Integration Using the Substitution Technique

©Alan H. Stein
The University of Connecticut at Waterbury
stein@math.uconn.edu http://www.math.uconn.edu/∼stein

Introduction

The substitution technique (sometimes referred to as the chain rule for integration) is used to integrate functions which are derivatives of functions which require the chain rule in order to differentiate. (Remember that, when trying to integrate, the proper technique, if any, is not always readily apparent, so you must always be prepared to try something else if the technique you are using does not work.)

The Procedure

The substitution technique may be divided into the following steps. Every step but the first is purely mechanical. With a little bit of practice (in other words, make sure you do the homework problems assigned), you should have no more difficulty carrying out a substitution than you should be having by now when you differentiate.

Note: In the following, we will assume that you are trying to calculate an integral $\int f(x)dx$. If the dummy variable is called something other than x, then some of the names you would use for variables might be different.

(1) Choose a substitution $u = g(x)$.
    Some suggestions on how to choose a substitution will be made later.
(2) Calculate the derivative $\frac{du}{dx} = g'(x)$.
(3) Treating the derivative *as if it were a fraction*, solve for dx:

$$\frac{du}{dx} = g'(x), \quad du = g'(x)dx, \quad dx = \frac{du}{g'(x)}.$$

(4) Go back to the original integral and replace $g(x)$ by u and replace dx by $\frac{du}{g'(x)}$.
(5) Simplify.
    Every incidence of x should cancel out at this step. If not, you will need to try another substitution. Make sure that you simplify properly—this is the easiest step to make mistakes during.
(6) Integrate.
(7) Replace u by $g(x)$ in your result.
(8) Check your answer (of course).

Choosing an Appropriate Substitution

This is the only non-routine part of carrying out a substitution, but should not be at all difficult for any student who has mastered the art of differentiation. There are two basic tactics for choosing a substitution. Each will work in the vast majority of cases where a routine substitution is needed. Since neither will work in all cases, you need to be comfortable with both. Therefore, you should try using both methods on the same problem wherever possible. (There are quite a few non-routine substitutions that are used in special situations. They need to be learned separately.)

The First Method. The method most students probably find easiest to use relies on familiarity with the chain rule for differentiation. In order to decide on a useful substitution, look at the integrand and pretend that you are going to calculate its derivative rather than its integral. (You usually don't actually have to write anything down—you can usually just visualize the steps.) Look to see if there is any step during which you would have to use the chain rule. If so, think of the decomposition you would have to make, i.e. the step where you would write down something like $y = f(u), \quad u = g(x)$. Try the substitution $u = g(x)$.

Examples.

(1) $\int x \sin(x^2 + 1)\, dx$.

If you were differentiating $x \sin(x^2 + 1)$, you would first use the product rule, but eventually you would have to calculate the derivative of $\sin(x^2 + 1)$. To do so, you would need to write something like $y = \sin u, \quad u = x^2 + 1$. Try the substitution $u = x^2 + 1$.

(2) $\int \dfrac{x^4}{(x^5 + 18)^{10}}\, dx$.

If you were differentiating $\dfrac{x^4}{(x^5 + 18)^{10}}$, you would eventually have to calculate the derivative of $(x^5 + 18)^{10}$ using the chain rule. To do so, you would need to write something like $y = u^{10}, \quad u = x^5 + 18$. Try the substitution $u = x^5 + 18$.

The Second Method. This method involves looking at parts of the integrand and observing whether the derivative of part of the integrand equals some other factor of the integrand. If so, u may be substituted for that part. (In deciding, you may ignore constant factors, since they are easy to manipulate around.)

Examples.

(1) $\int x \sin(x^2 + 1)\, dx$.

$\dfrac{d}{dx}(x^2 + 1) = 2x$. We ignore the constant factor 2 in $2x$ and look to see whether x is a factor in the integrand. Since it is, we make the substitution $u = x^2 + 1$.

(2) $\int \dfrac{x^4}{(x^5 + 18)^{10}}\, dx$.

