(s) = \frac{1}{1 - 2\cdot 3^{-s}}, \qquad \zeta_A(s) = \zeta(s)\frac{\pi^{-s}}{1 - 2\cdot 3^{-s}}$$


0
$\frac{\log 2}{\log 3}$
1

**Definition.** A post critically finite (p.c.f.) self-similar set $F$ is a compact connected metric space with a finite boundary $\partial F \subset F$ and contractive injections $\psi_i : F \to F$ such that

$$F = \Psi(F) = \bigcup_{i=1}^{k} \psi_i(F)$$

and

$$\psi_v(F) \bigcap \psi_w(F) \subseteq \psi_v(\partial F) \bigcap \psi_w(\partial F),$$

for any two different words $v$ and $w$ of the same length. Here for a finite word $w \in \{1, \ldots, k\}^m$ we define $\psi_w = \psi_{w_1} \circ \ldots \circ \psi_{w_m}$.

We assume that $\partial F$ is a minimal such subset of $F$. We call $\psi_w(F)$ an $m$-cell. ***The p.c.f. assumption is that every boundary point is contained in a single 1-cell.***

**Theorem (Kigami, Lapidus).** The spectral dimension of the Laplacian $\Delta_\mu$ is the unique solution of the equation

$$\sum_{i=1}^{k} (r_i \mu_i)^{d_s/2} = 1$$

**Conjecture.** On every p.c.f. fractal $F$ there exists a local regular Dirichlet form $\mathcal{E}$ which gives positive capacity to the boundary points and is self-similar in the sense that

$$\mathcal{E}(f) = \sum_{i=1}^{k} \rho_i \mathcal{E}(f \circ \psi_i)$$

for a set of positive refinement weights $\rho = \{\rho_i\}_{i=1}^{k}$.

**Definition.** The group $G$ of acts on a finitely ramified fractal $F$ if each $g \in G$ is a homeomorphism of $F$ such that $g(V_n) = V_n$ for all $n \geqslant 0$.

**Proposition.** Suppose a group $G$ of acts on a self-similar finitely ramified fractal $F$ and $G$ restricted to $V_0$ is the whole permutation group of $V_0$. Then there exists a unique, up to a constant, $G$-invariant self-similar resistance form $\mathcal{E}$ with equal energy renormalization weights $\rho_i$ and

$$\mathcal{E}_0(f, f) = \sum_{x,y \in V_0} \big(f(x) - f(y)\big)^2.$$

Moreover, for any $G$-invariant self-similar measure $\mu$ the Laplacian $\Delta_\mu$ has the spectral self-similarity property (a.k.a. spectral decimation).

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# Lecture 3
# Kigami's resistance forms on fractals and relation to quantum graphs

**Definition.** A compact connected metric space $F$ is called a *finitely ramified self-similar set* if there are injective contraction maps $\psi_1, \ldots, \psi_m : F \to F$ and a finite set $V_0 \subset F$ such that

$$F = \bigcup_{i=1}^{m} \psi_i(F) = \Psi(F)$$

and

$$F_w \cap F_{w'} = V_w \cap V_{w'}$$

for any two distinct words $w, w' \in W_n = \{1, \ldots, m\}^n$, where $F_w = \psi_w(F)$, $V_w = \psi_w(V_0)$ and $\psi_w = \psi_{w_1} \circ \ldots \circ \psi_{w_n}$.

The vertices of generation $n$ are defined by $V_n = \Psi(V_{n-1}) = \Psi^n(V_0)$.

The fractal $F$ can be uniquely reconstructed from its "combinatorial skeleton" or "ancestor": $\{\partial F = V_0,\ V_1,\ \Psi|_{V_0}\}$ [Kigami, 1993, Appendix A].

A symmetric vanishing on the diagonal function $c_0 : V_0^2 \longrightarrow \mathbb{R}_+$ (set of conductances) defines a discrete Dirichlet form

$$\mathcal{E}_0(f) = \sum_{x,y \in V_0} (f(y) - f(x))^2 c_0(x,y).$$

Its refinement by $\Psi$ is

$$\mathcal{E}_1(f) = \Psi_\rho(\mathcal{E}_0)(f) = \sum_{i=1}^{k} \rho_i \cdot \mathcal{E}_0(f \circ \psi_i).$$

and the trace map is

$$\mathrm{Tr}(\mathcal{E}_1)(f) = \inf\{\mathcal{E}_1(g) | g : V_1 \rightarrow \mathbb{R}, g|_{V_0} = f\}.$$

**Theorem (Kigami).** For given a set of positive refinement weights $\rho = \{\rho_i\}_{i=1}^{k}$ self-similar local regular Dirichlet forms $\mathcal{E}$ which gives positive capacity to the boundary points are in one-to-one correspondence with the fixed points $\mathcal{E}_0$ of the renormalization map

$$\Lambda_\rho = \mathrm{Tr} \circ \Psi_\rho.$$

**Definition.** A resistance form $\mathcal{E}$ is self-similar if

$$\mathcal{E}(f, f) = \sum_{i=1}^{m} \rho_i \mathcal{E}(f \circ \psi_i, f \circ \psi_i).$$

**Conjecture.** Any finitely ramified self-similar set has a self-similar resistance form. Any p.c.f. self-similar set has a regular self-similar resistance form.

