

# Change of Basis

©Alan H. Stein

The University of Connecticut at Waterbury

stein@math.uconn.edu <http://www.math.uconn.edu/~stein>

## THE CHANGE-OF-COORDINATES MATRIX

Consider a vector space  $V$  of dimension  $n$  with two bases,  $\mathcal{B}_1$  and  $\mathcal{B}_2$ . If we denote the coordinate representation of a vector  $\mathbf{v}$  with respect to a basis  $\mathcal{B}$  by  $[\mathbf{v}]_{\mathcal{B}}$ , the first question we need to resolve is how to find the coordinate representation with respect to one basis if we know the representation with respect to another basis.

The transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  defined by  $T([\mathbf{v}]_{\mathcal{B}_1}) = [\mathbf{v}]_{\mathcal{B}_2}$  is a well-defined (since every vector  $\mathbf{v} \in \mathbb{R}^n$  is the coordinate representation of some element of  $V$  with respect to  $\mathcal{B}_1$ ) and linear, so it has a matrix representation. We denote the matrix for  $T$  by  ${}_{\mathcal{B}_2} \overleftarrow{P}_{\mathcal{B}_1}$ .

The  $j^{\text{th}}$  column of  ${}_{\mathcal{B}_2} \overleftarrow{P}_{\mathcal{B}_1}$  is the image, under  $T$ , of the  $j^{\text{th}}$  basis vector in  $\mathcal{B}_1$ . This image is precisely the coordinate representation, with respect to  $\mathcal{B}_2$ , of the  $j^{\text{th}}$  basis vector in  $\mathcal{B}_1$ .

In other words, the  $j^{\text{th}}$  column of  ${}_{\mathcal{B}_2} \overleftarrow{P}_{\mathcal{B}_1}$  is simply the coordinate representation with respect to  $\mathcal{B}_2$  of the  $j^{\text{th}}$  basis vector in  $\mathcal{B}_1$ .

**Another Perspective.** Since  $[\mathbf{v}]_{\mathcal{B}_2} = {}_{\mathcal{B}_2} \overleftarrow{P}_{\mathcal{B}_1} [\mathbf{v}]_{\mathcal{B}_1}$ , it follows that  $[\mathbf{v}]_{\mathcal{B}_1} = {}_{\mathcal{B}_2} \overleftarrow{P}_{\mathcal{B}_1}^{-1} [\mathbf{v}]_{\mathcal{B}_2}$ . Thus  ${}_{\mathcal{B}_2} \overleftarrow{P}_{\mathcal{B}_1}^{-1}$  is the change-of-coordinates matrix from  $\mathcal{B}_2$  to  $\mathcal{B}_1$ . We now know the columns of that matrix are the coordinate representations of the basis vectors in  $\mathcal{B}_2$  in terms of  $\mathcal{B}_1$ .

It follows that we can find  ${}_{\mathcal{B}_2} \overleftarrow{P}_{\mathcal{B}_1}$  by forming the matrix whose columns are the coordinate representations of the basis vectors in  $\mathcal{B}_2$  in terms of  $\mathcal{B}_1$  and then inverting it!

## MATRICES OF LINEAR TRANSFORMATIONS UNDER DIFFERENT BASES

### NOTATION

We will begin with the general situation of a linear transformation from one finite-dimensional vector space to another where we change bases for both spaces and then look at the special case where the two spaces are identical and we are just changing the basis.

Let  $V$  and  $W$  be vector spaces with  $\dim(V) = n$  and  $\dim(W) = m$ .

Let  $\mathcal{V}_1 = \{\mathbf{v}_1^{(1)}, \mathbf{v}_2^{(1)}, \dots, \mathbf{v}_n^{(1)}\}$  and  $\mathcal{V}_2 = \{\mathbf{v}_1^{(2)}, \mathbf{v}_2^{(2)}, \dots, \mathbf{v}_n^{(2)}\}$  be bases for  $V$  and let  ${}_{\mathcal{V}_2} \overleftarrow{P}_{\mathcal{V}_1}$  be the change-of-coordinates matrix from  $\mathcal{V}_1$  to  $\mathcal{V}_2$ , so that  $[\mathbf{v}]_{\mathcal{V}_2} = {}_{\mathcal{V}_2} \overleftarrow{P}_{\mathcal{V}_1} [\mathbf{v}]_{\mathcal{V}_1}$ .

