Poincaré Duality and Bakry–Émery Gradient Estimates on Dirichlet Spaces

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Bakry–Émery Gradient Estimates

If $\Gamma$ is an appropriate notion of gradient, and $P_t$ is an associated heat kernel, the Bakry–Émery Gradient estimates

$$\sqrt{\Gamma(P_t f)} \leq P_t \sqrt{\Gamma(f)}.$$ 

Can be used to establish

1. Riesz-Transform Bounds
   (Coulhon and Duong et al.)

2. Isoperimetric inequalities
   (e.g. Baudoin–Bonnefont)

3. Wasserstein Control
   (Kuwada Duality)
Generalizations of Curvature

The Bakry–Émery estimate can be thought of as a curvature condition.

In the appropriate settings it is equivalent to

1. Curvature Dimension Inequalities of Bakry-Émery.

**Question** Can we find a situation which supports a Bakry–Émery gradient estimate, but neither of the above?
Setting

- $(X,d)$ is a locally compact Hausdorff space
- $\mu$ Borel regular measure with volume doubling, i.e. there is some constant $C_{vol}$
  \[ C_{vol}\mu(B_{2r}(x)) \leq \mu(B_r(x)) \quad \text{and} \quad \mu(B_1(x)) \geq c_{vol} \]
- $(\mathcal{E}, \text{dom} \mathcal{E})$ is a local regular Dirichlet form with heat semigroup $P_t$.
- Energy Measures $\nu_{f,g}$ such that
  \[ 2 \int \phi \: d\nu_{f,g} = \mathcal{E}(f\phi, g) + \mathcal{E}(g\phi, f) - \mathcal{E}(\phi, fg). \]
- $\mathcal{E}$ admits a Carré du Champ/\(\mu\) is energy dominant
  \[ \mu \ll \nu_{f,g} \text{ for all } f \text{ and define } \Gamma_\mu(f, g) = \frac{d\nu_{f,g}}{d\mu} \]
- Poincaré inequality
  \[ \int_{B_r(x)} \left| f - \bar{f}_{B_r(x)} \right| \: d\mu \leq \nu_f(B_{CPr}(x)) \]
General Results

**Reisz Transform:** $f \mapsto \Gamma_\mu(\Delta^{-1/2} f)$.

**Theorem**

If we have
- Locally compact Hausdorff metric space $(X, d)$.
- Upper and lower volume Doubling measure $\mu$.
- Dirichlet form $(E, \text{dom} E)$ which admits a Carré du Champ.

Which Satisfy
- Poincaré Inequality
- Bakry–Émery inequality

Then the **Riesz Transform** is bounded for $p \geq 1$, i.e.

$$
\left\| \Gamma_\mu(f, f)^{1/2} \right\|_p \leq C_p \left\| \Delta^{1/2} f \right\|_p
$$
We say $f$ is **bounded variation**, and write $f \in BV$, if

$$\lim_{t \to 0} \int \sqrt{P_t f} \, d\mu < \infty$$

and define $\text{Var}(f) = \lim_{t \to 0} \int \sqrt{P_t f} \, d\mu$.

If $1_E \in BV$, we then the **perimeter** is called $\text{Per } E = \text{Var}(1_E)$.

$E$ is called a **Caccioppoli set** if $1_E \in BV$. 
Isoperimetric Inequalities

Theorem (Baudoin-K.)

If we have

- Locally compact Hausdorff metric space \((X, d)\).
- Upper and lower volume Doubling measure \(\mu\).
- Dirichlet form \((\mathcal{E}, \text{dom} \mathcal{E})\).

Which Satisfy

- Poincaré Inequality and Bakry–Émery Inequality

Then Isoperimetric Inequality there exists \(Q\) and \(C_{iso}\) such that

\[ \mu(E)^{1-1/Q} \leq C_{iso} P(E). \]

and Gaussian Isoperimetric Inequality

\[ C\mu(E)\sqrt{\ln \left(1/\mu(E)\right)} \leq \text{Per}(E). \]
Kuwada Duality

Let

\[ W_p(\nu_1, \nu_2) = \inf_{\pi} \left( \int d(x, y)^p \, \pi(dx, dy) \right)^{1/p} \]

be the $p$-Wasserstein Distance between two probability measures on a metric measure space $(X, d)$.