Thus we are looking for nonlinear eigenvectors $\mathcal{E}_0 \in \mathbb{D} \cap \mathbb{P}^\circ$

$$\Lambda_\rho(\mathcal{E}_0) = \gamma \mathcal{E}_0$$

where $\mathbb{D}$ is the cone of Dirichlet forms on $V_0$ with and $\mathbb{P}$ is the cone of nonnegative quadratic forms. Its interior $\mathbb{P}^\circ$ consists of positive forms.

**Proposition.**

(1) $\Lambda_\rho : \mathbb{D} \longrightarrow \mathbb{D}, \ \ \mathbb{P} \longrightarrow \mathbb{P}, \ \ \mathbb{P}^\circ \longrightarrow \mathbb{P}^\circ.$
(2) $\Lambda_\rho$ is continuous on $\mathbb{D} \cup \mathbb{P}^\circ$
(3) $\Lambda_\rho(\alpha\mathcal{E}) = \alpha\Lambda_\rho(\mathcal{E})$ for all $\alpha \geq 0$
(4) $\Lambda_\rho(\mathcal{E} + \mathcal{F}) \geq \Lambda_\rho(\mathcal{E}) + \Lambda_\rho(\mathcal{F})$

Hilbert's projective metric (a pseudo distance on $\mathbb{P}^\circ$) is

$$h(\mathcal{E}/\mathcal{F}) = \ln \frac{M(\mathcal{E}/\mathcal{F})}{m(\mathcal{E}/\mathcal{F})}.$$

where $\mathcal{E}, \mathcal{F} \in \mathbb{P}^\circ$ is the biggest lower bound of $\mathcal{E}/\mathcal{F}$,

$$m(\mathcal{E}/\mathcal{F}) = \sup\{\alpha > 0 | \alpha\mathcal{F} \leq \mathcal{E}\} > 0$$

and $M(\mathcal{E}/\mathcal{F}) = m(\mathcal{F}/\mathcal{E})^{-1}$.

**Proposition.**

(1) $h(\alpha\mathcal{E},\beta\mathcal{F}) = h(\mathcal{E},\mathcal{F})$ for all $\alpha,\beta > 0$.

(2) Let $H = \{\mathcal{E} \in \mathbb{B} | \mathrm{trace}(\mathcal{E}) = 1\}$ (an affine hyperplane). Then $(H \cap \mathbb{P}^\circ, h)$ is a complete metric space.

(3) The $h$- and the $\|\cdot\|$-topology coincide on $H \cap \mathbb{P}^\circ$.

(4) $h(\mathcal{E},\mathcal{F}) = 0$ if and only if $\mathcal{E} = \alpha\mathcal{F}$.

(5) $\lim_{\mathcal{F}\to\partial\mathbb{P}} h(\mathcal{E},\mathcal{F}) = +\infty$

(6) $\Lambda_\rho$ is $h$-nonexpansive on $\mathbb{P}^\circ$, that is, lower $q_\rho$-level sets are $\Lambda_\rho$-invariant.

Let $\mathbb{H} = H \cap \mathbb{D} \cap \mathbb{P}^\circ$ and $q_\rho : \mathbb{H} \longrightarrow \mathbb{R}_+$,

$$q_\rho(\mathcal{E}) = h(\Lambda_\rho(\mathcal{E}),\mathcal{E}).$$

**Proposition.**

(1) $\Lambda_\rho$ has a unique eigenvector $\mathcal{F} \in \mathbb{H}$ if and only if $q_\rho|_{\mathbb{H}}$ vanishes only at $\mathcal{F}$.

(2) $\Lambda_\rho$ has multiple eigenvectors in $\mathbb{H}$ if and only if $q_\rho$ vanishes on a connected set which accumulates at $\partial\mathbb{P}$.

(3) When a $\Lambda_\rho$-forward orbit started in $\mathbb{H}$ is contained in $B_r(\mathcal{E})$ for some $r > 0$ and $\mathcal{E} \in \mathbb{H}$, then there exists a $\Lambda_\rho$-eigenvector in $B_{3r}(\mathcal{E}) \cap \mathbb{H}$.

