Gaussian space-time fields and (S)PDE on fractals

Martina Zähle

(Institute of Mathematics, Friedrich Schiller University Jena)

Spring Eastern Sectional Meeting, Stony Brook New York
Analysis, Probability and Mathematical Physics on Fractals, III
March 19-20, 2016
0. Introduction

References:

M. Hinz, M.Z.: Semigroups, potential spaces and applications to (S)PDE, Potential Anal. 2012,

E. Issoglio, M.Z.: Regularity of the solutions to SPDEs in metric measure spaces, Stoch. PDE: Anal. Comp. 2015,

M.Z.: (S)PDE on fractals and Gaussian noise, Preprint,

(J. Hu, M.Z.: Potential spaces on fractals, Studia Math. 2005,

1. Parabolic (S)PDE in metric measure spaces

Related literature

Elliptic and some parabolic PDE on **self-similar sets** with respect to fractal Laplace operators (without noise terms): Falconer, Grigoryan, Hu, Khoshnevisan, Strichartz, ...

Abstract parabolic problems with Brownian and fractional Brownian noise: Dalang, DaPrato/Zabzcyk, Gubinelli/Lejay/Tindel, Maslowski/Nualart, Prévot/Röckner, Tindel/Tudor/Viens, ...

Abstract porous media equations (with additive Brownian noise) Hinz/Röckner/Teplyaev

Euclidean Burger type equations with **white noise in time** Hairer/Voss, Hairer/Weber, Hairer/Maas/Weber, Röckner/Zhu/Zhu

formal Cauchy problem on bounded smooth domains in $\mathbb{R}^n$

$$\frac{\partial u}{\partial t}(t, x) = -Au(t, x) + F(u(t, x)) + \langle G(u(t, x)), \frac{\partial}{\partial t} \nabla Z(t, x) \rangle, \ t \in (0, t_0],$$

A elliptic operator, $\frac{\partial}{\partial t} \nabla Z$ fractional space-time noise, initial value and Neumann boundary conditions, pathwise approach
Extension to non-linear spaces

$(X, d)$ locally compact separable metric space with Radon measure $\mu$

Consider the formal Cauchy problem on $[0, t_0] \times X$

\[
\frac{\partial u}{\partial t} = -A^\theta u + F(u) + G(u) \cdot \dot{z}, \quad t \in (0, t_0],
\]

with initial condition $u(0, x) = f(x)$, where

- $-A$ is the generator of a strongly continuous Markovian symmetric semigroup $(T_t)_{t \geq 0}$ on $L_2(\mu)$ satisfying natural heat kernel estimates HKE($\beta$) (e.g. the Neumann Laplacian in the above models)
- $A^\theta$, $\theta \leq 1$, is a fractional power of $A$
- $F$ and $G$ are sufficiently regular functions on $\mathbb{R}$
- $z$ is a random element in $C^{1-\alpha}([0, t_0], H^{\theta \beta}_p(\mu)^*)$ for some $\alpha < 1/2$
- $f \in H^{2\gamma+\theta \max(\delta S/4, \delta)+\varepsilon}_2,\infty(\mu)$, where $H^{\sigma}_2,\infty(\mu) := H^{\sigma}_2(\mu) \cap L_{\infty}(\mu)$

Aim: pathwise mild function solution $u \in C^\gamma([0, t_0], H^{\theta \delta}_2(\mu))$
HKE(\(\beta\)):

\[
t^{-\frac{dH}{w}} \Phi_1(t^{-\frac{1}{w}} d(x,y)) \leq p(t, x, y) \leq t^{-\frac{dH}{w}} \Phi_2(t^{-\frac{1}{w}} d(x,y))
\]

\[
\int_0^\infty s^{dH+\beta w/2-1} \Phi_2(s) ds < \infty
\]

(Barlow/Bass, Hambly, Fitzsimmons/Hambly/Kumagai, Hambly/Kumagai, Grigor’yan/Kumagai, Grigor’yan/Hu/Lau, Kigami, Stricharts, … for any \(\beta > 0\))

**Fractional Sobolev (or Bessel potential) spaces:**

\[
H_\sigma^p(\mu) := (A+I)^{-\sigma/2}(L_p(\mu))
\]