Then there is a dual form of the Bakry–Émery inequality called $p$-Wasserstein control:

\[ W_p(P_t^*\nu_1, P_t^*\nu_2) \leq e^{-kt} W_p(\nu_1, \nu_2). \]

Where

\[ \int f \, dP_t^*\nu = \int P_t f \, d\nu. \]
Theorem (Kuwada)

\[ p\text{-Wasserstein control:} \]

\[ W_p(P_t^*\nu_1, P_t^*)\nu_2 \leq e^{-kt} W_p(\nu_1, \nu_2). \]

is Equivalent to

\[ \sqrt{\Gamma(P_t f)} \leq e^{-kt} (P_t(\Gamma(f))^{p/2})^{1/p} \]
Goals

- To classify the differential forms on one-dimensional Dirichlet spaces, particularly on the Sierpinski Gasket.
- Relate the heat equation on differential forms to that on scalars.
Idea: to deal with these problems by developing a differential geometry for Dirichlet spaces and hence fractals

based on

*Differential forms on the Sierpinski gasket* and other papers by Cipriani–Sauvageot

*Derivations and Dirichlet forms on fractals*
by Ionescu–Rogers–Teplyaev, JFA 2012

*Vector analysis on Dirichlet Spaces*
by Hinz–Röckner–Teplyaev, SPA 2013
Differential forms on Dirichlet spaces

Let $X$ be a locally compact second countable Hausdorff space and $m$ be a Radon measure on $X$ with full support. Let $(\mathcal{E}, \mathcal{F})$ be a regular symmetric Dirichlet form on $L_2(X, m)$.

Write $\mathcal{C} := C_0(X) \cap \mathcal{F}$.

The space $\mathcal{C}$ is a normed space with

$$\|f\|_\mathcal{C} := \mathcal{E}_1(f)^{1/2} + \sup_{x \in X} |f(x)|.$$
We equip the space $\mathcal{C} \otimes \mathcal{C}$ with a bilinear form, determined by

$$\langle a \otimes b, c \otimes d \rangle_\mathcal{H} = \int_X bd \, d\Gamma(a, c).$$

This bilinear form is nonnegative definite, hence it defines a seminorm on $\mathcal{C} \otimes \mathcal{C}$.

$\mathcal{H}$: the Hilbert space obtained by factoring out zero seminorm elements and completing.
Differential forms on Dirichlet spaces

In the classical setting, this norm

$$\int |b|^2 |
\nabla a|^2 \, d\mu$$

Where $\mu$ is Lebesgue measure in the appropriate dimension.

And, any simple tensor $a \otimes b = \sum_{i=1}^{d} x^i \otimes b \frac{\partial a}{\partial x^i}$.

Think of $x^i \otimes 1$ as $dx^i$,
We call $\mathcal{H}$ the \textit{space of differential 1-forms} associated with $(\mathcal{E}, \mathcal{F})$.

The space $\mathcal{H}$ can be made into a $\mathcal{C}$-$\mathcal{C}$-bimodule by setting

\[ a(b \otimes c) := (ab) \otimes c - a \otimes (bc) \quad \text{and} \quad (b \otimes c)d := b \otimes (cd) \]

and extending linearly.

$\mathcal{C}$ acts on both sides by uniformly bounded operators.
we can introduce a derivation operator by defining \( \partial : \mathcal{C} \to \mathcal{H} \) by \( \partial a := a \otimes 1 \).

\[ \| \partial a \|^2 \leq 2 \mathcal{E}(a) \] and the \textbf{Leibniz rule} holds,

\[ \partial(ab) = a \partial b + b \partial a, \quad a, b \in \mathcal{C}. \]
The operator $\partial$ extends to a closed unbounded linear operator from $L_2(X, m)$ into $\mathcal{H}$ with domain $\mathcal{F}$.

Let $\partial^*$ denote its adjoint, such that

$$\langle \partial^* \omega, g \rangle_{L^2} = \langle \omega, \partial g \rangle_{\mathcal{H}}$$

(1)

Let $\mathcal{C}^*$ be the dual space of the normed space $\mathcal{C}$. Then $\partial^*$ defines a bounded linear operator from $\mathcal{H}$ into $\mathcal{C}^*$.

In this talk we shall consider $\partial^* : \mathcal{H} \rightarrow L^2(X)$ by restricting to the domain

$$\text{dom } \partial^* = \{ \eta \in \mathcal{H} \mid \exists f \in L^2(X) \text{ with } \partial^* \eta(\phi) = \langle f, \phi \rangle_{L^2} \}$$
PDE on fractals

We can think of $\partial$ as something like a gradient or an exterior derivative.

And think of $\partial^*$ as div or as the co-differential.

This allows for a lot of new differential equations to but represented on fractals

For instance, we now have a divergence form

$$\partial^* a(\partial u) = 0$$
Magnetic Schrödinger operators

Classically

\[ i \frac{\partial u}{\partial t} = (-i \nabla - A)^2 u + V u \]

becomes

\[ i \frac{\partial u}{\partial t} = (-i \partial - a)^* (-i \partial - a) u + V u \]

Where \( a \in \mathcal{H} \) and \( V \in L_\infty(X, m) \).
A result of *Hinz–Röckner–Teplyaev* shows that (with some technical conditions) there is a “fibrewise” inner product and norm on $\mathcal{H}$. Call the fibres $\mathcal{H}_x$ and the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H},x}$.

Note

$$\langle \partial f, \partial g \rangle_{\mathcal{H},x} = \Gamma_\mu(f, g)(x)$$

almost everywhere.
Definition of Poincaré Duality

Theorem (Baudoin–K.)