**Proposition.** Let $\{\rho_n\}$ be such that $\Lambda_{\rho_n}$ converges to $\Lambda$ in $(C(\mathbb{H}), \|\cdot\|_\infty)$. If $q = h(\Lambda(\cdot), \cdot) : \mathbb{H} \longrightarrow \mathbb{R}_+$ vanishes only at a single point, then there exists an $m \in \mathbb{N}$ such that $\Lambda_{\rho_n}$ has a unique eigenvector in $\mathbb{H}$, for $n \geq m$.

**Definition.** A collection of refinement weights $\rho$ is admissible if and only if

$$\Lambda_\rho(\mathcal{E}_0) = \gamma \mathcal{E}_0$$

has a solution $\mathcal{E}_0 \in \mathbb{D} \cap \mathbb{P}^\circ$.

**Proposition.** The set of admissible weights is open.

**Theorem. (Hambly, Metz, T.)** Let $\rho_n \nearrow \rho_\infty \in (0, \infty]^k$ and $\Lambda_{\rho_\infty}$ has a unique eigenvector in $\mathbb{H}$. Then there exist finite admissible refinement weights.

This result can be summarized as follows: *If, by collapsing a subset of cells of $F$, one can obtain a structure which has admissible weights, then $F$ also has admissible finite weights.*

**Proposition.** If $\#V_0 = 3$ then admissible weights exist.

**Definition.** The group $G$ of acts on a finitely ramified fractal $F$ if each $g \in G$ is a homeomorphism of $F$ such that $g(V_n) = V_n$ for all $n \geqslant 0$.

**Proposition.** Suppose a group $G$ of acts on a self-similar finitely ramified fractal $F$ and $G$ restricted to $V_0$ is the whole permutation group of $V_0$. Then there exists a unique, up to a constant, $G$-invariant self-similar resistance form $\mathcal{E}$ with equal energy renormalization weights $\rho_i$ and $\mathcal{E}_0(f, f) = \sum_{x,y \in V_0} \big(f(x) - f(y)\big)^2$.

**Theorem (Hambly, Metz, T.)** Suppose a self-similar finitely ramified fractal $F$ has connected interior and a group $G$ acts on $F$ such that its action on $V_0$ is transitive. Then there exists a $G$-invariant self-similar resistance form $\mathcal{E}$ on $F$.

**Theorem (Hambly, Metz, T.)** Suppose a self-similar finitely ramified fractal $F$ has connected interior and a symmetric boundary. Then there exists a $G$-invariant self-similar resistance form $\mathcal{E}$ on $F$.

**Examples.**

Generalized non-symmetric Sierpiński gaskets in $\mathbb{R}^2$:

$$\begin{aligned} \rho_2^{-1} + \rho_3^{-1} &> \rho_1^{-1} \\ \rho_1^{-1} + \rho_2^{-1} &> \rho_3^{-1} \\ \rho_1^{-1} + \rho_3^{-1} &> \rho_2^{-1} \end{aligned}$$

"Cut" Sierpiński gasket:

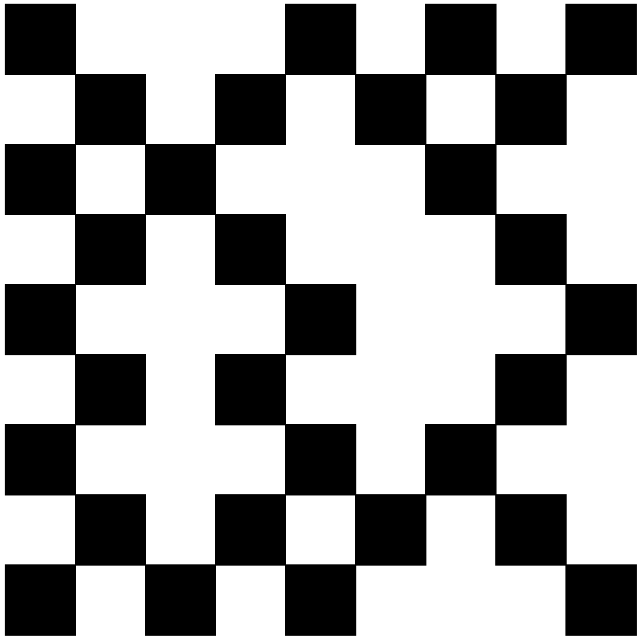
$$\rho_1 + \rho_2 = 1$$
$$\rho_3 + \rho_2 = 1$$

Unit interval:

