Sequences and Series

©Alan H. Stein
The University of Connecticut at Waterbury
stein@math.uconn.edu http://www.math.uconn.edu/∼stein

Sequences

A sequence is simply a function whose domain is the set of positive integers. We generally use slightly different notation for sequences than for other functions, so that we may write $a_n = n^2$ rather than $f(x) = x^2$. This helps us recognize when we are dealing with a sequence rather than another type of function. To further distinguish sequences, we generally call the independent variable an index.

When dealing with sequences, we are generally most interested in answering two questions.

(1) Does the sequence converge?
(2) What limit does the sequence converge to?

Often, we can determine whether a sequence converges, and what it converges to, by methods analogous to those used to determine limits at infinity for other functions. For example, if we consider the sequence $\{a_n\}$ defined by the formula $a_n = \dfrac{n + 5\ln(n^n)}{\sqrt{n} - n\ln(n)}$ and want to find $\lim_{n\to\infty} a_n$ we will, *or at least we should*, get exactly the same answer as we would if we considered the function $f(x) = \dfrac{x + 5\ln(x^x)}{\sqrt{x} - x\ln(x)}$ and wanted to find $\lim_{x\to\infty} f(x)$.

This similarity is emphasized by the following lemma.

Lemma 1. *Consider a monotonic function $f : \mathbb{R} \to \mathbb{R}$ and a sequence $a_n : \mathbb{Z}^+ \to \mathbb{R}$. If $a_n = f(n)$ for all $n \in \mathbb{Z}$ and $\lim_{x\to\infty} f(x)$ exists, then so does $\lim a_n$ and the latter is equal to the former.*

Based on this lemma, we can often even make use of L'Hopital's Rule to obtain the limit of a sequence, *even though we can't actually use L'Hopital's Rule on the sequence itself!* We were able to make use of this to determine the following important limits.

(1) $\lim \dfrac{\ln n}{n} = 0$
  (a) More generally, $\lim \dfrac{(\ln n)^\alpha}{n^\beta} = 0$ whenever $\beta > 0$.
(2) $\lim \dfrac{n}{\exp n} = 0$
  (a) More generally, $\lim \dfrac{n^\alpha}{\beta^n} = 0$ whenever $\beta > 1$.

We also, without the use of L'Hopital's Rule, make use of another limit in a similar vein.

(3) $\lim \dfrac{\alpha^n}{n!} = 0$ for any $\alpha \in \mathbb{R}$.

We can easily see this last by writing $\dfrac{\alpha^n}{n!} = \dfrac{\alpha}{1} \cdot \dfrac{\alpha}{2} \cdots \dfrac{\alpha}{n}$. When n is very large, most of these factors will be extremely small, making the product very small regardless of how large

α is. One quick way of verifying that is to choose some integer $N \geq 2\alpha$, let $B = \dfrac{\alpha^N}{N!}$ and observe that $\dfrac{\alpha^n}{n!} \leq \dfrac{B}{2^{n-N}}$, which obviously goes to zero as $n \to \infty$.

The upshot of these particular limits is the rule of thumb that logarithms are much smaller than powers, which in turn are much smaller than exponentials, which then in turn are much smaller than factorials.

We have one other key lemma which sometimes enables us to answer the first question in the affirmative even when we can't answer the second question.

Lemma 2. *If a sequence $a_n : \mathbb{Z}^+ \to \mathbb{R}$ is monotonic and bounded, then it is convergent.*

It is rare that we actually use this theorem on specific sequences, but it is a useful tool for proving various tests for convergence of infinite series.

Sequence Summary. When actually calculating a limit $\lim_{n\to\infty} a_n$, we work almost exactly as if we were calculating a limit $\lim_{x\to\infty} f(x)$. We also use a theorem about the convergence of monotonic sequences for theoretical purposes when examining series.

Note that just as not every function is called f and not every independent variable is called x, not every sequence is called a and not every index is called n.

Series

A series is an expression $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$.

Actually, a series is more generally an expression $\sum_{n=\alpha}^{\infty} a_n$, but it's easier to pretend for now that α is always 1. Indeed, we'll often write merely write $\sum a_n$, assuming that n goes from some integer α to ∞ and recognizing that the value of α is irrelevant.

Remember that a series is not a sequence. A series actually involves two separate sequences, the sequence $\{a_n\}$ of its terms and the sequence $\{s_n\}$ of its partial sums, where $s_n = \sum_{k=1}^{n} a_k$. We are primarily interested in the convergence of its sequence of partial sums, not in the convergence of its terms! That, indeed, is how the convergence of a series is series whose terms approach zero. This very useful lemma merely eliminates many series from contention in the *convergence sweepstakes*, but does not show that a series converges in and of itself. To do that, we need to use other tests.

Most convergence tests are actually tests for the convergence of *positive term series.*

Theorem 4. *If* $\sum |a_n| < \infty$*, then* $\sum a_n$ *converges.*

In other terminology, if a series is *absolutely convergent*, then it is convergent. Thus, even if a series $\sum a_n$ is not a positive term series, we can test the associated positive term series $\sum |a_n|$ for convergence as a substitute.

