

Lines and Planes

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LINES

The key idea related to equations of lines is you always need to have a vector, say \mathbf{x}_0 , from the origin to a point of the line and a vector, say \mathbf{v} , parallel to the line. The simplest form, conceptually, for an equation of such a line is then the vector (parametric) form

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{v}t.$$

For example, if have a vector $2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ from the origin to a point on the line, and a vector $4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ parallel to the line, then we have an equation $x = 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k} + t(4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})$ for the line.

Other forms for equations of the same line are the scalar parametric form

$$\begin{aligned}x &= 2 + 4t \\y &= 3 - 3t \\z &= -5 + 2t\end{aligned}$$

and the scalar parametric form

$$\frac{x - 2}{4} = \frac{y - 3}{-3} = \frac{z + 5}{2}.$$

The main thing to remember is that, whatever information about a line is known, *you need to somehow or other find a point on the line and a vector parallel to the line.*

For example, if you know two points on the line, you can use the vector from one to the other as the vector parallel to the line. If you know a line parallel to the line you're interested in, you can use any vector parallel to that line.

PLANES

Every plane has a vector orthogonal (perpendicular) to it, called a normal vector and usually denoted by the letter \mathbf{n} . (Actually, each plane has infinitely many normal vectors, but each is a scalar multiple of every other one and any one of them is just as useful as any other one.) The useful fact about normal vectors is that if you draw a vector connecting any two points in the plane, then the normal vector will be orthogonal to it. Since it is easy to check whether two vectors are orthogonal using the dot product, this fact can be used to check whether a given point is on the plane, provided that at least one point on the plane is known.

Suppose we have a plane Π , a normal vector $\mathbf{n} \perp \Pi$ and a point $P_0(x_0, y_0, z_0) \in \Pi$ and we want to determine whether another point $P(x, y, z)$ is on the plane. We merely have to consider the vector \mathbf{v} with initial point P_0 and endpoint P and calculate the dot product $\mathbf{n} \cdot \mathbf{v}$. Clearly, the point P will be on the plane if and only if the dot product is zero.

To carry out the calculations, let $\mathbf{x}_0 = (x_0, y_0, z_0)$ and $\mathbf{x} = (x, y, z)$. Then $\mathbf{v} = \mathbf{x} - \mathbf{x}_0$ and P will be on the plane if and only if $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$ or, equivalently, if $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{x}_0$ and this gives an equation for the plane.

For example, suppose we have a plane with normal vector $\mathbf{n} = (2, -3, 5)$ containing the point $(4, 7, 8)$. Then the plane will have the equation $(2, -3, 5) \cdot \mathbf{x} = (2, -3, 5) \cdot (4, 7, 8)$. Since $(2, -3, 5) \cdot (4, 7, 8) = 20$, this can be written in either of the forms

$$(2, -3, 5) \cdot \mathbf{x} = 20 \text{ or } 2x - 3y + 5z = 20.$$

Thus, the main fact to remember is that, whenever you need an equation of a plane, you need to obtain a point on the plane and a normal vector.

SUMMARY

If you remember to do the following, you will not have any difficulty finding equations of points and planes.

To find an equation of a line, first find a point on the line and a vector parallel to the line.

To find an equation of a plane, first find a point on the plane and a vector normal to the plane.