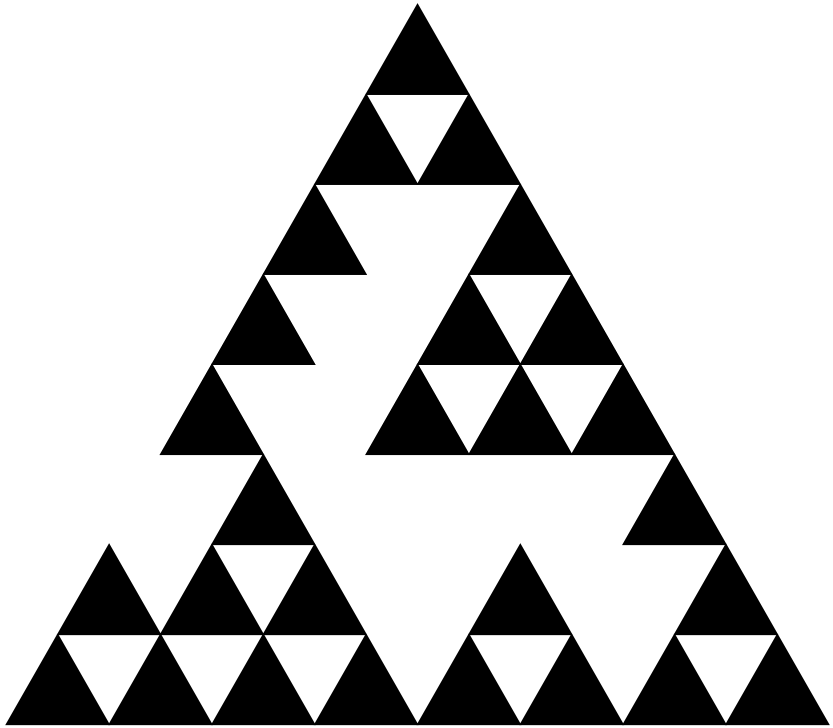
$$\rho_1 + \rho_2 = 1$$

Vicsek set:

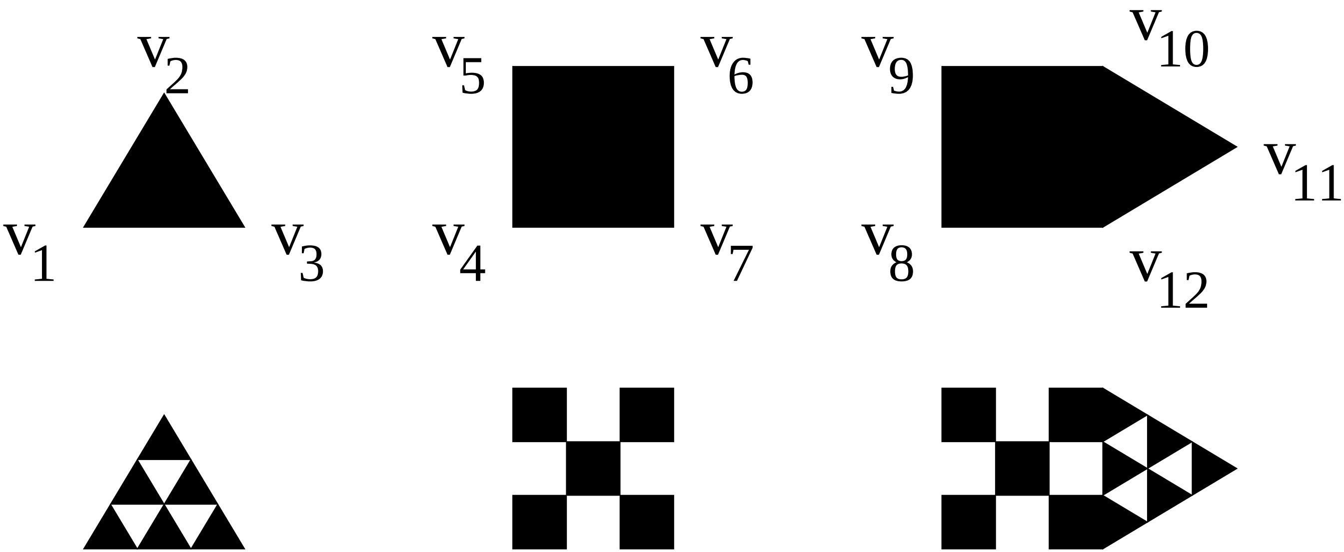
$$\rho_1 + \rho_3 + \rho_5 = 1$$
$$\rho_2 + \rho_4 + \rho_5 = 1$$



A generalized Vicsek set



A generalized Sierpiński gasket




The house fractal.