Similarly, let  $\mathcal{W}_1 = \{\mathbf{w}_1^{(1)}, \mathbf{w}_2^{(1)}, \dots, \mathbf{w}_m^{(1)}\}$  and  $\mathcal{W}_2 = \{\mathbf{w}_1^{(2)}, \mathbf{w}_2^{(2)}, \dots, \mathbf{w}_m^{(2)}\}$  be bases for  $W$  and let  ${}_{\mathcal{W}_2} \overleftarrow{P}_{\mathcal{W}_1}$  be the change-of-coordinates matrix from  $\mathcal{W}_1$  to  $\mathcal{W}_2$ , so that  $[\mathbf{w}]_{\mathcal{W}_2} = {}_{\mathcal{W}_2} \overleftarrow{P}_{\mathcal{W}_1} [\mathbf{w}]_{\mathcal{W}_1}$ .

Let  $T : V \rightarrow W$ , let  $T_1$  be the matrix representation for  $T$  with respect to the bases  $\mathcal{V}_1$  and  $\mathcal{W}_1$  and let  $T_2$  be the matrix representation for  $T$  with respect to the bases  $\mathcal{V}_2$  and  $\mathcal{W}_2$ .

## THE MAIN THEOREM

**Theorem** (Change of Basis Theorem).  $T_2 = {}_{w_2}P_{{}_{w_1}} T_1 {}_{v_2}P_{{}_{v_1}}^{-1}$

## PROOFS

We will prove this theorem two different ways.

**Proof Based on a Sequence of Linear Transformations:** We consider the sequence of linear transformations sending  $[\mathbf{v}]_{v_2} \xrightarrow{{}_{v_2}P_{{}_{v_1}}^{-1}} [\mathbf{v}]_{v_1} \xrightarrow{T_1} [T\mathbf{v}]_{w_1} \xrightarrow{{}_{w_2}P_{{}_{w_1}}} [T\mathbf{v}]_{w_2}$ .

The matrix for each transformation, looked at as a transformation  $\mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $\mathbb{R}^n \rightarrow \mathbb{R}^m$  or  $\mathbb{R}^m \rightarrow \mathbb{R}^m$  with standard basis, is shown above each arrow. In other words,

$$[\mathbf{v}]_{v_1} = {}_{v_2}P_{{}_{v_1}}^{-1} [\mathbf{v}]_{v_2}, [T\mathbf{v}]_{w_1} = T_1 [\mathbf{v}]_{v_1} \text{ and } [T\mathbf{v}]_{w_2} = {}_{w_2}P_{{}_{w_1}} [T\mathbf{v}]_{w_1}.$$

It follows that  $[T\mathbf{v}]_{w_2} = {}_{w_2}P_{{}_{w_1}} T_1 {}_{v_2}P_{{}_{v_1}}^{-1} [\mathbf{v}]_{v_2}$ . Since  $[T\mathbf{v}]_{w_2} = T_2 [\mathbf{v}]_{v_2}$ ,  $T_2$  must equal  ${}_{w_2}P_{{}_{w_1}} T_1 {}_{v_2}P_{{}_{v_1}}^{-1}$ .  $\square$

**Proof Based on the Images of the Basis Vectors:** Here we use the fact that the matrix for a linear transformation with respect to given basis has columns consisting of the images of the basis vectors. Thus, the  $j^{\text{th}}$  column for  $T_2$  is  $[T\mathbf{v}_j^{(2)}]_{w_2}$ . We then note the following sequences of equalities:

$$[T\mathbf{v}_j^{(2)}]_{w_2} = {}_{w_2}P_{{}_{w_1}} [T\mathbf{v}_j^{(2)}]_{w_1} = {}_{w_2}P_{{}_{w_1}} T_1 [\mathbf{v}_j^{(2)}]_{v_1} = {}_{w_2}P_{{}_{w_1}} T_1 {}_{v_2}P_{{}_{v_1}}^{-1} [\mathbf{v}_j^{(2)}]_{v_2}$$