$\dfrac{d}{dx}(x^5 + 18) = 5x^4$. We ignore the constant factor 5 in $5x^4$ and look to see whether x^4 is a factor in the integrand. Since it is, we make the substitution $u = x^5 + 18$.

Note that if we calculated $\dfrac{d}{dx}(x^4) = 4x^3$, we would not find x^3 as a separate factor of the integrand and so would not try the substitution $u = x^4$ (except if we became desperate). We might also take a look at $\dfrac{d}{dx}((x^5 + 18)^{10})$, but simpler substitutions are usually more effective than more complex substitutions.

A Complete Example

$$I = \int \tan(2x+1)\sec^2(2x+1)\,dx.$$

(Note how I have given the integral a label, I, so that I can refer to it without completely rewriting it. This is not absolutely necessary, but can be helpful and save time.)

We will calculate the integral following the steps outlined.

(1) Choose a substitution $u = g(x)$.

Observing that, were we to calculate the derivative of the integrand, we would eventually need to use the chain rule to differentiate $\tan(2x+1)$ and need to write $y = \tan u, \quad u = 2x+1$, we choose the substitution $u = 2x+1$.

(2) Calculate the derivative $\dfrac{du}{dx} = g'(x)$.

$\dfrac{du}{dx} = 2.$

(3) Treating the derivative *as if it were a fraction*, solve for dx:

$du = 2dx, \quad dx = \dfrac{du}{2}.$

(4) Go back to the original integral and replace $g(x)$ by u and replace dx by $\dfrac{du}{g'(x)}$.

$$I = \int \tan(2x+1)\sec^2(2x+1)\,dx = \int \tan u \sec^2 u \frac{du}{2}.$$

(5) Simplify.

$$I = \frac{1}{2}\int \tan u \sec^2 u\,du.$$

(6) Integrate.

To integrate, we need to make another substitution. Once again, I will follow precisely the steps outlined above, until I have calculated the integral in terms of u.

(a) Choose a substitution $v = g(u)$.

If we were going to differentiate the integrand, we would need the chain rule to differentiate $\sec^2 u$ and would write $y = v^2, \quad v = \sec u$, we make the substitution $v = \sec u$.

Note that, using the other method, we might have observed that $\dfrac{d}{du}(\tan u) = \sec^2 u$ and made the substitution $v = \tan u$. That would also work—try it.

(b) Calculate the derivative $\dfrac{dv}{du} = g'(u)$.

$\dfrac{dv}{du} = \dfrac{d}{du}(\sec u) = \sec u \tan u.$

(c) Treating the derivative *as if it were a fraction*, solve for du:

$$dv = \sec u \tan u du, \quad du = \frac{dv}{\sec u \tan u}.$$

(d) Go back to the original integral and replace $g(u)$ by v and du by $Fracdvg'(u)$.

$$I = \frac{1}{2}\int \tan u \sec^2 u\, du = \frac{1}{2}\int (\tan u) v^2 \frac{dv}{\sec u \tan u}.$$

(e) Simplify.
The terms involving $\tan u$ cancel, so we obtain

$$I = \frac{1}{2}\int \frac{v^2}{\sec u} dv.$$

At first glance, not all the terms involving u have cancelled out. However, a second glance reminds us that $v = \sec u$, so we may write

$$I = \frac{1}{2}\int \frac{v^2}{v} dv = \frac{1}{2}\int v dv.$$

(f) Integrate.
$I = \frac{1}{2}\int v dv = \frac{1}{2} \cdot v^2/2 = v^2/4.$

(g) Replace v by $g(u)$ in your result.
$I = \sec^2 u/4.$

(h) Check your answer.
We skip this step here, since we can check our final answer later. We can now go back to the original procedure.

(7) Replace u by $g(x)$ in your result.

$$I = \sec^2 u/4 = \frac{1}{4}\sec^2(2x+1).$$

(8) Check your answer.

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{4}\sec^2(2x+1)\right) &= \frac{1}{4} \cdot 2\sec(2x+1) \cdot \sec(2x+1)\tan(2x+1) \cdot 2 \\ &= \tan(2x+1)\sec^2(2x+1). \end{aligned}$$