**Definition.** A pair $(\mathcal{E}, \operatorname{Dom}\mathcal{E})$ is a resistance form on a countable set $V_*$ if

- $\operatorname{Dom}\mathcal{E}$ is a linear subspace of $\ell(V_*)$ containing constants, $\mathcal{E}$ is a nonnegative symmetric quadratic form on $\operatorname{Dom}\mathcal{E}$, and $\mathcal{E}(u,u) = 0$ if and only if $u$ is constant.
- Let $\sim$ be an equivalence relation on $\operatorname{Dom}\mathcal{E}$ defined by $u \sim v$ if and only if $u - v$ is constant on $V_*$. Then $(\mathcal{E}/\!\sim, \operatorname{Dom}\mathcal{E})$ is a Hilbert space.
- For any finite subset $V \subset V_*$ and for any $v \in \ell(V)$ there exists $u \in \operatorname{Dom}\mathcal{E}$ such that $u\big|_V = v$.
- For any $p, q \in V_*$ there exists the effective resistance between metric

$$R(p,q) = \sup\left\{\frac{\big(u(p)-u(q)\big)^2}{\mathcal{E}(u,u)} : u \in \operatorname{Dom}\mathcal{E}\right\} < \infty$$

  Hence any $u \in \operatorname{Dom}\mathcal{E}$ has a unique $R$-Hölder continuous extension to $\Omega$, the $R$-completion of $V_*$.
- Markov property: for any $u \in \operatorname{Dom}\mathcal{E}$ we have that $\mathcal{E}(\bar{u},\bar{u}) \leqslant \mathcal{E}(u,u)$, where

$$\bar{u}(p) = \begin{cases} 1 & \text{if } u(p) \geqslant 1, \\ u(p) & \text{if } 0 < u(p) < 1, \\ 0 & \text{if } u(p) \leqslant 1. \end{cases}$$

For any finite subset $U \subset V_*$ the finite dimensional Dirichlet form $\mathcal{E}_U$ on $U$ is

$$\mathcal{E}_U(f,f) = \inf\{\mathcal{E}(g,g) : g \in \mathrm{Dom}\,\mathcal{E}, g\big|_U = f\}$$

and is called the **trace** of $\mathcal{E}$ on $U$.

If $U_1 \subset U_2$ then $\mathcal{E}_{U_1}$ is the trace of $\mathcal{E}_{U_2}$ on $U_1$.

**Theorem (Kigami).** Suppose that $V_n$ are finite subsets of $V_*$ and that $\bigcup_{n=0}^{\infty} V_n$ is $R$-dense in $V_*$. Then

$$\mathcal{E}(f,f) = \lim_{n\to\infty} \mathcal{E}_{V_n}(f,f)$$

for any $f \in \mathrm{Dom}\,\mathcal{E}$, where the limit is non-decreasing.

**Theorem (Kigami).** Suppose that $V_n$ are finite sets, and the finite dimensional resistance forms $\mathcal{E}_{V_n}$ on $V_n$ are compatible: each $\mathcal{E}_{V_n}$ is the trace of $\mathcal{E}_{V_{n+1}}$ on $V_n$.

Then there exists a resistance form $\mathcal{E}$ on $V_* = \bigcup_{n=0}^{\infty} V_n$ such that

$$\mathcal{E}(f,f) = \lim_{n\to\infty} \mathcal{E}_{V_n}(f,f)$$

for any $f \in \mathrm{Dom}\,\mathcal{E}$, and the limit is non-decreasing.

**Definition.** A ***finitely ramified fractal*** $F$ is a compact metric space with a ***cell structure*** $\mathcal{F} = \{F_\alpha\}_{\alpha\in\mathcal{A}}$ and a ***boundary (vertex) structure*** $\mathcal{V} = \{V_\alpha\}_{\alpha\in\mathcal{A}}$ such that the following conditions hold.

- $\mathcal{A}$ is a countable index set;
- each $F_\alpha$ is a distinct compact connected subsets of $F$;
- each $V_\alpha$ is a finite subset of $F_\alpha$ with at least two elements;
- if $F_\alpha = \bigcup_{j=1}^k F_{\alpha_j}$ then $V_\alpha \subset \bigcup_{j=1}^k V_{\alpha_j}$;
- there exists a filtration $\{\mathcal{A}_n\}_{n=0}^\infty$ such that
  (1) $\mathcal{A}_n$ are finite subsets of $\mathcal{A}$, $\mathcal{A}_0 = \{0\}$, $F_0 = F$;
  (2) $\mathcal{A}_n \cap \mathcal{A}_m = \varnothing$ if $n \neq m$;
  (3) for any $\alpha\in\mathcal{A}_n$ there are $\alpha_1, ..., \alpha_k\in\mathcal{A}_{n+1}$ such that $F_\alpha=\bigcup_{j=1}^k F_{\alpha_j}$;
- $F_{\alpha'} \bigcap F_\alpha = V_{\alpha'} \bigcap V_\alpha$ for any two distinct $\alpha, \alpha' \in \mathcal{A}_n$;
- for any strictly decreasing infinite sequence of cells there exists $x \in F$ such that $\bigcap_{n\geqslant 1} F_{\alpha_n} = \{x\}$.

If these conditions are satisfied, then

$$(F, \mathcal{F}, \mathcal{V}) = (F, \{F_\alpha\}_{\alpha\in\mathcal{A}}, \{V_\alpha\}_{\alpha\in\mathcal{A}})$$

is called a ***finitely ramified cell structure.***

**Definition.** A function is harmonic if it minimizes the energy for the given set of boundary values. A function is $n$-harmonic if it minimizes the energy for the given set of values on $V_n$.

