Fractal interpolation functions on p.c.f. self-similar sets

Huo-Jun Ruan
Zhejiang University

Partially based on the jointed work with S.-G. Ri
Cornell, September 10-13, 2011
Motivation

- Harmonic functions and piecewise harmonic splines have finite energy, but beyond that it is not easy to verify that explicit functions have finite energy.
- For example, nonconstant linear functions on Sierpinski gasket (SG for short) have infinity energy.
- We introduce a class of fractal interpolation functions (FIFs for short) on p.c.f. fractals, extending the definition of Bransley in 1986 and Çelik, Koçak and Özdemir in 2008.
- We give a sufficient condition for linear FIFs to have finite energy. This shows that the class of FIFs provides a large collections of explicit functions with finite energy.
- We also study other properties of these FIFs:
  - normal derivative
  - Laplacian
  - Min-max property
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Classical fractal interpolation functions

Let \( \{(x_j, y_j)\}_{j=0}^{N} \subset \mathbb{R}^2 \) be a given data set, where \( x_0 < x_1 < \cdots < x_N \). We can construct a fractal function \( f \) satisfying \( f(x_j) = y_j, \forall j \) as follows.

- Let \( L_j(x) \) be a contractive homeomorphism satisfying
  \[
  L_j(x_0) = x_{j-1}, \quad L_j(x_N) = x_j, \quad j = 1, 2, \ldots, N.
  \]

- Denote \( K = [x_0, x_N] \times \mathbb{R} \).

- For \( j = 1, 2, \ldots, N \), define a map \( \Psi_j : K \to \mathbb{R} \) be continuous with, for a constant \( 0 < \alpha_j < 1 \),
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  \Psi_j(x_0, y_0) = y_{j-1}, \quad \Psi_j(x_N, y_N) = y_j,
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\( \{K, W_j : j = 1, 2, \ldots, N\} \) is an IFS.
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Classical fractal interpolation functions

- Clearly,

\[(x_0, y_0) \xrightarrow{W_j} (x_{j-1}, y_{j-1}), \quad (x_N, y_N) \xrightarrow{W_j} (x_j, y_j),\]

**Theorem (Barnsley’1986)**

The IFS \(\{K, W_j : j = 1, 2, \ldots, N\}\) defined above has a unique attractor \(G\), i.e., \(G = \bigcup_{j=1}^{N} W_j(G)\). Furthermore, \(G\) is the graph of a continuous function \(f : [x_0, x_N] \to \mathbb{R}\) which obeys

\[f(x_j) = y_j, \quad j = 0, 1, 2, \ldots, N.\]

We call such a function a fractal interpolation function or FIF for short.

Notice that \((x, f(x)) \xrightarrow{W_j} (L_j(x), \Psi_j(x, f(x))).\) We have

\[f(L_j(x)) = \Psi_j(x, f(x)), \quad \text{for all } x \in [x_0, x_N], \ j = 1, 2, \ldots, N.\]
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A typical case is that we choose $L_j$ and $\Psi_j$ to be all linear:
$L_j(x) = a_jx + e_j, \Psi_j(x, y) = c_jx + d_jy + f_j$. Then

$$W_j \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_j & 0 \\ c_j & d_j \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_j \\ f_j \end{pmatrix}, \quad j = 1, 2, \ldots, N.$$

- We call such an FIF to be an **affine FIF**. In this case,
  $$f(L_j(x)) = d_jf(x) + c_jx + f_j.$$
- $a_j$ and $e_j$ are determined by $\{x_j\}_{j=0}^N$.
- Given $\{x_j, y_j\}_{j=0}^N$. $\{d_j\}$ can be freely chosen in $(-1, 1)$. And all $c_j$ and $f_j$ are determined by $\{x_j, y_j\}_{j=0}^N$ and $\{d_j\}_{j=1}^N$.
- $d_j$ are called **vertical scaling factors**.
A typical case is that we choose \( L_j \) and \( \Psi_j \) to be all linear: 
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Classical fractal interpolation functions

Figure: vertical scaling factor $d_j$
Example

\[ W_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \]

\[ W_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix}, \]

\[ W_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 \\ -1 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix}. \]
Figure: An FIF with $d_1 = d_2 = d_3 = \frac{1}{3}$. 
Another typical case:

\[ L_j(x) = a_j x + e_j, \quad \Psi_j(x, y) = d_j y + \varphi_j(x), \quad \forall j. \]

\( d_j \) is also called a vertical scaling factor. In this case,

\[ f(L_j(x)) = d_j f(x) + \varphi_j(x). \]
Classical fractal interpolation functions

Works on classical FIFs:

- Box dimension and Hausdorff dimension;
- Using affine FIFs to fit discrete data;
- Hölder continuity;
- Calculus and fractional calculus;
- The relationship between the range of affine FIF and vertical scaling factors,
- Higher dimensional case: fractal interpolation surfaces.
- ...
We assume that $X$ is a p.c.f. self-similar set with boundary $V_0 = \{q_1, \ldots, q_{N_0}\}$, where

- $X$: self-similar set determined by IFS $\{\mathbb{R}^d; F_1, \ldots, F_N\}$.
- $F_j(q_j) = q_j$ for all $j = 1, \ldots, N_0$ and $N_0 \leq N$.

We denote:

- $\Sigma = \{1, 2, \ldots, N\}$.
- $\Sigma^* = \bigcup_{m=1}^{\infty} \Sigma^m$. 
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Huo-Jun Ruan | FIFs on p.c.f. self-similar sets
\[ \forall m \in \mathbb{Z}^+, V_m = \bigcup_{\omega \in \Sigma^m} F_\omega(V_0). \]
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P.C.F. self-similar sets

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Given a function $B : V_1 \to \mathbb{R}$, we will construct a fractal function $f$ on $X$ such that $f|_{V_1} = B$. We call $B$ the basic function of $f$.