In the above situation, choose \( \omega \in \mathcal{H} \) such that \( \| \omega \|_{\mathcal{H},x} = 1 \) \( \mu \)-a.e.
then \( \ast L^2(X, \mu) \rightarrow \mathcal{H} \) defined by

\[
\ast f = \omega \cdot f
\]

is an isometry both globally and fiberwise with inverse

\[
\ast \eta(x) = \langle \omega, \eta \rangle_{\mathcal{H},x}.
\]

In particular \( L^2(X, \mu) \cong \mathcal{H} \) as Hilbert spaces.

Proof Hino index 1 implies that \( \dim \mathcal{H}_x = 1 \) almost everywhere.
Consider

$$\tilde{\Delta} = \partial \partial^*$$

with domain

$$\text{dom} \, \tilde{\Delta} = \{ \omega \in \mathcal{H} \mid \partial^* \omega \in \text{dom} \, \partial \}.$$
Hodge Decomposition

When restricted to topologically 1-dimensional fractals, there is a Hodge decomposition with

\[ \mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1 \]

where

\[ \mathcal{H}^0 = \text{Im} \partial \quad \text{are Exact Forms} \]

and

\[ \mathcal{H}^1 = \text{ker} \partial^* \quad \text{are Harmonic Forms} \]
The co-differential has the following product rule

$$\partial^*(\eta \cdot f) = \langle \partial f, \eta \rangle_{\mathcal{H}_x} + f \partial^* \eta.$$ 

Thus if $\omega \in \mathcal{H}^1$ is harmonic, then the second term on the right disappears and we get.

$$\partial^* \star f = \star \partial f.$$ 

Note: It is not true that

$$\star \partial^* \eta = \partial \star \eta$$
Theorem (Baudoin–K.)

Consider the self-similar energy form $\mathcal{E}$ on $SG$, with respect to a borel measure $\mu$,

1. $\mu = \nu_h$ is the energy measure associated to the harmonic $h$ with boundary $V_0$.
2. $\star$ is the Hodge Star with respect to $\partial h$.
3. $\Delta_0$ is the Dirichlet Laplacian with boundary $V_0$.

Then $\tilde{\Delta}$ restricted to exact forms $\mathcal{H}^0$ is equal to $-\star \Delta_0 \star$ as operators.

If $\Delta_\mu = -\partial^* \partial$ is the generator of $\mathcal{E}$ with respect to $\mu$, this implies that

$$\text{dom } \Delta_\mu = \{ f \in \text{dom } \mathcal{E} \mid \star \partial f = \Gamma(f, h) \in \text{dom}_0 \mathcal{E} \}$$
Energy measures can be extended to elements of $\mathcal{H}$ by

$$\int \phi \, d\nu_\omega := \langle \omega \cdot \phi, \omega \rangle_{\mathcal{H}}.$$

**Theorem (Baudoin–K.)**

Consider the self-similar energy form $\mathcal{E}$ on $SG$, with respect to a borel measure $\mu$,

1. $\mu = \nu_\omega$ is the energy measure associated to the harmonic form $\omega \in \mathcal{H}^1$.
2. $\star$ is the Hodge Star with respect to $\omega$.
3. $\Delta_\omega$ is the generator of $\mathcal{E}$.

Then $\tilde{\Delta}$ restricted to exact forms $\mathcal{H}^0$ is equal to $-\star \Delta \star$ as operators.
Theorem (Baudoin–K.)

In either of the settings of the above theorems, the Bakry–Émery inequality is satisfied.

That is if $\mu$ is either $\nu_h$ for some harmonic function $h$, or $\nu_\omega$ for some harmonic form $\omega$, then

$$\sqrt{\Gamma_\mu(e^{-t\Delta_\mu}f)} \leq e^{-t\Delta_\mu}\sqrt{\Gamma_\mu(f)}.$$
Proof of Bakry–Émery inequality

Idea:

$$e^{t\tilde{\Delta}} \partial = \partial e^{-t\Delta}$$

Then because $\tilde{\Delta} = -\ast \Delta \ast$

$$\ast e^{-t\Delta} \ast \partial = \partial e^{-t\Delta}.$$

Thus

$$|e^{-t\Delta} \ast \partial f(x)| = \|\partial e^{-t\Delta} f\|_{\mathcal{H},x} = \sqrt{\Gamma(e^{-t\Delta} f)(x)}.$$

The Inequality follows from the fact that

$$|e^{-t\Delta} \ast \partial f(x)| \leq e^{-t\Delta} |\ast \partial f| = e^{-t\Delta} \sqrt{\Gamma(f)}.$$
We can build a fractafold by gluing copies of $SG$ together.

**Theorem (Baudoin–K.)**

The fractafold $X$ admits a Poincaré duality, and satisfies the Bakry–Emery inequality.

The inequality also is preserved by taking products.