**Theorem.** Suppose that all $n$-harmonic functions are continuous. Then any continuous function is $R$-continuous, and any $R$-Cauchy sequence converges in the topology of $F$. Also, there is a continuous injective map $\theta : \Omega \longrightarrow F$ which is the identity on $V_*$.

Then we can (and will) consider $\Omega$ as a subset of $F$. Then $\Omega$ is the $R$-closure of $V_*$. In a sense, $\Omega$ is the set where the Dirichlet form $\mathcal{E}$ "lives".

**Theorem.** Suppose that all $n$-harmonic functions are continuous. Then $\mathcal{E}$ is a local regular Dirichlet form on $\Omega$ (with respect to any measure that charges every nonempty open set).

**Definition.** We fix a complete, up to constant functions, energy orthonormal set of harmonic functions $h_1, ..., h_k$, where $k = |V_0| - 1$, and define the Kusuoka energy measure by

$$\nu = \nu_{h_1} + ... + \nu_{h_k}.$$

If $F_{\alpha'} \subset F_\alpha$, then

$$M_{\alpha,\alpha'} : \ell(V_\alpha) \to \ell(V_{\alpha'})$$

is the linear map which is defined as follows. If $f_\alpha$ is a function on $V_\alpha$ then let $h_{f_\alpha}$ be the unique harmonic function on $F_\alpha$ that coincides with $f_\alpha$ on $V_\alpha$. Then we define

$$M_{\alpha,\alpha'} f_\alpha = h_{f_\alpha}\big|_{V_{\alpha'}}.$$

**Proposition.** If $F_\alpha = \bigcup F_{\alpha_j}$ then $D_\alpha = \sum M^*_{\alpha,\alpha_j} D_{\alpha_j} M_{\alpha,\alpha_j}$ and

$$\nu(F_\alpha) = \operatorname{Tr} M^*_\alpha D_\alpha M_\alpha$$

where $M_\alpha = M_{0,\alpha}$ and $D_\alpha$ is the matrix of the Dirichlet form $\mathcal{E}_\alpha$ on $V_\alpha$.

We denote $Z_\alpha = \dfrac{M_\alpha^* D_\alpha M_\alpha}{\nu(F_\alpha)}$ if $\nu(F_\alpha) \neq 0$. Then we define matrix valued functions $Z_n(x) = Z_\alpha$ if $\nu(F_\alpha) \neq 0$, $\alpha \in \mathcal{A}_n$ and $x \in F_\alpha \backslash V_\alpha$. Note that $\operatorname{Tr} Z_n(x) = 1$ by definition.

**Theorem.** For $\nu$-almost all $x$ there is a limit $Z(x) = \lim_{n\to\infty} Z_n(x)$.

*Proof.* One can see, following Kusuoka's idea, that $Z_n$ is a bounded $\nu$-martingale. □

The energy measures $\nu_h$ are the same as the energy measures in the general theory of Dirichlet forms. The matrix $Z$ is the matrix whose entries are the densities

$$Z_{ij} = \frac{d\nu_{h_i,h_j}}{d\nu}$$

It has been recently proved by Hino that $\nu$ is singular with respect to any product measure $\mu$ for a large class of fractals.

**Theorem.** If the space of piecewise harmonic functions is dense in $\mathrm{Dom}\,\mathcal{E}$ then any $f \in \mathrm{Dom}\,\mathcal{E}$ has a weak gradient $\nabla f$ such that

$$\mathcal{E}(f,f) = \int_F \langle \nabla f, Z \nabla f \rangle d\nu$$

**Conjecture.** For any finitely ramified fractal

$$\mathrm{rank} Z(x) = 1$$

for $\nu$-almost all $x$.

This has been recently proved by Hino for a large class of p.c.f. fractals.

# GRADIENT IN HARMONIC COORDINATED

Let $V_0 = \{v_1, ..., v_m\}$ and let $h_j$ be the unique harmonic function with boundary values $h_j(v_i) = \delta_{i,j}$.

Kigami's harmonic coordinate map $\psi : F \longrightarrow \mathbb{R}^m$ is

$$\psi(x) = (h_1(x), ..., h_m(x)).$$

*In what follows we assume that* $\psi : F \longrightarrow F_H = \psi(F)$ *is a homeomorphism,* $F = F_H$, $\psi(x) = x$ *and identify* $\ell(V_0)$ *with* $\mathbb{R}^m$ *in the natural way.*

**Theorem.** If $f$ is the restriction to $F$ of a $C^1(\mathbb{R}^m)$ function then $f \in \mathbf{Dom}\,\mathcal{E}$, and such functions are dense in $\mathbf{Dom}\,\mathcal{E}$. Moreover,

$$\mathcal{E}(f, f) = \int_F \langle \nabla f, Z \nabla f \rangle d\nu$$

for any $f \in C^1(\mathbb{R}^m)$.

