Differentials and Tangent Approximations

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Differentials

Differentials are used to estimate how much a function changes when the value of the independent variable changes as small amount.

A function $y = f(x)$ should be thought of as a dynamic object, with the value of the dependent variable y changing when the value of the independent variable x changes. The derivative may be thought of as the rate at which the dependent y changes in relation to the independent variable x. For example, if $\frac{dy}{dx} = 5$, then y will change about 5 times as fast as x, while if $\frac{dy}{dx} = -3$, then y will change about three times as fast as x, but in the opposite direction, so that if x increases by .2, then y will decrease by about .6, while if x decreases by .4, then y will increase by about 1.2. This idea leads to the concept of a differential.

Consider a function $y = f(x)$ defined and differentiable at a point x_0. Let $y_0 = f(x_0)$. Suppose x changes by an amount Δx, from x_0 to $x_1 = x_0 + \Delta x$. Then y will change by an amount we denote by $\Delta y = f(x_0 + \Delta x) - f(x_0)$. (Note that this is the numerator in the difference quotient used to define a derivative.) We then define two entities, dx and dy, with dx being identical to Δx and dy being an approximation for Δy.

Definition 1. *$dx = \Delta x$ is called the differential of x.*
$dy = f'(x_0)dx$ is called the differential of y.

The key fact about differentials is that dy is a very good approximation to Δy if Δx is small. This is stated precisely in the following theorem.

Theorem 1. $\lim_{\Delta x \to 0} \frac{\Delta y - dy}{\Delta x} = 0.$

Example. We will estimate how much the function $y = f(x)$ defined by the formula $f(x) = \sqrt{x}$ changes as x increases from 4 to 4.03. We can do it by calculating dy for $x_0 = 4$ and $\Delta x = .03$.

Since $f'(x) = \frac{1}{2\sqrt{x}}$, we find that $f'(4) = 1/4$ and thus $dy = (1/4)(.03) = .12$. We thus can estimate that $f(x)$ increases by approximately .12.

A Note About Notation. It is easy to remember the formula $dy = \frac{dy}{dx} \cdot dx$. However, remember that $\frac{dy}{dx}$ really stands for $\left.\frac{dy}{dx}\right|_{x_0} = f'(x_0)$. In other words, you must evaluate $\frac{dy}{dx}$, not just write down a formula for it.

Tangent Approximations

Tangent approximations can be used to approximate the value of a function at a point in the domain that is close to a point at which both the function and its derivative are easily evaluated. The idea is really quite simple, provided that you know how to (1) write an equation of a line and (2) use the derivative to find the slope of a tangent line.

Consider a function f which is defined and differentiable at a point x_0 and suppose we need to approximate $f(z)$ for some value z close to x_0.

Start by calculating $y_0 = f(x_0)$. This gives the coordinates (x_0, y_0) of the point of tangency.

Next find the derivative $f'(x)$ and evaluate $m = f'(x_0)$. This gives the slope of the tangent line.

Armed with a point and the slope, find an equation of the tangent line. (The simplest method is to use the point-slope formula $y - y_0 = m(x - x_0)$, but you may use any method that works.) After you have an equation, solve explicitly for y to put the equation in the form $y = T(x)$.

You can then approximate $f(z)$ with $T(z)$.

Example. Suppose $f(x) = \sqrt{x}$ and you wish to approximate $f(4.01)$. Observe that 4 is close to 4.01, so use the tangent to the graph of f at 4.

Start by calculating $f(4) = \sqrt{4} = 2$, so use the point $(4, 2)$.

Next, find $f'(x) = \dfrac{1}{2\sqrt{x}}$ and use it to obtain the slope $m = f'(4) = \dfrac{1}{2\sqrt{4}} = 1/4$ of the tangent line.

Use the point $(4, 2)$ and the slope $1/4$ to obtain an equation $y - 2 = \dfrac{1}{4}(x - 4)$ of the tangent line. Solve explicitly for y to obtain $y = T(x) = 2 + \dfrac{1}{4}(x - 4)$.

Finally, approximate $f(4.01)$ by $T(4.01) = 2 + \dfrac{1}{4}(x - 4) = 2 + \dfrac{1}{4}(.01) = 2.04$.