(agrees with the Fourier analytic approach in the Euclidean case)
for $\theta = 1$ mild solution defined by

$$u(t, x) = T_t f(x) + \int_0^t T_{t-s} F(u(s, \cdot))(x) \, ds$$

$$+ \int_0^t T_{t-s} (G(u(s, \cdot)) \cdot \dot{z}(s))(x) \, ds$$

the last formal integral to be determined,

(for $\theta < 1$ use the subordinated semigroup $T_t^\theta$ with generator $-A^\theta$ instead of $T_t$)

Main ideas:

- the smoothness is measured in terms of fractional Sobolev (or Bessel potential) spaces $H^\sigma_2(\mu)$ generated by the semigroup, and the latter lifts certain dual spaces to function spaces,

- the product of functions and ”distributions” is introduced by means of duality relations, the Dirichlet form, and the heat kernel estimates

- the time integral is realized via Banach space valued fractional calculus

(our approach is independent of series expansions)
for $\theta = 1$ mild solution defined by

$$u(t, x) = T_tf(x) + \int_0^t T_{t-s}F(u(s, \cdot))(x) \, ds$$

$$+ \int_0^t T_{t-s} (G(u(s, \cdot)) \cdot \dot{z}(s))(x) \, ds$$

the last formal integral to be determined,
(for $\theta < 1$ use the subordinated semigroup $T_t^\theta$ with generator $-A^\theta$ instead of $T_t$)

**Main ideas:**

- the smoothness is measured in terms of fractional Sobolev (or Bessel potential) spaces $H_2^\sigma(\mu)$ generated by the semigroup, and the latter lifts certain dual spaces to function spaces,
- the product of functions and "distributions" is introduced by means of duality relations, the Dirichlet form, and the heat kernel estimates
- the time integral is realized via *Banach space valued fractional calculus*

(our approach is independent of series expansions)
Rigorous definition of the integral and solution of the stochastic partial (pseudo) differential equation

Rewrite the above formal integral as

\[
\int_0^t T_{t-s} (G(u(s, \cdot)) \cdot \dot{z}(s)) (x) \, ds = \int_0^t \Phi_t(s) (\dot{z}(s)) (x) \, ds
\]

where \( \Phi_t(s)(w) := T_{t-s} (G(u(s, \cdot)) \cdot w) \) is for \( u \in H^\delta_{2, \infty}(\mu) \) shown to be an operator valued mapping

\[
\Phi_t : [0, t_0] \to L \left( H^\rho_{2, \infty}(\mu)^*, H^\delta_{2, \infty}(\mu) \right)
\]

(for certain \( \rho \)) with fractional order of smoothness \( \alpha' \) slightly larger than \( \alpha \). By assumption \( z(s) \) has fractional order of smoothness \( 1 - \alpha' \), so that we can define

\[
\int_0^t \Phi_t(s) (\dot{z}(s)) \, ds := \int_0^t D^{\alpha'}_{0+} \Phi_t(s) \left( D^{1-\alpha'}_{t-} z_t(s) \right) \, ds
\]

for left and right sided fractional derivatives \( D^{\alpha'}_{0+} \) and \( D^{1-\alpha'}_{t-} \) (and \( z_t := z - z(t) \)).
Rigorous definition of the integral and solution of the stochastic partial (pseudo) differential equation

Rewrite the above formal integral as

\[ \int_0^t T_{t-s} (G(u(s, \cdot)) \cdot \dot{z}(s)) (x) \, ds = \int_0^t \Phi_t(s) (\dot{z}(s)) (x) \, ds \]

where \( \Phi_t(s)(w) := T_{t-s} (G(u(s, \cdot)) \cdot w) \) is for \( u \in H_{2,\infty}^\delta(\mu) \) shown to be an operator valued mapping

\[ \Phi_t : [0, t_0] \rightarrow L \left( H_2^\rho(\mu)^*, H_{2,\infty}^\delta(\mu) \right) \]