We have the analog of the Gauss-Green formula:

$$\mathcal{E}(f,g) = -\int_F g\Delta_\nu f d\nu,$$

for any function $g \in \mathrm{Dom}\,\mathcal{E}$, vanishing on the boundary $V_0$, and any function $f \in \mathrm{Dom}\,\Delta_\nu$, where $\Delta_\nu$ is the **energy Laplacian**.

**Theorem.** If $f$ is the restriction to $F$ of a $C^2(\mathbb{R}^m)$ function then $f \in \mathrm{Dom}\,\Delta_\nu$, and such functions are dense in $\mathrm{Dom}\,\Delta_\nu$. Moreover, $\nu$-almost everywhere

$$\Delta_\nu f = \mathrm{Tr}\,(ZD^2 f)$$

where $D^2 f$ is the matrix of the second derivatives of $f$.

**Conjecture.** On the Sierpiński gasket, if $f \in \mathrm{Dom}\,\Delta_\nu$ then $f$ is the restriction to $F$ of a $C^1(\mathbb{R}^m)$ function.

We also can define a different sequence of approximating energy forms. In various situations these forms are associated with so called ***quantum graphs, photonic crystals and cable systems.*** If $f \in C^1(\mathbb{R}^m)$ then

$$\mathcal{E}_n^Q(f,g) = \sum_{x,y \in V_n} c_{n,x,y} \mathcal{E}_{x,y}^Q(f,f)$$

where

$$\mathcal{E}_{x,y}^Q(f,f) = \int_0^1 \left(\tfrac{d}{dt} f\big(x(1-t)+ty\big)\right)^2 dt$$

is the integral of the square of the derivative

$$\tfrac{d}{dt} f\big(x(1-t)+ty\big) = \langle \nabla f\big(x(1-t)+ty\big), y-x\rangle$$

of $f$ along the straight line segment connecting $x$ and $y$. Thus $\mathcal{E}_{x,y}^Q(f,f)$ is the usual one dimensional energy of a function on a straight line segment.

If $f$ is linear then $\mathcal{E}_{x,y}^Q(f,f) = \big(f(x) - f(y)\big)^2$. Therefore if $f$ is piecewise harmonic then $\mathcal{E}_n^Q(f,f) = \mathcal{E}_n(f,f)$ for all large enough $n$.

Therefore for any $C^1(R^m)$-function we have

$$\lim_{n\to\infty} \mathcal{E}_n^Q(f,f) = \mathcal{E}(f,f)$$

It is easy to see that if $g$ is a $C^1(R^m)$-function vanishing on $V_0$ and $f$ is a $C^2(R^m)$-function then

$$\mathcal{E}_n^Q(f,g) = \sum_{x,y\in V_n} c_{n,x,y} \int_0^1 g\big(x(1-t)+ty\big)\Big(\tfrac{d^2}{dt^2} f\big(x(1-t)+ty\big)\Big)dt$$

because after integration by parts all the boundary terms are canceled. Then if $\alpha \in \mathcal{A}_n$ then

$$\sum_{x,y\in V_\alpha} c_{n,x,y} \tfrac{d^2}{dt^2} f\big(x(1-t)+ty\big) =$$

$$\sum_{x,y\in V_\alpha} c_{n,x,y} \sum_{i,j=1}^{m} D_{ij}^2 f\big(x(1-t)+ty\big)(y_i-x_i)(y_j-x_j) =$$

$$\mathrm{Tr}\Big(M_\alpha^* D_\alpha M_\alpha\big(D^2 f(x_\alpha) + R_n(x,y,t,f,\alpha,x_\alpha)\big)\Big)$$

where $x_\alpha \in V_\alpha$ and

$$\lim_{n\to\infty} |R_n(x,y,t,f,\alpha,x_\alpha)| = 0$$

uniformly.

Let $\mathcal{H}_x$ be the space of harmonic functions on $F$ that vanishes at $x$.

**Definition.** If $h \in \mathcal{H}_x$ then the ***intrinsic derivative*** $\frac{df}{dh}(x) \in \mathbb{R}$ exists if

$$f(y) = f(x) + h(y)\frac{df}{dh}(x) + o\big|h(y)\big|_{y\to x}.$$

The ***intrinsic gradient*** $\mathrm{Grad}_x f \in \mathcal{H}_x$ exists if for any non constant $h \in \mathcal{H}_x$

$$f(y) = f(x) + \mathrm{Grad}_x(y) + o\big|h(y)\big|_{y\to x}.$$

**Theorem (Pelander, T).** Let $\mu$ be a self-similar measure on a p.c.f. s-s set with weights $\mu_j$. Let $\gamma^+$ and $\gamma^-$ be the upper and lower Lyapunov exponents of the matrices $M_j$ with respect to the measure $\mu$ and $\log\gamma = \sum_{j=1}^m \mu_j \log(r_j\mu_j)$.