- For $j \in \Sigma$, let $\psi_j : X \times \mathbb{R} \to \mathbb{R}$ be continuous such that, for some constant $\alpha_j < 1$,
  $$\psi_j(q_k, B(q_k)) = B(q_{jk}) \quad \text{for all } q_k \in V_0,$$
  $$|\psi_j(x, z') - \psi_j(x, z'')| \leq \alpha_j |z' - z''|$$
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- Define $W_j : X \times \mathbb{R} \to X \times \mathbb{R}$ for $j = 1, 2, \ldots, N$ by
  $$W_j(x, z) = (F_j(x), \psi_j(x, z)).$$

- $\{X \times \mathbb{R}, W_j : j = 1, 2, \ldots, N\}$ is an IFS.
Given a function $B : V_1 \to \mathbb{R}$, we will construct a fractal function $f$ on $X$ such that $f|_{V_1} = B$. We call $B$ the basic function of $f$.

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Similarly as the classical case, we have the following theorem.

**Theorem (R’2010)**

The IFS \( \{ X \times \mathbb{R}, W_j : j = 1, 2, \ldots, N \} \) defined above has a unique attractor \( G \), i.e., \( G = \bigcup_{j=1}^{N} W_j(G) \). Furthermore, \( G \) is the graph of a continuous function \( f : X \to \mathbb{R} \) which obeys

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f|_{V_1} = B.
\]

We call such a function \( f \) a fractal interpolation function, or FIF for short, on \( X \).

In case that for any \( j \),

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The figure displays the FIF on SG where

\[ B(q_1) = 0, \quad B(q_2) = 1, \quad B(q_3) = 0, \]
\[ B(q_{12}) = 1, \quad B(q_{13}) = 0.8, \quad B(q_{23}) = 0.5. \]

and \( \psi_j(x, z) = d_j z + \varphi_j(x) \) where \( d_1 = 0.3, \ d_2 = 0.2, \ d_3 = 0.3 \)

and \( \varphi_j \) is a harmonic function for any \( j \).
The following theorem presents a sufficient condition such that a linear FIF has finite energy.

**Theorem (R’2010)**

Let $f$ be the linear FIF determined by $\{(F_j(x), \Psi_j(x, z))\}_{j=1}^N$, where $\Psi_j(x, z) = d_j z + \varphi_j(x)$, $\forall j$. Let $\{r_1, \ldots, r_N\}$ be the renormalization factors in energy form, i.e.

$$\mathcal{E}(u) = \sum_{j=1}^N r_j^{-1} \mathcal{E}(u \circ F_j),$$

for all $u : X \to \mathbb{R}$. Then $f \in \text{dom} \mathcal{E}$ if $\sum_{j=1}^N r_j^{-1} d_j^2 < 1/2$ and $\varphi_j \in \text{dom} \mathcal{E}$ for all $j \in \Sigma$.

Let $f$ be the linear FIF defined in the above example.
Let $\mathcal{E}$ be the standard energy on SG, then $f \in \text{dom} \mathcal{E}$. 
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Now, we will focus on linear FIFs on SG.

**Theorem**

Let \( B : V_1 \to \mathbb{R} \) be a given function. For any given numbers \( d_j \in (-1, 1), \ j = 1, 2, 3 \), there exists a unique continuous function \( f : SG \to \mathbb{R} \), such that \( f|_{V_1} = B \) and

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f(F_j(x)) = d_j f(x) + h_j(x)
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for \( x \in SG \), where \( h_j \) are all harmonic functions on SG.

- Denote by \( B_0 \) the function on \( V_1 \) satisfying \( B_0|_{V_0} = 0 \) and \( B_0|_{V_1 \setminus V_0} = 1 \).
- We will focus on FIFs with basic function \( B_0 \) and with same vertical scaling factors \( d \in (-1, 1) \), i.e., \( d_i = d \) for all \( i \).
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Theorem (Ri&R’2011)

Let \( f \) be the uniform FIF with vertical scaling factor \( d \). Then

- \( \mathcal{E}(f) < \infty \) if and only if \( |d| < \frac{1}{\sqrt{5}} \). Furthermore, in case that \( |d| < \frac{1}{\sqrt{5}} \), we have \( \mathcal{E}(f) = \frac{10}{1-5d^2} \).
- \( \partial_n f(q_j) \) exists for some (then for all) \( j = 1, 2, 3 \) if and only if \( |d| < \frac{3}{5} \). Furthermore, in case that \( |d| < \frac{3}{5} \), we have
  
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Let \( f \) be the uniform FIF with vertical scaling factor \( d \). Then \( 0 \leq f(x) \leq 1 \) for all \( x \in SG \) if and only if \( d \in \left[ -\frac{3}{5}, \frac{1}{5} \right] \).
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**Theorem (Ri&R’2011)**

Let $f$ be the uniform FIF with vertical scaling factor $d$. Then, for any $x \in V_* \setminus V_1$, we have the following property:

- $\Delta f(x) = 0$ if $d = 0$,
- $\Delta f(x) = -15$ if $d = \frac{1}{5}$, and
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**Corollary (Ri&R’2011)**

Let $f$ be the uniform FIF with vertical scaling factor $d = \frac{1}{5}$. Let $\alpha$ be a given real number. Then $-\frac{\alpha f}{15}$ is the unique solution of the following Dirichlet problem:

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\begin{align*}
\Delta f(x) &= \alpha, \quad \forall x \in SG \setminus V_0, \\
\end{align*}
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Thank you!