Since  $[\mathbf{v}_j^{(2)}]_{v_2}$  is a column vector for which the  $j^{\text{th}}$  entry is 1 while every other entry is 0, the product of any matrix with  $[\mathbf{v}_j^{(2)}]_{v_2}$  is simply the  $j^{\text{th}}$  column of that matrix. Thus  ${}_{w_2}P_{{}_{w_1}} T_1 {}_{v_2}P_{{}_{v_1}}^{-1} [\mathbf{v}_j^{(2)}]_{v_2}$  is the  $j^{\text{th}}$  column of  ${}_{w_2}P_{{}_{w_1}} T_1 {}_{v_2}P_{{}_{v_1}}^{-1}$ .

In other words, we have shown that the  $j^{\text{th}}$  column of  $T_2$  is the same as the  $j^{\text{th}}$  column of  ${}_{w_2}P_{{}_{w_1}} T_1 {}_{v_2}P_{{}_{v_1}}^{-1}$ . It follows that  $T_2 = {}_{w_2}P_{{}_{w_1}} T_1 {}_{v_2}P_{{}_{v_1}}^{-1}$ .  $\square$

## SPECIAL CASES

**One Vector Space.** Suppose there is just one vector space  $V$  with bases  $\mathcal{B}_1$  and  $\mathcal{B}_2$ ,  $T : V \rightarrow V$  and  ${}_{\mathcal{B}_2}P_{{}_{\mathcal{B}_1}}$  is the change-of-coordinates matrix from  $\mathcal{B}_1$  to  $\mathcal{B}_2$ . Let  $T_{\mathcal{B}_1}$  be the matrix representation for  $T$  with respect to the basis  $\mathcal{B}_1$  and let  $T_{\mathcal{B}_2}$  be the matrix representation for  $T$  with respect to the basis  $\mathcal{B}_2$ . Then the Change of Basis Theorem reduces to

**Theorem.**  $T_{\mathcal{B}_2} = {}_{\mathcal{B}_2}P_{{}_{\mathcal{B}_1}} T_{\mathcal{B}_1} {}_{\mathcal{B}_2}P_{{}_{\mathcal{B}_1}}^{-1}$ .

**One Vector Space and One Basis is the Standard Basis.** Suppose  $V = \mathbb{R}^n$ ,  $A$  is the matrix with respect to the standard basis for a linear transformation  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $\mathcal{B}$  is another basis. Let  $P$  be the matrix whose  $j^{\text{th}}$  column is the coordinate representation of the  $j^{\text{th}}$  element of  $\mathcal{B}$  with respect to the standard basis. Then  $P$  is the change-of-coordinates matrix from  $\mathcal{B}$  to the standard basis and  $P^{-1}$  is the change of coordinates matrix from the standard basis to  $\mathcal{B}$ .

In this case, the Change of Basis Theorem says that the matrix representation for the linear transformation is given by  $P^{-1}AP$ . We can summarize this as follows.

**Theorem.** *Let  $A$  and  $B$  be the matrix representations for the same linear transformation  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  for the standard basis and a basis  $\mathcal{B}$  and let  $P$  be the matrix for which the  $j^{\text{th}}$  column is the coordinate representation of the  $j^{\text{th}}$  element of  $\mathcal{B}$  with respect to the standard basis. Then  $B = P^{-1}AP$  and  $A = PBP^{-1}$ .*

**Corollary** (Diagonal Matrix Representation). *Suppose  $A = PDP^{-1}$ , where  $D$  is a diagonal  $n \times n$  matrix. If  $\mathcal{B}$  is the basis for  $\mathbb{R}^n$  formed from the columns of  $P$ , then  $D$  is the  $\mathcal{B}$ -matrix of the transformation  $\mathbf{v} \mapsto A\mathbf{v}$ .*