If $\gamma^+ > \gamma$ then $\frac{df}{dh}(x)$ exists for any $f \in \mathbf{Dom}\,\mathbf{\Delta}_\mu$, any non constant $h \in \mathcal{H}$ and $\mu$-almost all $x$.

If $\gamma^- > \gamma$ then $\mathrm{Grad}_x f$ exists for any $f \in \mathbf{Dom}\,\mathbf{\Delta}_\mu$, any non constant $h \in \mathcal{H}$ and $\mu$-almost all $x$.

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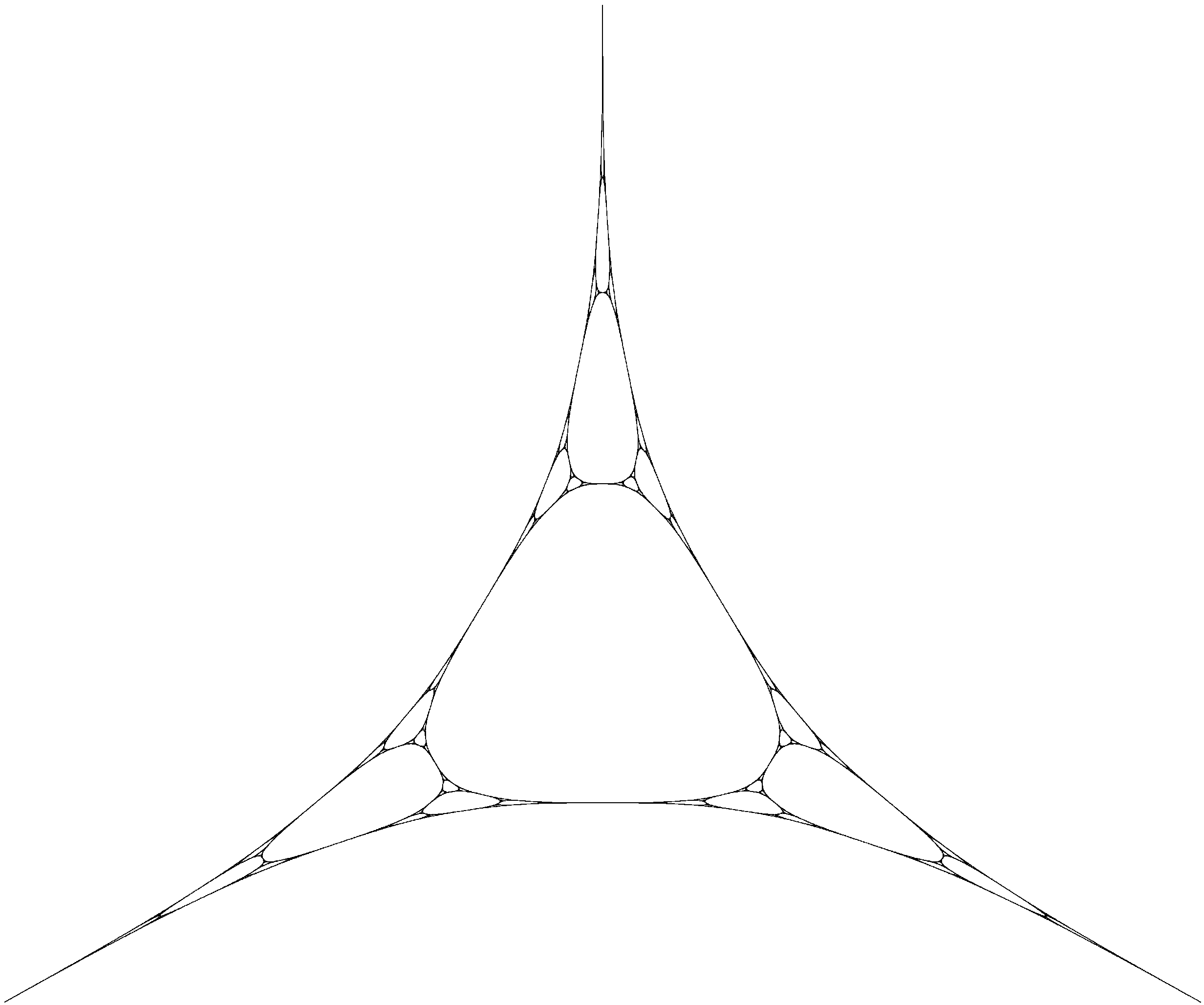

**Sierpiński gasket**
**in harmonic coordinates**