Limits

©Alan H. Stein
The University of Connecticut at Waterbury
stein@math.uconn.edu   http://cc.math.uconn.edu/~stein

The first concept that sets Calculus apart from the mathematics you have studied before is that of a limit. Limits may be approached on at least three levels.

- A purely computational level.
- An intuitive level.
- A theoretical level.

We will be most concerned with the first two levels. Those are the only levels we will deal with in these notes. You will need to be able to compute certain types of limits routinely and you will also need to develop some intuitive feeling for the meaning of limits in order to understand other concepts, such as derivatives and integrals, based on them. You will be able to squeak by (at this level) without being able to deal with the formal definitions of various types of limits, or being able to prove the properties of limits, but you must make an attempt to understand the definitions and do the proofs.

The intuitive idea of a limit is the following:

Definition 1 (Intuitive Definition of a Limit). *A function $f(x)$ has limit β as x approaches α if the values of $f(x)$ tend to be very close to β when x is very close to α. (Note that the value of $f(\alpha)$ is irrelevant.)*

Caution: Try not to be sidetracked by the appearance of Greek letters, which are used often as a convenience in mathematics. Concentrate on *substance* rather than *form*.

Notation 1. *We write* $\lim_{x\to\alpha} f(x) = \beta$. *The number α is called the* limit point*, while the number β is referred to as the* limit*.*

Examples

Let $f(x) = x^2+3$. Clearly, if x is close to 4, then $f(x)$ is close to 19, so we say $\lim_{x\to 4} f(x) = 19$ or $\lim_{x\to 4} x^2 + 3 = 19$.

Caution: Many students incorrectly write things like $\lim = 19$ or $\lim_{x\to 4} = 19$. Of course, neither makes any sense. It is important to use the same notation as the rest of the world. This is not because the notation the rest of the world uses is necessarily any better than the notation you may wish to use, but because nobody else will understand you unless you use accepted notation.

You may have noticed that $f(4) = 19$, which is the same as the limit, and wonder what the big deal about limits is. However, recall that the value of a function at its *limit point* is supposed to be irrelevant. So consider the following two examples.

Suppose $f(x) = \begin{cases} x^2+3 & \text{if } x \neq 4 \\ 0 & \text{if } x = 4 \end{cases}$. Since the value of f at the *limit point* is irrelevant, we can say that $f(x)$ is clearly close to 19 if x is close to 4 (since $x^2 + 3$ is clearly close to 19 if x is close to 4), and thus conclude here that $\lim_{x\to 4} f(x) = 19$ even though $f(4) \neq 19$.

Since the meaning of a limit should not depend on either the name of the function (which does not always have to be f) or the name of the independent variable (which does not always have to be x), in the next example we will vary the names to protect the innocent.

Suppose $g(t) = \frac{t^2-9}{t-3}$ and consider $\lim_{t\to 3} g(t)$.

Note that $g(t)$ is undefined when $t = 3$, but that is irrelevant since the limit should not depend on the value or even the existence of the function at the limit point.

Note further that, since $t^2 - 9 = (t+3)(t-3)$, we can simplify the formula for g and write $g(t) = t + 3$ if $t \neq 3$. It is thus obvious that $g(t)$ is close to 6 if t is close to 3, and thus $\lim_{t\to 3} g(t) = 6$ even though $g(3)$ itself is undefined.

Most important limits that come up in practice are similar to the one above, which demonstrates the single strategy that is used for the calculation of almost all limits you will come across:

If the limit of a function is not *obvious*, then simplify the formula for that function until the limit *is obvious*.

Most often, the simplification will involve factoring and cancelling, as above. Sometimes it will involve rationalizing. (Ironically, it is almost always the *numerator* that you will need to rationalize. In practice, you will almost never need to rationalize a denominator. Fortunately, the technique is the same.) Sometimes it will involve simplifying complex rational expressions.

In any event, the key to calculating almost all limits is the routine simplification of algebraic or, on occasion, trigonometric, expressions. You will spend 90% of your time on these manipulations, which you studied extensively in high school algebra and trigonometry. If you do them correctly, you will find the *Calculus* part routine.