Second Order Linear Differential Equations

©Alan H. Stein
The University of Connecticut at Waterbury
stein@math.uconn.edu   http://www.math.uconn.edu/~stein

Let L be a linear differential operation of the form

$$L[y] = ay'' + by' + cy, \tag{1}$$

where a, b and c are constants. A differential equation of the form

$$L[y] = 0 \tag{2}$$

is said to be homogeneous, whereas a differential equation of the form

$$L[y] = g(x), \tag{3}$$

where $g(x) \neq 0$, is said to be nonhomogeneous.

The Homogeneous Equation

Homogeneous differential equations of the form (2) can be solved easily using the characteristic equation

$$ar^2 + br + c = 0. \tag{4}$$

There are three cases to consider, depending on the nature of the solutions of the characteristic equation.

Case 1: The characteristic equation has two distinct real roots, r_1 and r_2.

In this case, the differential equation (2) has a fundamental set of solutions $\{e^{r_1 x}, e^{r_2 x}\}$, and its general solution is

$$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}, \tag{5}$$

where c_1 and c_2 are arbitrary constants. Every solution is in this form, and the particular values of the arbitrary constants can be found by plugging in the initial conditions.

Case 2: The characteristic equation has a single, double real root r.

In this case, the differential equation has a fundamental set $\{e^{rx}, xe^{rx}\}$ of solutions, and its general solution is in the form

$$y_h = (c_1 + c_2 x)e^{rx}.$$

Case 3: The characteristic equation has two complex solutions of the form $\alpha \pm i\beta$.

In this case, the differential equation has a fundamental set $\{e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x\}$ of solutions, and its general solution is in the form

$$y_h = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x).$$

The Nonhomogeneous Equation

Theory. The difference between any two solutions of the nonhomogeneous equation (3) is a solution of the homogeneous equation (2). As a consequence, if we can find a single particular solution y_p of (3), and we can find the general solution y_h of (2), then $y = y_p + y_h$ will be the general solution of the nonhomogeneous equation (3). Thus, if we can solve the homogeneous equation (2), we need only find any solution of the nonhomogeneous equation (3) in order to find all its solutions.

Furthermore, because of the linearity of L, if y_1 is a solution of $L[y] = g_1(x)$ and y_2 is a solution of $L[y] = g_2(x)$, then $L[y_1 + y_2] = L[y_1] + L[y_2] = g_1(x) + g_2(x)$, so $y_1 + y_2$ is a solution of $L[y] = g_1(x) + g_2(x)$. The consequence of this is that, if $g(x)$ is a sum $\sum g_i(x)$ of terms, then we can solve each of the differential equations $L[y] = g_i(x)$ individually and add their solutions together to obtain a solution of (3).

Thus, in studying how to solve (3), we can assume that $g(x)$ contains only a single term, since if it contains more than one term we can work on one term at a time.

We have learned two basic methods of finding particular solutions of the nonhomogeneous differential equation (3). They are the *Method of Undetermined Coefficients* and *Variation of Parameters.*

Undetermined Coefficients. The Method of Undetermined Coefficients works when $g(x)$ is a polynomial, an exponential function (such as ae^{kx}), a sine or cosine function (such as $a\sin bx$ or $a\cos bx$)), or a product of such functions.

If $g(x)$ is a polynomial, then we assume that the solution y_p is a polynomial of the same degree, with unknown coefficients. We then plug y_p (and its derivatives) into the original nonhomogeneous differential equation (3), getting polynomials on each side. We then use the same technique learned in *partial fractions* to determine the coefficients, and thus find y_p.

If $g(x)$ is an exponential e^{kx}, then we assume y_p is an exponential ce^{kx}, and use the same method to determine c.

If $g(x)$ is one of the trigonometric functions listed, then we assume that $y_p = c_1 \cos bx + c_2 \sin bx$ and use the same method to determine c_1 and c_2.

If $g(x)$ is a product of such functions, then we assume that y_p is a product of functions of the types described, and use the same method to determine the unknown constants.

The Catch. There is one potential problem with the Method of Undetermined Coefficients (besides the possibility of making an algebraic error). Sometimes, one of the terms in the assumed form of y_p happens to be a solution of the associated homogeneous differential equation (2). If that is the case, simply multiply the entire expression that you are assuming for y_p by x. You may have to repeat this more than once, if the resulting expression still contains a term which is a solution of the associated homogeneous equation.

Variation of Parameters. The method of Variation of Parameters requires that a fundamental set $\{y_1, y_2\}$ of solutions of the associated homogeneous differential equation (2) be found first. When this can be done, Variation of Parameters is guaranteed to yield a solution of the nonhomogeneous equation, even in cases where the Method of Undetermined Coefficients doesn't apply. This is a significant advantage. Another advantage is that it applies to any second order linear differential equation; it is not necessary that the coefficients be constant.

There is, as might be expected, a downside. The solution involves the calculation of two integrals, each of which may be difficult, if not impossible, to obtain in closed form in terms of elementary functions.

The idea behind this method is to assume that there is some particular solution y_p of the form $u_1y_1 + u_2y_2$, where y_1 and y_2 are, as described above, solutions of the associated homogeneous differential equation (2). One then plugs y_p into (3), makes the simplifying assumption that

$$u_1'y_1 + u_2'y_2 = 0, \tag{6}$$

and finds that y_p will satisfy (3) provided that

$$u_1'y_1' + u_2'y_2' = g(x). \tag{7}$$

One can solve (6) and (7) simultaneously for u_1' and u_2' using Cramer's Rule, obtaining

$$u_1' = -\frac{y_2 g(x)}{W(y_1, y_2)}, \quad u_2' = \frac{y_1 g(x)}{W(y_1, y_2)}, \tag{8}$$

where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2.

Therefore, to solve the nonhomogeneous differential equation (3), we can first find u_1' and u_2' using (8), integrate to find u_1 and u_2, and let $y_p = u_1 y_1 + u_2 y_2$.

Reminder

This handout is not complete. It does not include any theory, and it does not include some special cases discussed in class or in the text.

Make sure you understand the reasoning that led to the methods described here, and that you are able to solve both homogeneous and nonhomogeneous differential equations, obtaining both general solutions and particular solutions satisfying given initial conditions, without referring to these notes or any other outside resource.