Positive Term Series

Here we consider the tests for convergence of positive term series, all of which can then be used to test for absolute convergence of a series if it's not a positive term series. We will assume that each of the series considered in this section are positive term series.

Most of the tests are based on the following relatively straightforward consequence of the lemma about the convergence of monotone sequences.

Lemma 5. *A positive term series is convergent if and only if its sequence of partial sums is bounded.*

Based on this lemma, we are able to prove two convergence theorems and the Integral Test.

Theorem 6. *(Convergence Test I) Consider positive terms series* $\sum a_n$ *and* $\sum b_n$*, where* $0 \le a_n \le b_n$ *for all large enough integers* n*.*

(1) *If* $\sum b_n < \infty$ *then* $\sum a_n < \infty$*.*
(2) *If* $\sum a_n = \infty$ *then* $\sum b_n = \infty$*.*

Theorem 7. *(Convergence Test II) Consider positive terms series* $\sum a_n$ *and* $\sum b_n$ *where* $\lim \dfrac{a_n}{b_n} = \gamma \neq 0$ *for some* $\gamma \in \mathbb{R}$*. Then either both series converge, or both series diverge.*

Theorem 8. *(Integral Test) Suppose* $a_n = f(n)$ *for all large enough* $n \in \mathbb{Z}^+$ *for some monotonic function* $f : \mathbb{R} \to \mathbb{R}$*. Then* $\sum a_n < \infty$ *if and only if the improper integral* $\int^\infty f(x)\,dx < \infty$*.*

Note that, in the integral, the lower limit was omitted. That is, technically, improper notation. It was used to make the point that the lower limit is irrelevant and just has to be large enough to be able to try to evaluate the improper integral. Also note the similarity between these two comparison tests and the two comparison tests for convergence of improper integrals. That similarity is, of course, no accident.

In trying to determine the convergence of a positive term series, the first thing we need to do, unless the series is either geometric or a P-Series, is that see if there is some baseline series we can compare it to. Of course, in order to do that, we need a collection of baseline series we are familiar with. These are provided by the following.

Theorem 9. *(P-Test)*

$$\sum \frac{1}{n^p} \begin{cases} < \infty & \text{if } p > 1 \\ = \infty & \text{if } p \le 1 \end{cases}$$

Theorem 10. *(Geometric Series) A geometric series* $\sum ar^{n-1} = a + ar + ar^2 + \ldots$ *converges if and only if* $|r| < 1$*, in which case it converges to* $\dfrac{a}{1-r}$.

Note how the P-Test is very similar to the test for the convergence of improper integrals that goes by the same name, and that it can easily be verified using the Integral Test.

When confronted with a series, we start by trying to see if it similar to a P-Series, generally by using the same idea we use to intuitively guess at the limit of a sequence with the same terms. That is, if we are looking at a fraction, we ignore all but the most significant terms of both the numerator and the denominator.

For example, if we were interested in the series $\sum \dfrac{n^2 - n}{n^3\sqrt{n} + 5n + 3}$, we would look at $\sum \dfrac{n^2}{n^3\sqrt{n}} = \sum \dfrac{1}{n^{3/2}}$. Since the latter series is a convergent P-Series, we expect the former to converge as well.

This, however, is not actually a valid argument. It needs to be massaged into a valid argument. That can be done in this case, since it is obvious that $\dfrac{n^2 - n}{n^3\sqrt{n} + 5n + 3} < \dfrac{1}{n^{3/2}}$, but not always. In such cases, we generally resort next to the ratio test.

Theorem 11. *(Ratio Test) Consider a positive term series* $\sum a_n$ *such that* $\lim \dfrac{a_{n+1}}{a_n}$ *exists. If that limit is less than 1, then the series converges, while the series diverges if that limit is greater than 1.*

The Ratio Test generally works well on series that are almost geometric, such as $\sum \dfrac{n}{2^n}$, or which are much smaller than geometric series but may be difficult to apply the comparison tests to, such as $\sum \dfrac{2^n}{n!}$, but does not work on series that are close to being P-Series, such as the example above.

If the Comparison Tests don't seem to work, and the Ratio Test doesn't work, the last thing to try is the Integral Test.

Positive Term Series Summary. Start examining a positive term series by checking whether its terms approach zero. If they do, first check whether it's either a geometric series or a P-Series. If it's not, try to apply one of the comparison tests. The next thing to try is generally the ratio test, followed by the integral test.

Other Series

If a series $\sum a_n$ is not a positive term series, try checking for absolute convergence. This can be done by the strategy above, applied to the associated positive term series $\sum |a_n|$.

If the series is not absolutely convergent, then the best hope for showing that it's conditionally convergent comes if it's an alternating series. If so, one can often determine its convergence almost at sight, based on the Alternating Series Test.

Theorem 12. *(Alternating Series Test) If* $a_n \geq 0$ *for all large enough integers* n *and the sequence* $\{a_n\}$ *eventually converges monotonically to zero, then the alternating series* $\sum(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \ldots$ *converges.*

This test works easily on alternating series such as the Alternating Harmonic Series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \pm \ldots$.

Final Summary

When analyzing both sequences and series–which you need to remember are different creatures, it's generally a good idea to try to use rough estimates to make a preliminary decision about whether they converge or diverge, and then use more precise methods to confirm that preliminary decision.