(for certain \( \rho \)) with fractional order of smoothness \( \alpha' \) slightly larger than \( \alpha \). By assumption \( z(s) \) has fractional order of smoothness \( 1 - \alpha' \), so that we can define

\[ \int_0^t \Phi_t(s) (\dot{z}(s)) \, ds := \int_0^t D_{0+}^{\alpha'} \Phi_t(s) \left( D_{t-}^{1-\alpha'} z_t(s) \right) \, ds \]

for left and right sided fractional derivatives \( D_{0+}^{\alpha'} \) and \( D_{t-}^{1-\alpha'} \) (and \( z_t := z - z(t) \)).
Main result (for $\theta = 1$)

Define

$$||u||_{W^\gamma([0,t_0],H)} := \sup_{0 \leq t \leq t_0} \left( ||u(t)||_H + \int_0^t \frac{||u(t) - u(s)||_H}{(t-s)^{\gamma+1}} \, ds \right).$$

Let $0 < \alpha < \gamma$, $0 < \beta < \delta < \min(d_S/2, 1)$, $\gamma < 1 - \alpha - \beta/2 - d_S/4$, $p := d_S/(d_S - \delta)$,

$$z \in C^{1-\alpha} \left( [0, t_0], H^\beta_p(\mu)^* \right).$$

Then under HKE($\beta$) problem (1) has a unique mild solution

$$u \in W^\gamma \left( [0, t_0], H^{\theta\delta}_2, \infty (\mu) \right).$$

We proved a contraction principle in this space. Moreover,

$$u \in C^{\gamma} \left( [0, t_0], H^{\theta\delta}_2 (\mu) \right).$$

If the noise coefficient function $G$ is linear, the $L_\infty$-norms can be omitted. This leads to solutions for all spectral dimensions of the semigroup. (Above: $d_S < 4$)
Main result (for $\theta = 1$)

Define

$$\|u\|_{W^{\gamma}([0,t_0],H)} := \sup_{0 \leq t \leq t_0} \left( \|u(t)\|_H + \int_0^t \frac{\|u(t) - u(s)\|_H}{(t-s)^{\gamma+1}} \ ds \right).$$

Let $0 < \alpha < \gamma$, $0 < \beta < \delta < \min(d_S/2, 1)$, $\gamma < 1 - \alpha - \beta/2 - d_S/4$, $p := d_S/(d_S - \delta)$,

$$z \in C^{1-\alpha} ([0, t_0], H_p^\beta (\mu)^*) .$$

Then under HKE($\beta$) problem (1) has a unique mild solution

$$u \in W^{\gamma} ([0, t_0], H_{2,\infty}^{\theta\delta} (\mu)) .$$

We proved a contraction principle in this space. Moreover,

$$u \in C^{\gamma} ([0, t_0], H_2^{\theta\delta} (\mu)) .$$

If the noise coefficient function $G$ is linear, the $L_\infty$-norms can be omitted. This leads to solutions for all spectral dimensions of the semigroup. (Above: $d_S < 4$)
2. Gaussian space-time noise

**Aim:** Examples for the space-time noise $z$ in random version, pathwise approach

**Assumptions:** compact metric measure space $(X, d, \mu)$, semigroup $T_t$ as above and $T_t 1 = 1$, i.e., conservaive

**Auxiliary tool:**

**Lemma** $C^\sigma(X)$ is embedded in $H^\sigma'_{2/w}(\mu)$ for any $q$ and $0 < \sigma' < \sigma$ such that upper HKE($\sigma' 2/w$) for the semigroup $T_t$ are satisfied.

**Corollary**
If $Y$ is a random space-time field with sample paths in $C^{1-\alpha}([0, t_0], C^\sigma(X))$ then

$$z(t) := (A + I)^{(\beta - \sigma' 2/w)/2} Y(t, \cdot)$$

with $\sigma' < \sigma$ satisfies the above conditions on the noise.

**Gaussian case: without semigroup assumptions**
2. Gaussian space-time noise

Aim: Examples for the space-time noise $z$ in random version, pathwise approach

Assumptions: compact metric measure space $(X, d, \mu)$, semigroup $T_t$ as above and $T_t1 = 1$, i.e., conservative

Auxiliary tool:

Lemma $C^\sigma(X)$ is embedded in $H_{q^\prime/2/w}(\mu)$ for any $q$ and $0 < \sigma^\prime < \sigma$ such that upper HKE($\sigma^\prime 2/w$) for the semigroup $T_t$ are satisfied.

