Demonstrating a Line is a Tangent

If one is interested in the slope of the line tangent to the graph of the equation $y = x^2$ at the point $(3, 9)$, one cannot find it directly. However, it seems reasonable that if one takes a point (x, x^2) on the graph which is close to the point $(3, 9)$, then the slope of the line l through $(3, 9)$ and (x, x^2) will be close to the slope of the tangent.

The slope of l is $\dfrac{x^2 - 9}{x - 3}$. Since $x^2 - 9 = (x+3)(x-3)$ and $x - 3 \neq 0$ (since (x, x^2) is close to but not coincident with $(3, 9)$ and thus $x \neq 3$), the slope of l can be written as $x + 3$.

Since (x, x^2) is taken to be close to $(3, 9)$, x must be close to 3 and it would appear that $x + 3$, and hence the slope of l, must be close to 6.

Note: Fairly soon, we will study the concept of a limit and observe that we actually calculated $\lim_{x\to 3} \dfrac{x^2 - 9}{x - 3} = 6$. *Later, we will study the concept of a derivative and observe that we have actually calculated* $\dfrac{d}{dx}(x^2)\,|_{x=3} = 6$.

Since it would appear that the slope of l is also close to the slope of the tangent line, the slope of the tangent line is probably also close to 6 and if one had to guess at the slope of the tangent line the most reasonable guess would be 6. Indeed, it can be demonstrated, using concepts from algebra and analytic geometry, that the slope is 6. The following is such a demonstration.

The line through the point $(3, 9)$ with slope 6 has equation $y - 9 = 6(x - 3)$. Simplifying, we can obtain the Slope-Intercept form for the equation of the tangent: $y - 9 = 6x - 18$, $y = 6x - 9$. This is slightly more convenient than the Point-Slope form to use for the purpose at hand.

To demonstrate the tangent line has slope 6, it suffices to show the equation obtained is the equation of the tangent line.

If one graphs the equation $y = x^2$ and visualizes the line tangent at the point $(3, 9)$, it is obvious that the tangent line lies *below* the graph except at the point $(3, 9)$ and also that the tangent line is the only line with that property.

We can thus demonstrate that the slope of the tangent line is 6 by demonstrating the graph of $y = 6x - 9$ has that property.

Certainly, $(3, 9)$ falls on the graph of $y = 6x - 9$, so we need only show that all other points on the line fall below the graph of $y = x^2$.

To do this, it suffices to show that for any real number $x \neq 3$, the point on the line $y = 6x - 9$ with first coordinate x lies below the point on the graph of $y = x^2$ with first coordinate x.

This can be shown by demonstrating that the point on the line $y = 6x - 9$ with first coordinate x has a second coordinate smaller than the second coordinate of the point on the graph of $y = x^2$ with first coordinate x.

Since the second coordinate of the point on the line is $6x - 9$ and the second coordinate of the point on the curve is x^2, this boils down to showing that $6x - 9$ is smaller than x^2.

Of course, $6x-9$ is smaller than x^2 if and only if $x^2-(6x-9)$ is positive. But $x^2-(6x-9) = x^2 - 6x + 9 = (x - 3)^2$, which is certainly positive when $x \neq 3$.

This completes the demonstration (it does not quite qualify as a proof) that the line tangent to the graph of $y = x^2$ has slope 6.