Line Integrals

©Alan H. Stein
The University of Connecticut at Waterbury
stein@math.uconn.edu http://www.math.uconn.edu/∼stein

Line integrals take the general idea of a Riemann Sum and an integral one step further.

For ordinary integrals, we take a function f defined on an interval $[a, b]$ in the real line, partition the interval into subintervals, choose a point c_k from each interval, multiply the value $f(c_k)$ of the function at that point by the basic dimension of that subinterval, its length Δx_k, add them up to get the Riemann Sum $\sum_k f(c_k)\Delta x_k$ and in some sort of limiting process come up with the definite integral $\int_a^b f(x)dx$.

For double integrals, we take a function f defined on a plane region $\mathcal{E}$, partition the region into subregions, choose a point $P_k = (x_k, y_k)$ from each subregion, multiply the value $f(P_k)$ of the function at that point by the basic dimension of that subregion, its area ΔA_k, add them up to get the Riemann Sum $\sum_k f(P_k)\Delta A_k$ and in some sort of limiting process come up with the double integral $\int_{\mathcal{E}} f(x, y)dA$.

For triple integrals, we take a function f defined on a solid region $\mathcal{E}$, partition the region into subregions, choose a point $P_k = (x_k, y_k, z_k)$ from each subregion, multiply the value $f(P_k)$ of the function at that point by the basic dimension of that subregion, its volume ΔV_k, add them up to get the Riemann Sum $\sum_k f(P_k)\Delta V_k$ and in some sort of limiting process come up with the triple integral $\int_{\mathcal{E}} f(x, y, z)dV$.

For line integrals, we take a function f defined on a curve $\mathcal{C}$, partition the curve into subarcs, choose a point $P_k = (x_k, y_k)$ from each subarc, multiply the value $f(P_k)$ of the function at that point by the basic dimension of that subarc, its length Δs_k, add them up to get the Riemann Sum $\sum_k f(P_k)\Delta s_k$ and in some sort of limiting process come up with the line integral $\int_{\mathcal{C}} f(x, y)ds$.

For a variation, we replace the length Δs_k of the subarc by the length Δx_k of the projection of the subarc on the $x-$ axis, getting the Riemann Sum $\sum_k f(P_k)\Delta x_k$ and the line integral $\int_{\mathcal{C}} f(x, y)dx$, or perhaps by the projection on the $y-$axis to get a line integral $\int_{\mathcal{C}} f(x, y)dy$, or even a sum of terms of each type to get a line integral of the form $\int_{\mathcal{C}} Pdx + Qdy$.

To actually calculate a line integral, we need a parametrization of the curve $\mathcal{C}$, say $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, and mechanically replace the differential ds by $\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$, the differential dx by $\frac{dx}{dt}dt$ and the differential dy by $\frac{dy}{dt}dt$, and put in limits from a to b to get an ordinary integral in terms of t.

Important: Try to understand why that mechanical substitution actually works.