Corollary

If $Y$ is a random space-time field with sample paths in $C^{1-\alpha}([0, t_0], C^\sigma(X))$ then

$$z(t) := (A + I)^{(\beta - \sigma^\prime 2/w)/2}Y(t, \cdot)$$

with $\sigma^\prime < \sigma$ satisfies the above conditions on the noise.

Gaussian case: without semigroup assumptions
2. Gaussian space-time noise

**Aim:** Examples for the space-time noise $z$ in random version, pathwise approach

**Assumptions:** compact metric measure space $(X, d, \mu)$, semigroup $T_t$ as above and $T_t1 = 1$, i.e., conservative

**Auxiliary tool:**

Lemma $C^\sigma(X)$ is embedded in $H^{\sigma'2/w}_q(\mu)$ for any $q$ and $0 < \sigma' < \sigma$ such that upper HKE($\sigma'2/w$) for the semigroup $T_t$ are satisfied.

**Corollary**

If $Y$ is a random space-time field with sample paths in $C^{1-\alpha}([0, t_0], C^\sigma(X))$ then

$$z(t) := (A + I)^{(\beta-\sigma'2/w)/2}Y(t, \cdot)$$

with $\sigma' < \sigma$ satisfies the above conditions on the noise.

*Gaussian case: without semigroup assumptions*
Theorem (Extension of a classical result of Kolmogorov and Chentsov for the Euclidean case)

Let \((X, d, \mu)\) be an arbitrary \(D\)-regular compact metric measure space and \(Y\) a Gaussian field on \([0, t_0] \times X\) with mean zero and

(a) \(\mathbb{E}(Y(s, x) - Y(t, x))^2 \leq c|s - t|^{2H}\)

(b) \(\mathbb{E}(Y(t, x) - Y(t, x))^2 \leq c d(x, y)^{2K}\)

(c) \(\mathbb{E}(Y(s, x) - Y(s, y) - (Y(t, x) - Y(t, y)))^2 \leq c|s - t|^{2H} d(x, y)^{2K}\)

for some \(0 < H < 1, K > 0\) and any \(s, t \in [0, t_0], x, y \in X\).

Then \(Y\) has a modification \(\tilde{Y}\) such that a.s. \(\tilde{Y} \in C^\alpha([0, t_0], C^\beta(X))\) for all \(\alpha < H\) and \(\beta < K\).

For fractional Brownian space-time sheets in the Euclidean case we have equalities in (a), (b) and (c).
Standard example for $Y$:

Let $\{e_i\}_{i \in \mathbb{N}}$ be a complete orthonormal system of eigenfunctions of $A$ in $L_2(\mu)$ (if exist) and $0 \leq \lambda_0 < \lambda_1 < \lambda_2 < \ldots$ be the corresponding eigenvalues, suppose that $|e_i(x) - e_i(y)| \leq c\lambda_i^{a/2}d(x, y)^b$, set $a' := \max(a, d_S/2)$. Choose i.i.d. fractional Brownian motions $\{B_i^H(t)\}_{i \in \mathbb{N}}$ with Hurst exponent $1/2 < H < 1$ and choose

$$Y(t, \cdot) := \sum_{i=1}^{\infty} B_i^H(t) q_i e_i \quad \text{with} \quad \sum_{i=1}^{\infty} q_i^2 \lambda_i^b < \infty,$$

for real coefficients $q_i$ and $b := \max(a, d_S/2)$, then there exists a version of $Y$ with a.s. sample paths in $C^{a\sigma}([0, t_0], C^{\tau}(X))$ for any $0 < \sigma < H$ and $0 < \tau < K$.

(the corresponding noise $z$ may be considered as a "distribution" valued fractional Brownian motion)
Special cases

If $d$ is the resistance metric on $X$, then the above condition on the eigenfunctions is always fulfilled (for $a = b = 1$).

Application to fractals

- certain p.c.f. fractals with regular harmonic structure (here $d_S < 2$), authors mentioned above
- generalized Sierpinski carpets (Barlow/Bass)
- certain products of fractals (Strichartz)